Computational Mathematics with SageMath
Prof. Ajit Kumar
Department of Mathematics
Institute of Chemical Technology, Mumbai

Lecture – 23
Application of integration using SageMath

Welcome to the 23rd lecture on Computational Mathematics with SageMath. In this lecture, we will look at applications of integrals.

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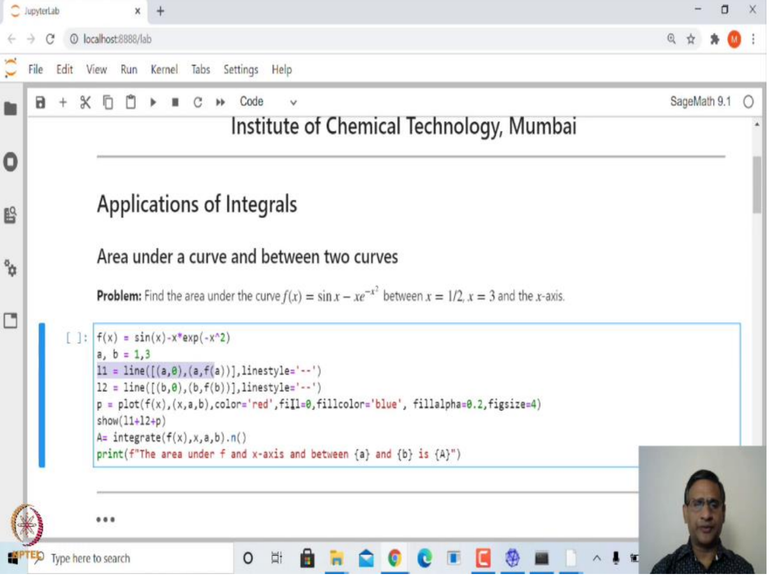

In the last lecture, we looked at the application of integral as an area. An integral is defined as a signed area under the curve. We look at applications like finding the area under a curve or between two curves, finding volume, surface area, and things like that. Let us look at, for example, if you have to find the area under a curve or between two curves, we can use this integral.

First, let us take an example. Suppose, you have a curve f(x) which is $sin(x) - xe^{-x^2}$, and you want to find the area under this curve between x equal to half and x equal to 3. This is simply the integral of this function between half and 3 that is all.

You can plot this shaded area using this plot function, inside this plot function itself there is something called to fill. Fill 0 will give you a filling between the graph of the function and the x-axis. And you can define what is the color of that fill and you can also define the opacity of that fill, that is what is demonstrated in this Sage code, you are defining the function.

Then you define the in the area as an integral of f(x) from a to b and take the numerical value, and then plot the graph of the function, plot the graph of the two lines x equal to a and x equal to b in dotted style, and then print the actual area, that is what you can see here that is the area under this curve plotted as a shading.

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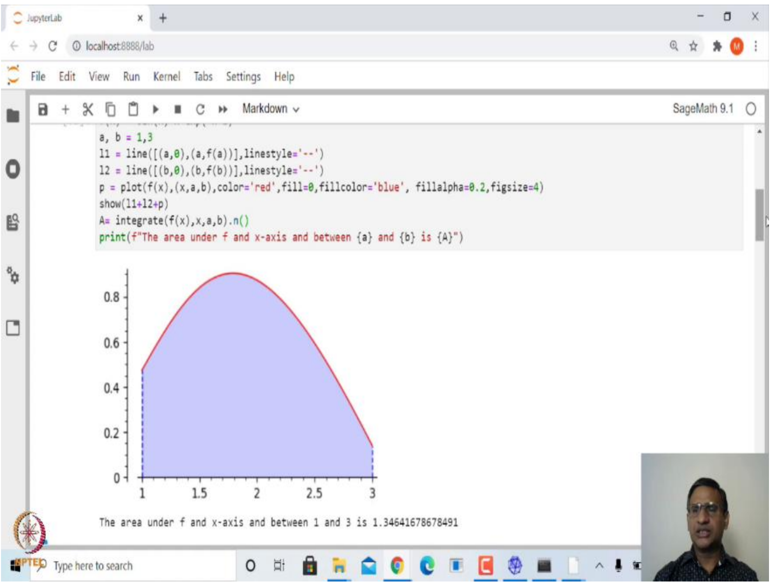

The actual area in this case the area under this curve is 1.3461 and so on.

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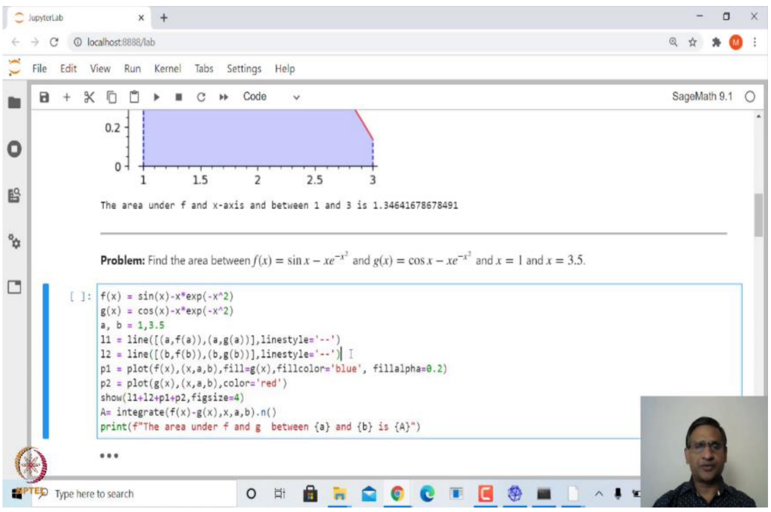

If you have to find the area between two curves say, for example, we want to define the area between the curve $f(x) = sin(x) - xe^{-x^2}$, and $g(x) = cos(x) - xe^{-x^2}$ again between 1 and 3.5.

Then again you can do the same thing. if you try to plot the graph of these two functions, one function may lie above the other function. In that case, you can simply take the difference between the two functions and find the area.

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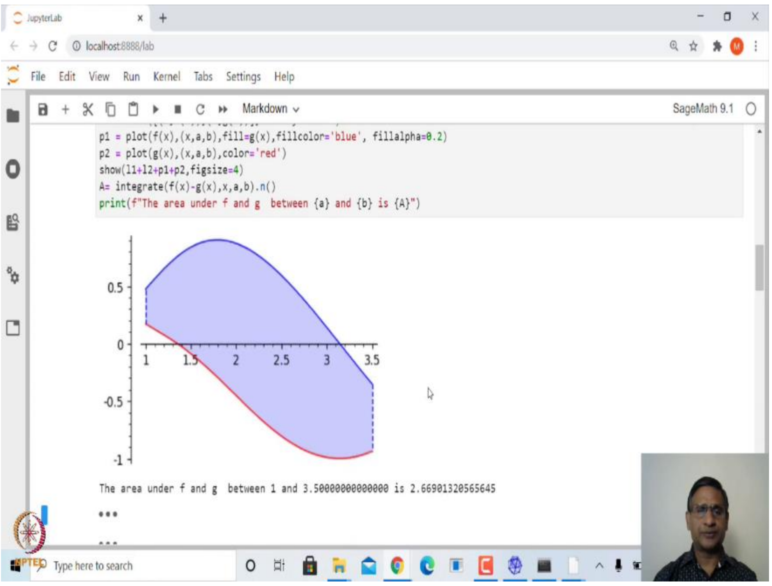

And in this case, if you look at the above one which is the f(x), below one, is the graph of g(x). And f(x) is always above g(x) between this limit. In that case, the area of this curve you can simply find by taking the difference between the two values $f(x) - g(x)$ and calculate its integral.

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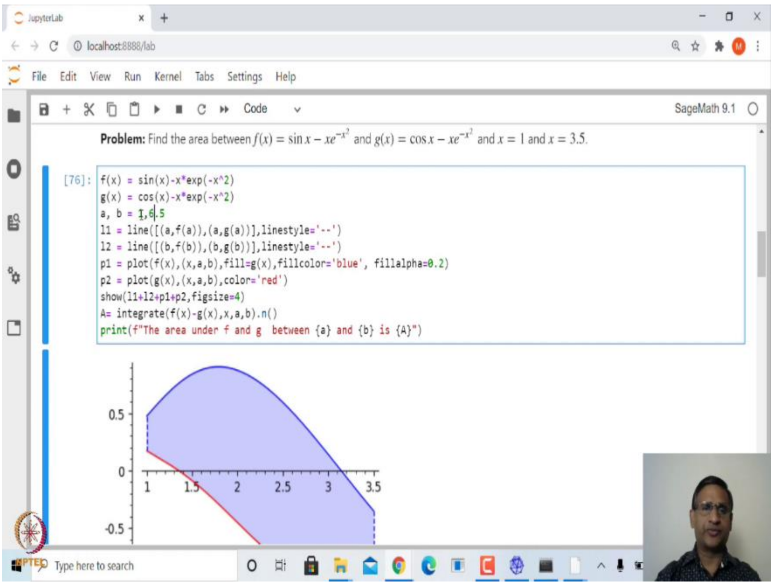

However, it can happen instead of, for example, let us say if I want to find out between let us say again 1 and 6.5.

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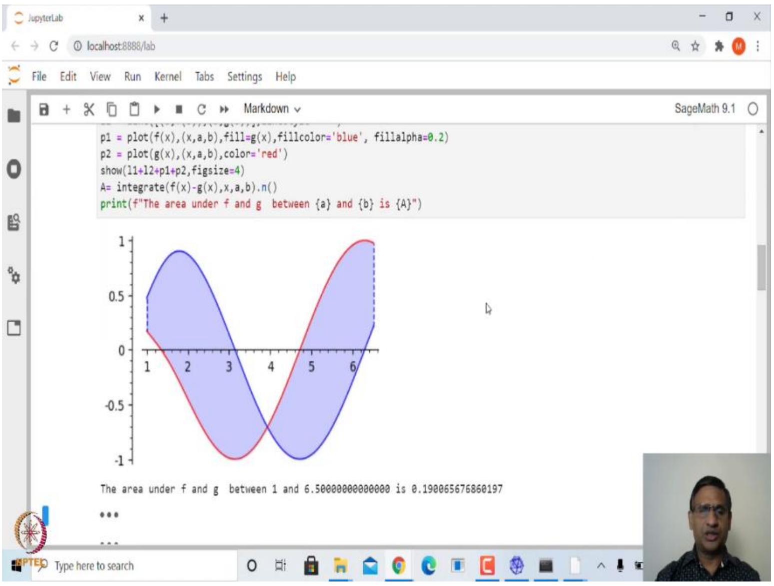

Now, in this case, you see here a bit at this point one has to find out what is the point of intersection. To the left of this the f(x) is bigger than g(x); whereas, to the right of this g(x) is less than f(x).

In that case, you need to find out this point of intersection of these two functions, and then calculate the area to the left of which, and the area to the right of which the first one will be to the left of this point will be $f(x) - g(x)$ the integral and the right-hand side will be g(x)-f(x) integral.

This is not the correct answer. one has to split this integral and then find out this is not the correct one. You can find out, for example, let us see how do I find out the intersection of these two curves.

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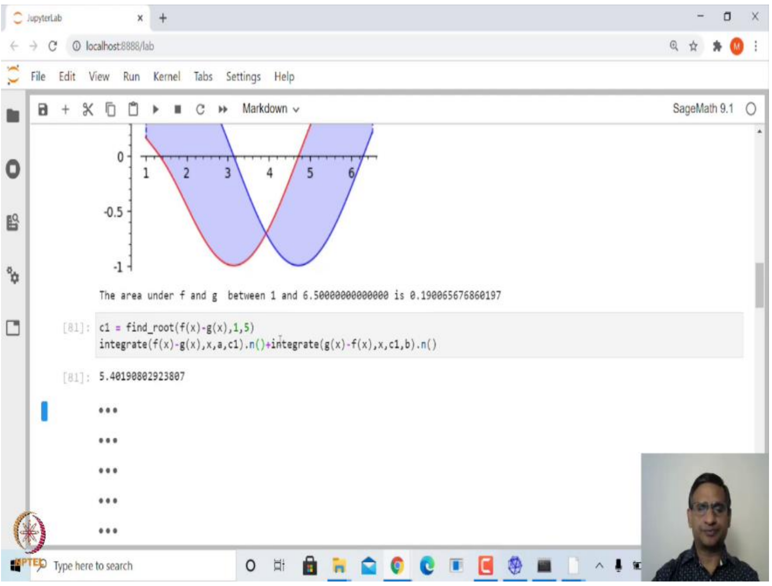

You can say find the underscore root of f(x) - g(x). I will say f(x) - g(x) between let us say 1 and 5, to begin with. This is 3.92699. If I store this in a value c1, then the actual integral is going to be integral of, let me write this. This will be the left-hand integral. This integral a to c1, and then the next one will be the integral of g(x) - f(x) from c1 to b.

If you add these two, take its numerical value dot n and here also dot n that is the actual integral. This is how you can find an area between two curves two intersecting curves.

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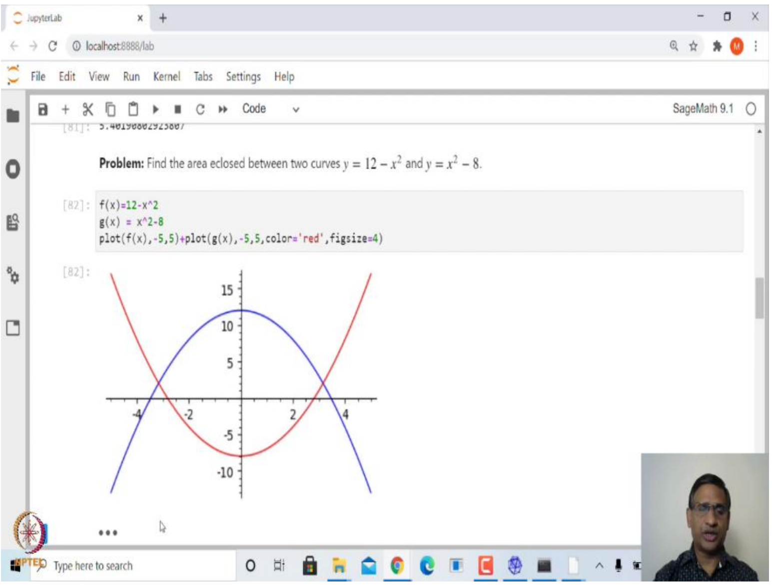

Let us take this example. Find the area enclosed between the two curves. If you try to plot the graph of these two functions together, then the enclosed area is only between this intersection point and this intersection point, which you need to find out first.

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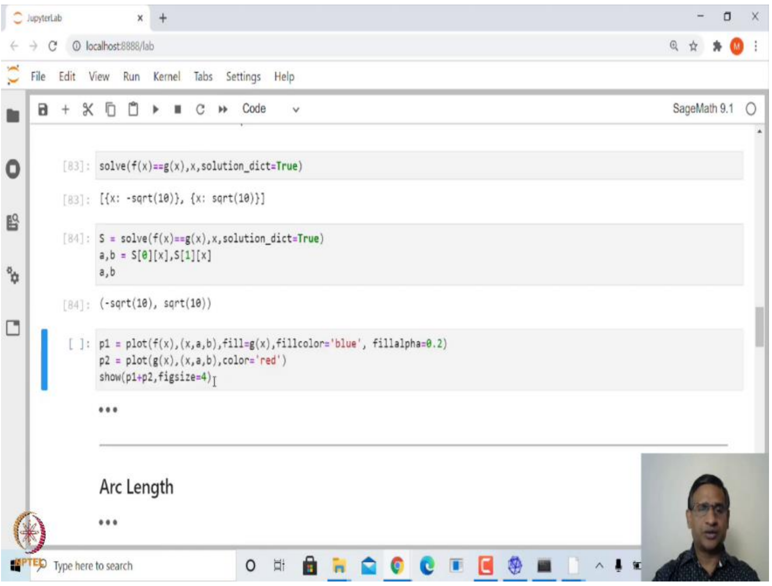

Let us find out the solution of this f(x) equal to g(x) solution dictionary equal to true. It says that the two curves intersect at square root 10 and minus square root 10. Let us store these two values in a and b. These are the a and b which is minus square root 10 is a, or square root 10 is b. And we want to find the area between these two curves. Here, in this case, you can see the blue one which is the graph of f(x) is above the graph of g(x) in this region.

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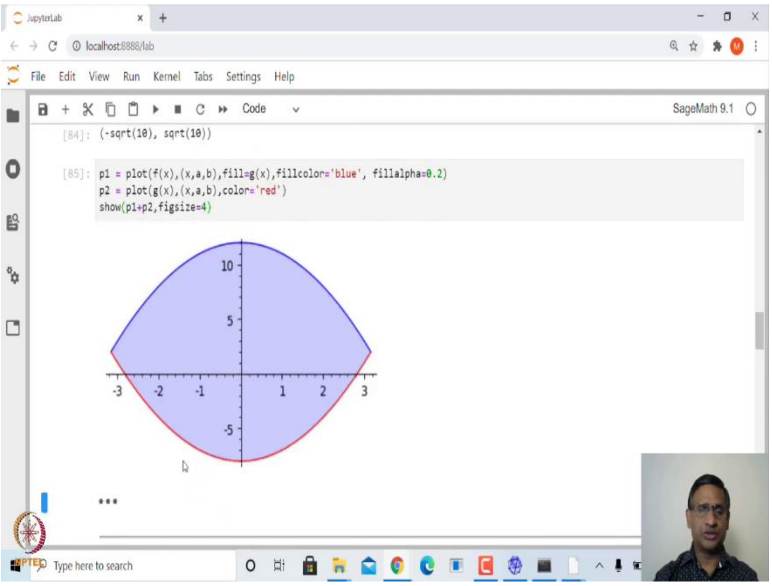

Therefore, to find the integral, we need to plot the shading of the area between these two curves.

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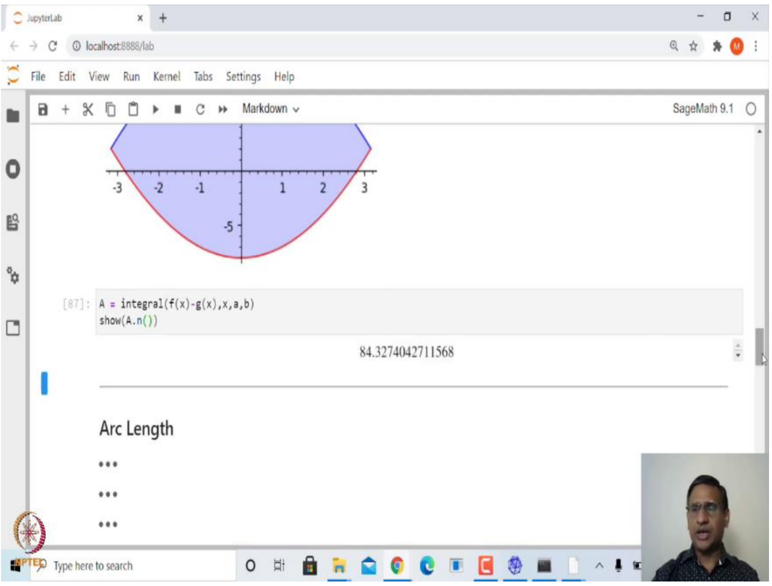

Then to find the integral that is an area, you simply need to say f(x) - g(x) integral of that from a to b, and that integral is $\frac{80}{3\sqrt{10}}$. If you want you can find dot n, and this will give you the numerical value.

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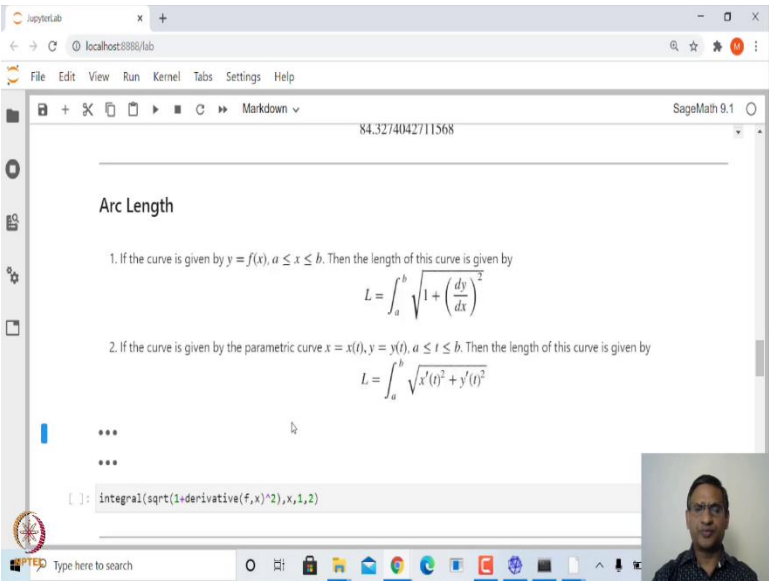

Another application of integral is in finding the arc length of an arc. Now, suppose you have an arc or curve which is defined by y is equal to f(x) function explicitly defined function between a and b. In that case, the arc length is given by the integral of $\sqrt{1 + (\frac{dy}{dx})^2}$ from a to b.

This $1 + \left(\frac{dy}{dx}\right)^2$ 1 is what is called length I mean the ds generally we denote this by ds and you are integrating ds from a to b that is a small arc length. A small change in the arc length that is the ds, and then you are integrating from a to b.

Similarly, if you have a curve defined by parameters say x equal to x(t), y is equal to y(t), and then in this case the arc length will be $x'(t)^2 + y'(t)^2$ and then integrate from a to b.

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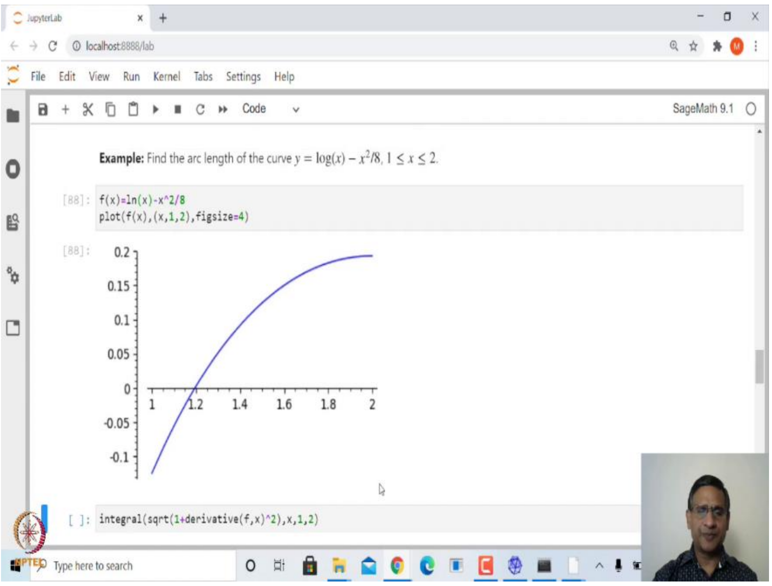

Let us take a simple example. Suppose you have curve $y = \frac{log(x) - x^2}{8}$ between 1 and 2, then its arc length. First, let us plot a graph of this function, this is how the graph looks like.

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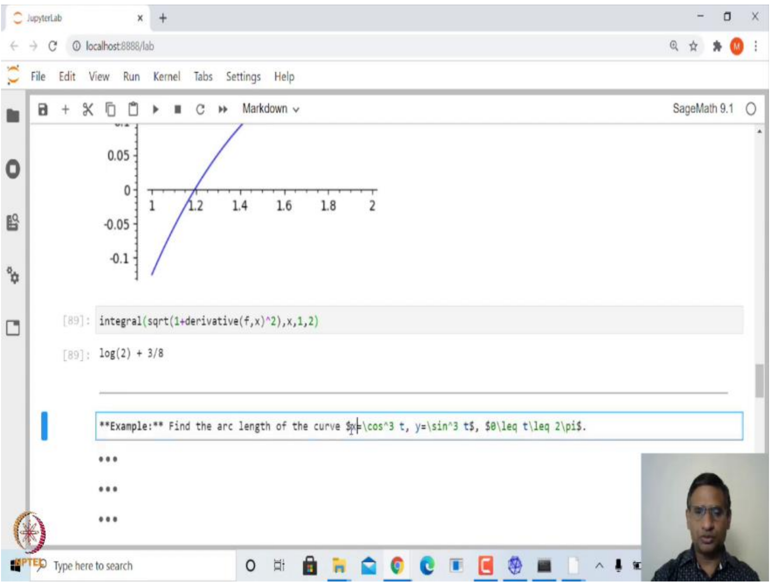

Its arc length is integral of square root 1 plus the derivative of f with respect to x square, and x varying from 1 to 2, this is $\frac{log2+3}{8}$. Similarly, if you have a curve defined by curve x $x = cos^3$(t), $y = sin^3$(t).

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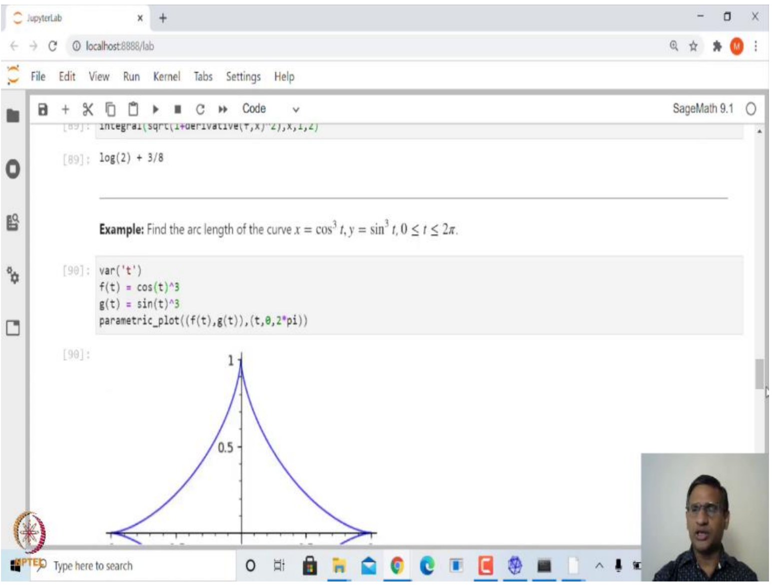

Here curve $x = cos^3$(t), $y = sin^3$(t). If you look at this x and y, they will satisfy a relation. If you look at $x^{\frac{2}{3}} + y^{\frac{2}{3}}$, you will get cos^2(t) $+ sin^2$(t) which is 1, that is the implicit relation.

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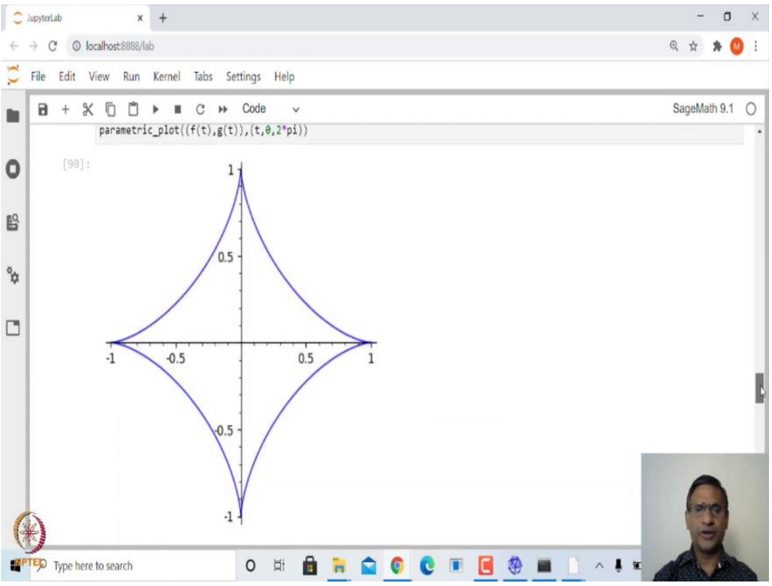

If you try to plot a graph of this function using a parametric plot, this is how it looks like. This is a very famous curve; I think it is known as an asteroid. You want to find the arc length of, this curve. This curve you can just see that is symmetric, whatever is the length in the first quadrant, the total length will be 4 times.

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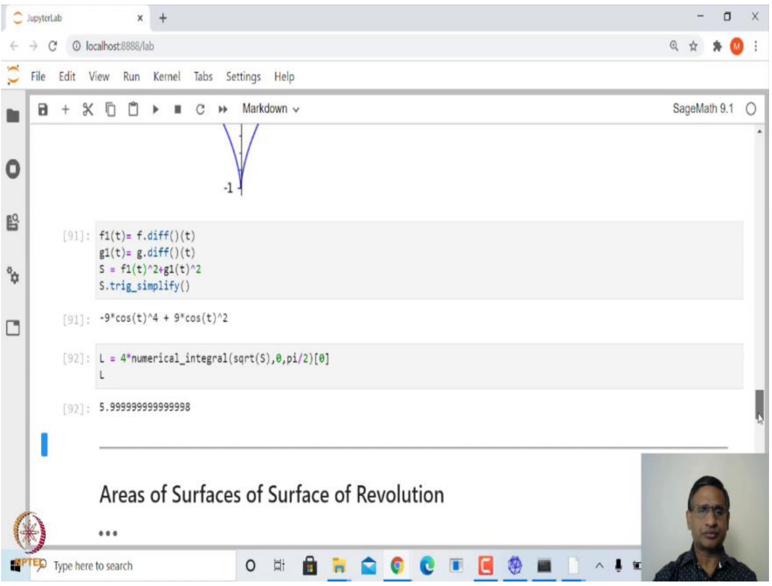

We can try to find out arc length in the first quadrant, and then multiply that by 4. This is the ds square which is ${f_1}^2(t) + {g_1}^2(t)$. And $f_1$and g_1 are derivatives of f and g. Here we have taken x coordinate as f(t), y coordinate as g(t), that is the area element.

Now, this takes the numerical integral of this from 0 to π by 2 because for the first quadrant the t varies between 0 to π by 2, and multiply this by 4, you will get the area this is 5.999. This is 6 the total length, total arc length for this curve is 6.

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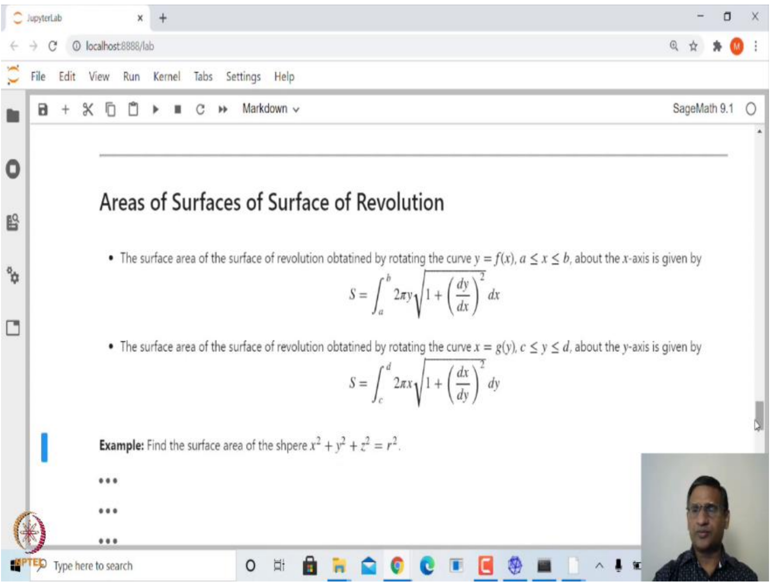

If you want to find the surface area of the surface of the revolution, we have already seen how to plot the surface of the revolution of any curve. And then the surface area of the surface of a revolution of a curve y = f(x) between the x equal to a and b about the x-axis is given by this formula. This is an integral from a to b, 2*π*y into this area element ds which is $1 + \left(\frac{dy}{dx}\right)^2$.

In case you rotate, this about the y-axis, then this formula is integral from c to d that is the domain in which y varies. Here it is, it should be x = g(y). This is integral from c to d 2*π*x into the integral square root of $1 + \left(\frac{dy}{dx}\right)^2$. Using this, let us try to find, the surface area of the sphere $x^2 + y^2 + g^2 = r^2$.

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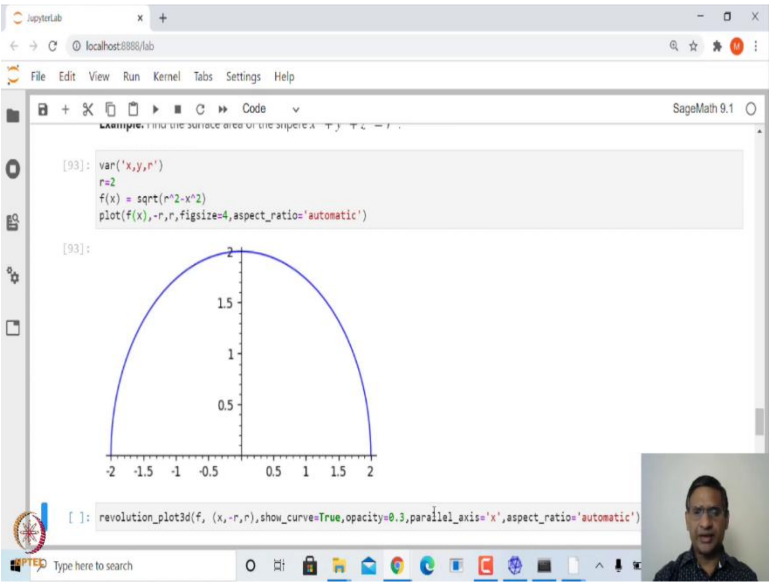

Now, this sphere you can obtain by the surface of a revolution by revolving let us say upper semicircle from minus r to r about the x-axis. If I define a function $f(x) = \sqrt{r^2 - x^2}$ and try to plot the graph, this is the graph of the semi-circle. And then if you revolve this, draw the revolution plot3d of this curve between -r and r and about the x-axis.

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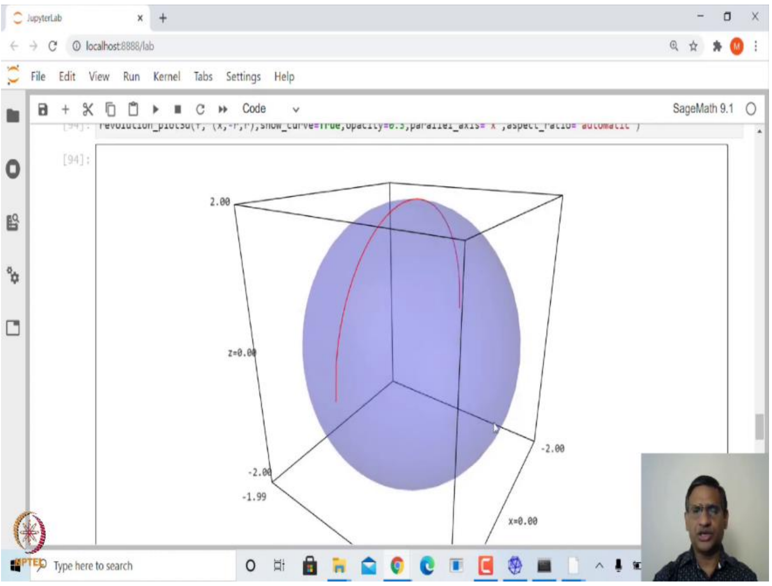

You will get a sphere. And this is the curve that is rotated like this about the x-axis.

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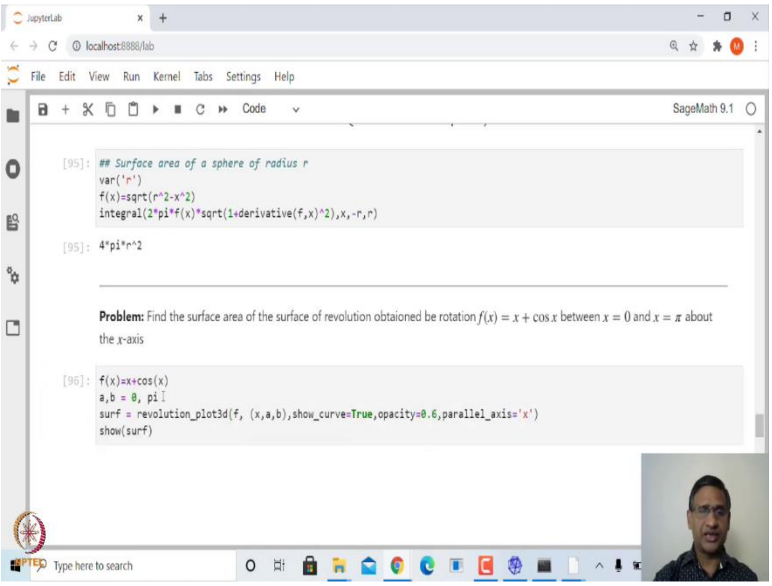

If you try to find the surface area of this, $f(x) = \sqrt{r^2 - x^2}$, and then it's integral that is the surface area this is by formula 2πf(x) that $y = f(x)\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$ that is the same as saying f'(x) the whole square between -r to r, you should get this as 4 $\pi\, r^2$square that is the surface area of the sphere.

If you want to find the surface area of the surface of a revolution about this curve f(x) equal to x + cos(x) between x equal to 0 and x equal to pi about the x-axis, then again you can do that. Let us plot the graph of this surface first.

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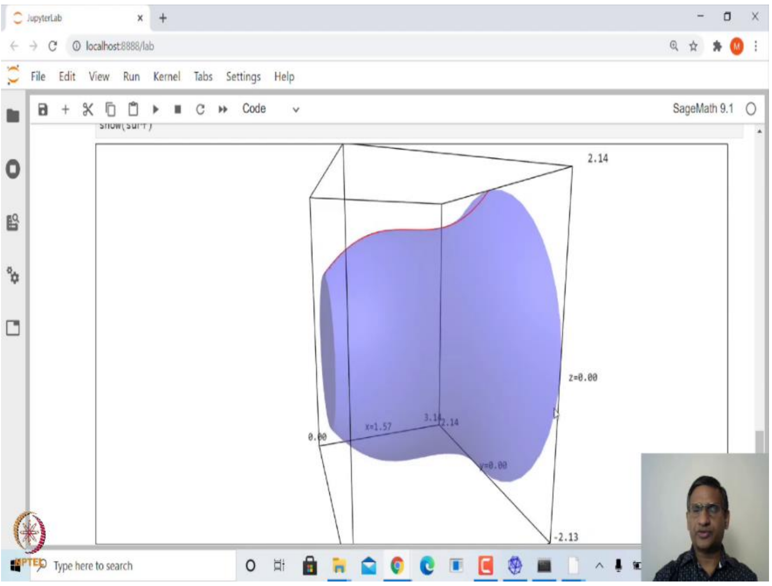

If you try to plot this surface, this is how it looks like.

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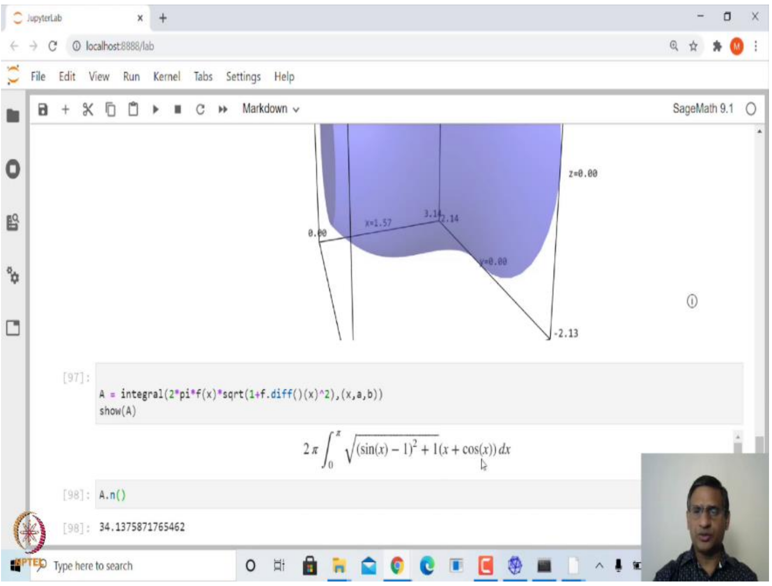

Whose surface area will be given by integral from a to b. a and b here are mentioned already 2πf(x) $\sqrt{1+f'(x)^2}$ from a to b, and this integral that is the surface area turns out to be this value.

You may need to compute what is its numerical value because it is just giving you the output as the integral which we have entered right. If you are unable to find the integral in closed form in numerical form, then you may use dot n or use numerical integral for arc length and surface area etcetera.

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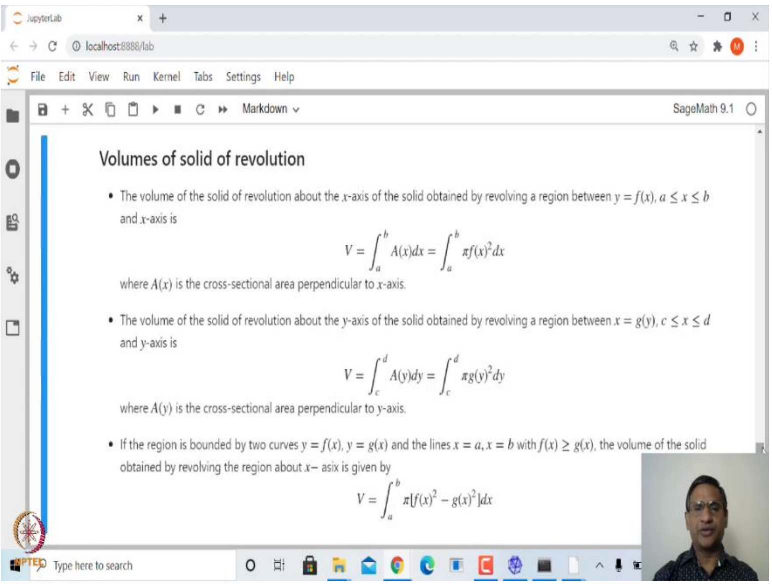

Let us look at how to find the volume of a solid or when we obtain this solid by revolving some region which is let us say bounded by y equal to f(x) between a and b about the x-axis, then this is given by formula V is equal to the integral of a to b A(x) dx, where A(x) is the cross-sectional area.

And which is the cross-section area, in this case, will be the pi and the height at x which is f(x) the whole square. Similarly, if you rotate about the y-axis, the x equal to the g(y) curve in this cross-section area, if it is A(y) this will be pi into g(y) the whole square integrated from c to d.

Whereas, if you have a region bounded by two curves y = f(x) and y = g(x), and let us assume that f(x) is greater than equal to g(x), in that case, you can take the difference of the outer surface minus the inner surface, and then find the volume by using the formula. I am sure all of you must have done these things in your regular classes.

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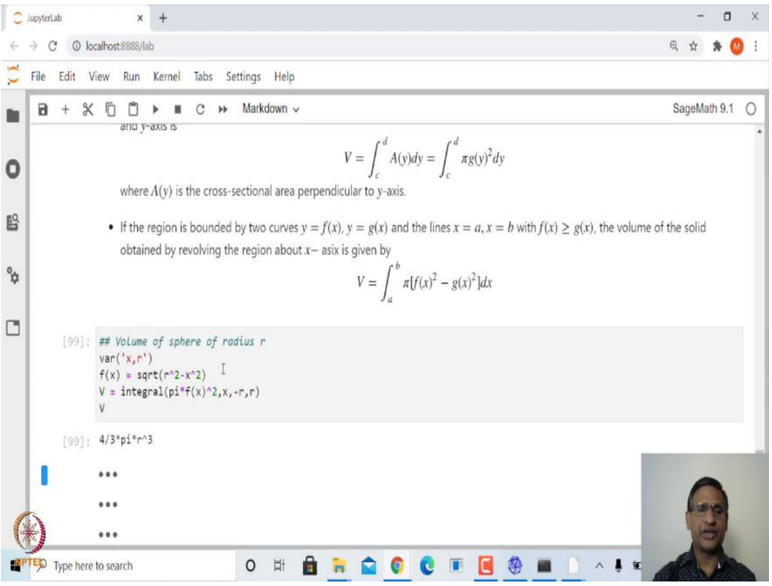

Let us take some examples. Again, if you look at what will the volume of the sphere, again we can obtain a sphere by revolving the semi-circle about the x-axis. And if you use the formula V is equal to integral π into f(x) the whole square x varying from - r to r this is $\frac{4}{3}\pi r^3$ that is the volume of a sphere of radius r.

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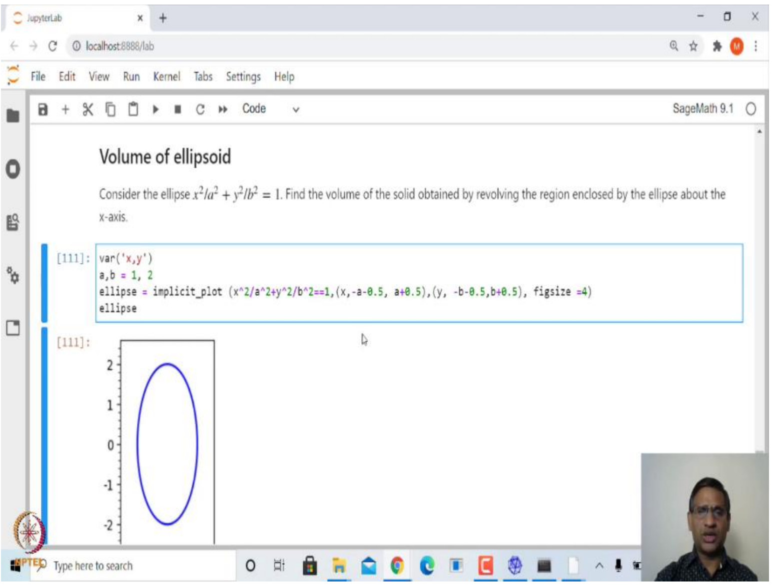

Let us take the next example as computing the volume of an ellipsoid. One can obtain an ellipsoid by the surface of the revolution of rotating or revolving this ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

about the x-axis. You can just take the upper part of this ellipse and rotate about the x-axis between - x equal to - a to x equal to a, you will get an ellipsoid. Let us try to first plot a graph of an ellipse between - a and a.

Using implicit underscore plot x square upon a square plus y square upon b square is equal to 1 x varying between a - 0.5 to a + 0.5. Here we have taken a as 1, b as 2; and b varying between b - 0.5 to b + 0.5.

This is the ellipse. Now, if you take the upper half ellipse that can be obtained as $\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$. Therefore, $y^2 = (1 - \frac{x^2}{a^2})b^2$, and then y will be the square root of that.

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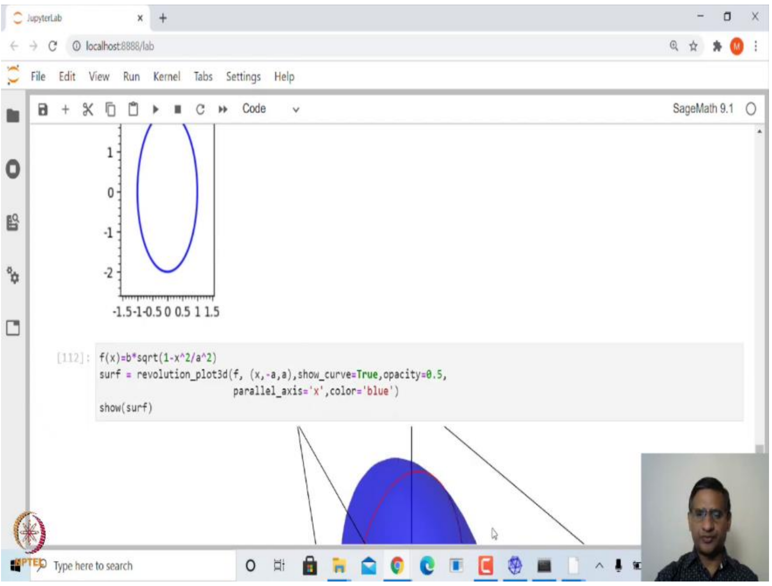

If you take that function $f(x) = b\sqrt{1 - \frac{x^2}{a^2}}$ and revolve this or rotate this about x-axis between - a and a, and then you will get an ellipsoid. That is what you have let me make it slightly smaller.

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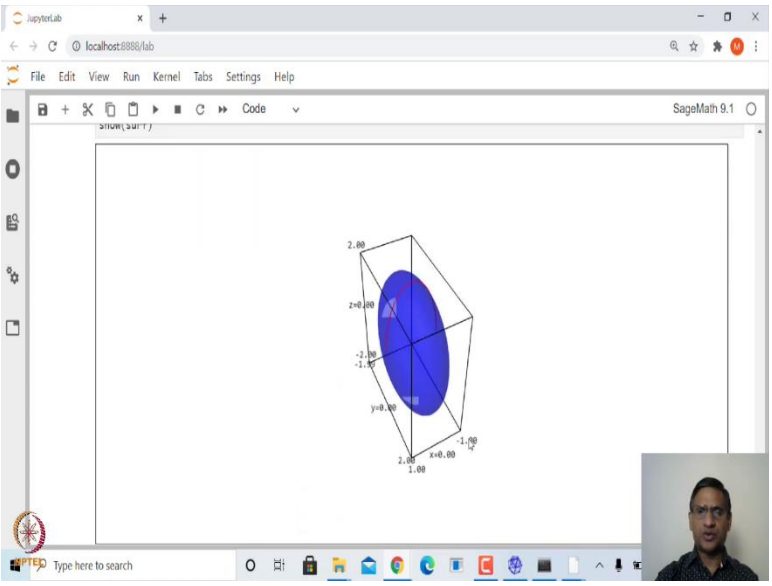

This is the ellipsoid. You can rotate this and see that is the semi-upper semi ellipse which we have rotated.

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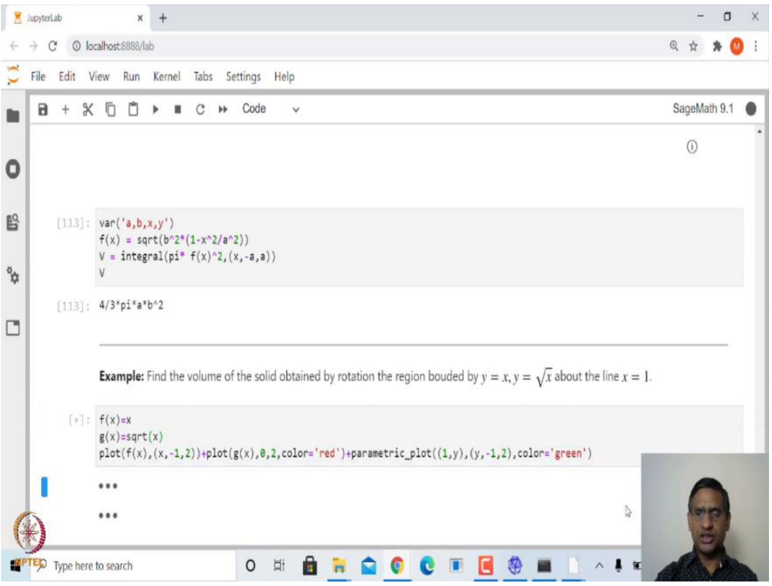

You can compute its volume by the formula integral of $\pi f(x)^2$, x varying between - a and a, and here $f(x) = b\sqrt{1-\frac{x^2}{a^2}}$. This volume is $\frac{3}{2}\pi ab^2$. You can see here in these are the symbolic integrals that Sage can find. But it is not just that it will find the numerical integral, but it can also find the symbolic integral in the majority of the cases wherever the integral is possible in closed form.

Next example, let us look at, suppose we want to find the volume of the lid obtained by the region y = x and $y = \sqrt{x}$ about the line x equal to 1. Let, us first plot the region or the two curves.

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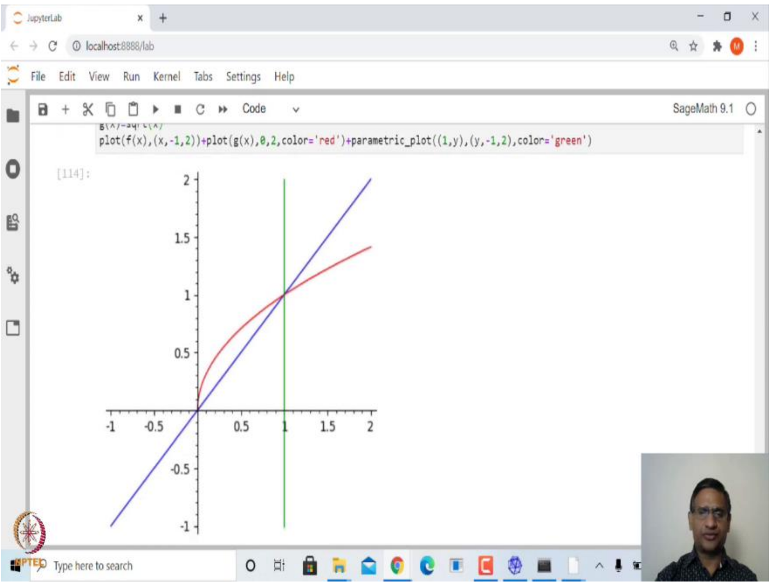

The red one is $y = \sqrt{x}$, and this blue one is y = x curve, and this is the green vertical line is x equal to 1. You want to rotate this portion of this region about this green line, that is the axis of revolution in this case.

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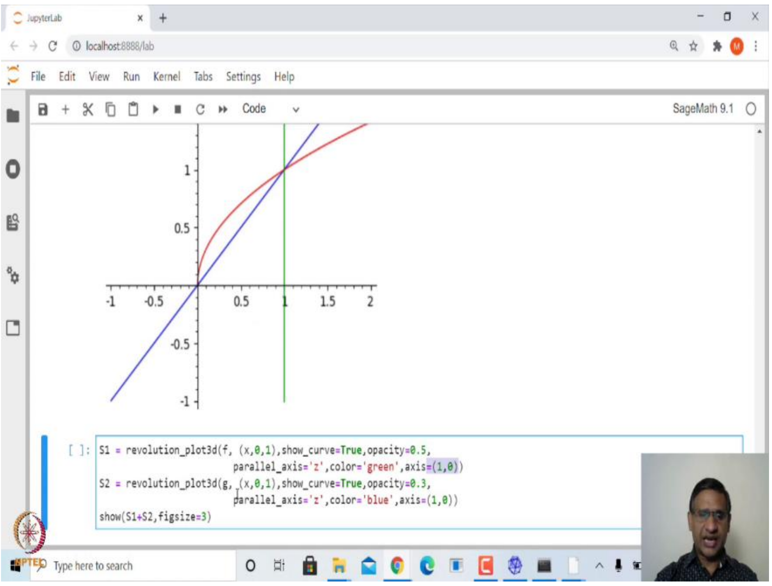

You can obtain that surface by surface of revolution. And in this case, you have two curves - one is the outer curve and another one is an inner curve. let us plot S1 as the revolution of f which is y = x, and the second one - S2 as a revolution of g(x) from 0 to 1. And here you are mentioning the axis of rotation as (1, 0). It is passing through (1, 0). And then you are rotating about the z-axis.

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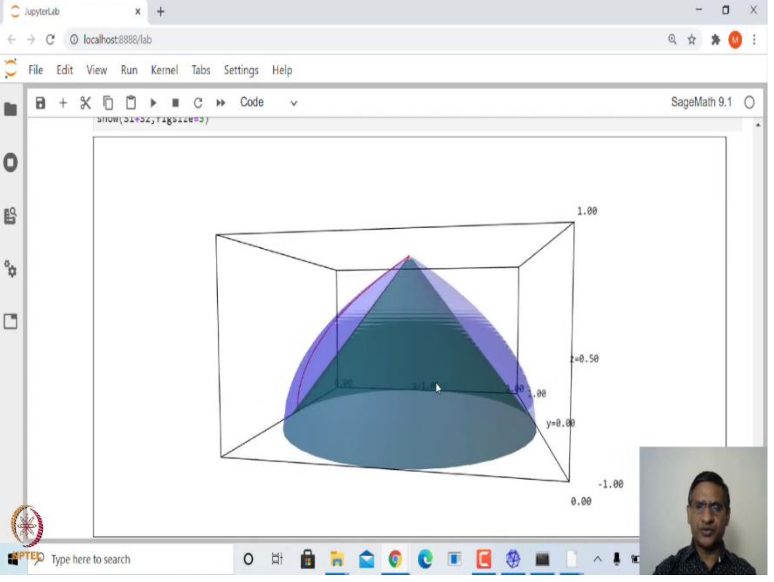

Then if you put all these things together, you will get a solid surface. Though here it is plotted and it may look hollow inside, but the hollow inside is that the solid you can think of because here we have taken the region inside this as a hollow one.

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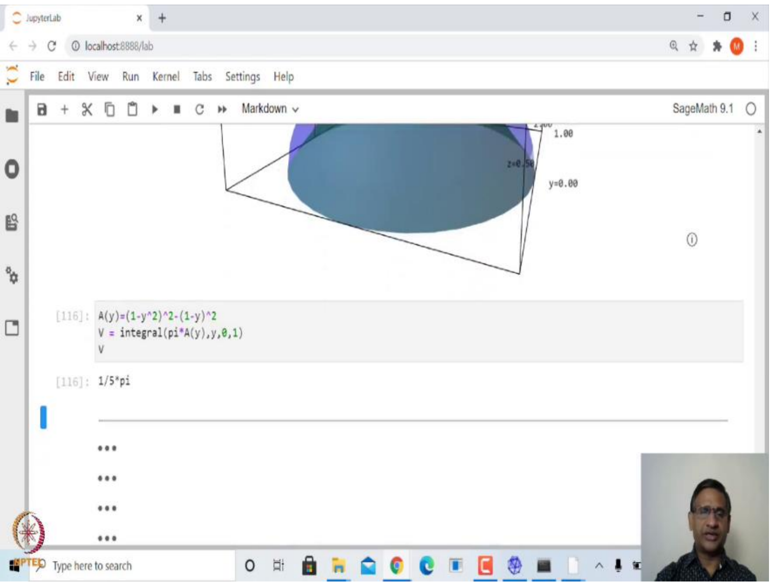

You can find now the volume of this surface of revolution. In this case, if you look at the cross-sectional area which will be parallel to the x-axis or perpendicular to the y-axis, you have to calculate from x equal to 1. The outer one is going to be 1 - y square the whole square of this minus the inner cross-sectional area which is 1 - y the whole square.

And therefore, this integral is going to be pi into A(y) integral varying from 0 to 1, that is $\frac{\pi}{5}$ that is the volume of this solid obtained by revolving the area between these two curves about x equal to 1. You could have rotated about x equal to 2, or you can rotate about the x-axis, about let us say x, y equal to 1, all such things you can do it quite efficiently and very easily.

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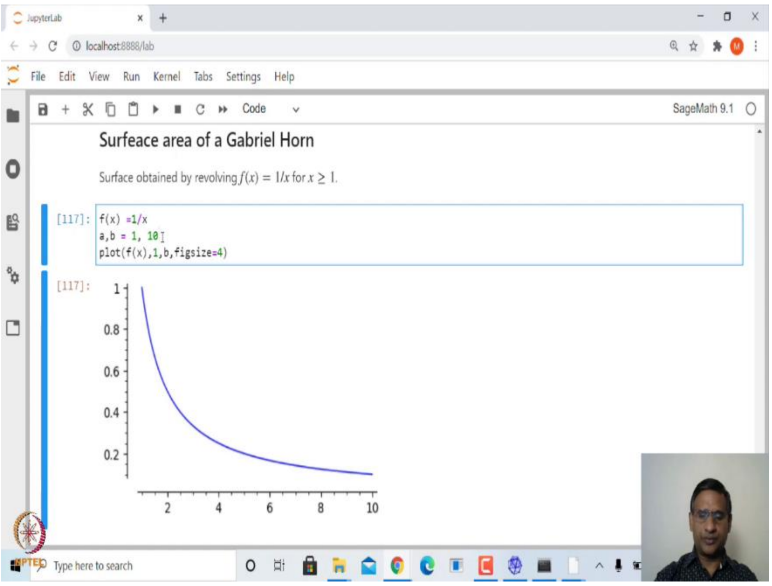

Let us look at another problem. This is a surface obtained by revolving this curve $y = \frac{1}{x}$ about x-axis from 1 to ∞. This is what is called the Gabriel horn surface. If you try to plot a graph of this function, this is how it looks between 1 and 10.

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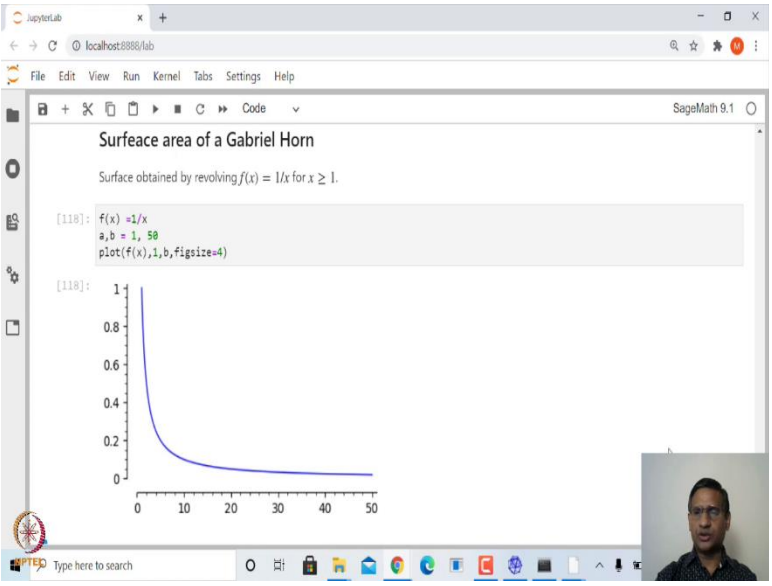

If you take 1 and 50, then this is how it will look like now this you want to rotate about the x-axis.

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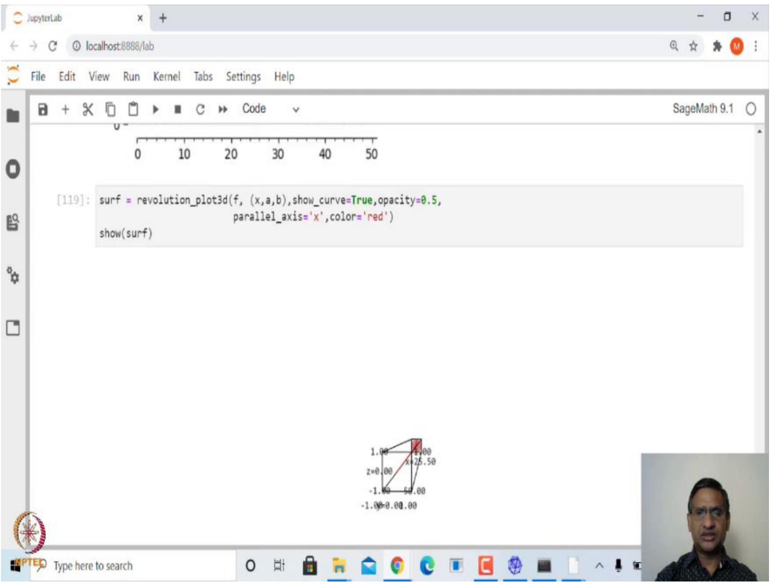

When you rotate about the x-axis, let us say then this surface will look like this. Let instead of 50, let me make it 10 because 50 will be too big. And then rotate about the x-axis, then this surface looks like this.

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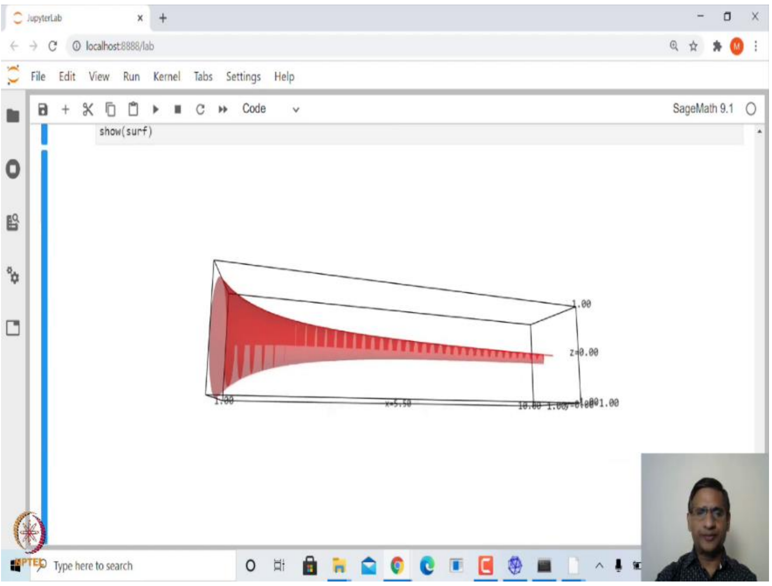

You can see here this is the beginning and then it is kind of becoming very thin and thin. This is what we call Gabriel horn.

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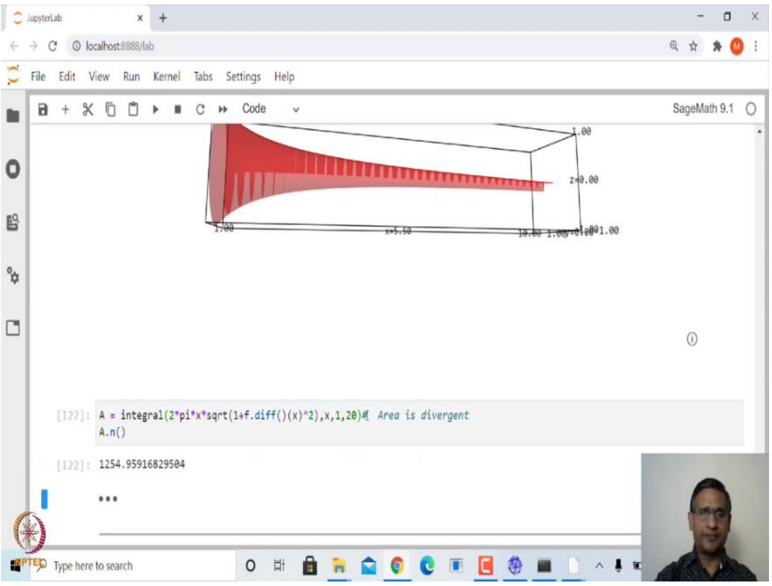

Now, the interesting thing about this surface is that if you try to find the surface area of the surface of revolution, for example, if I say surface area calculated between 1 and 20 using the formula for surface area, this is 1254.

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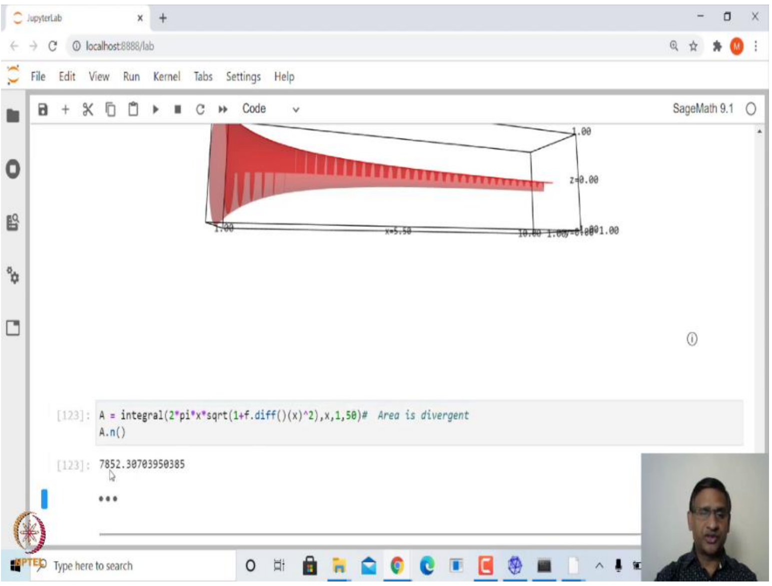

If I go up to let us say instead of 20, if I go up to 50, this will become 7852.

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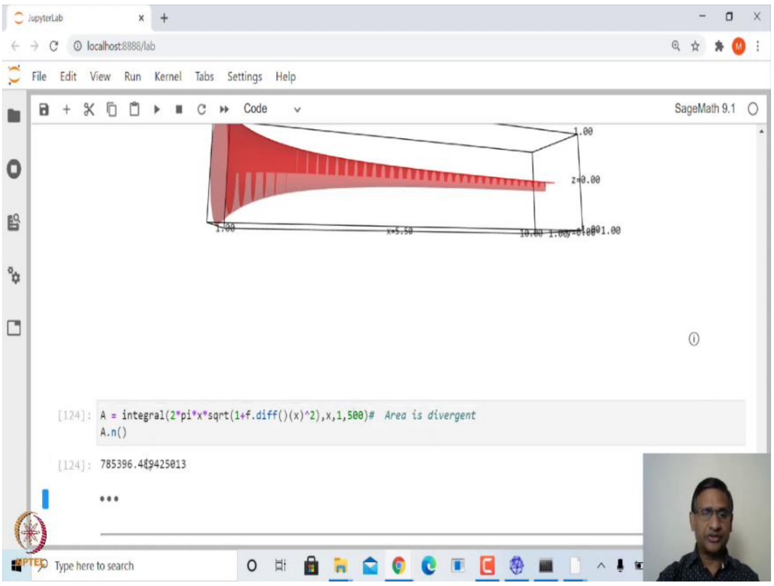

If I go up to 500 then this is.

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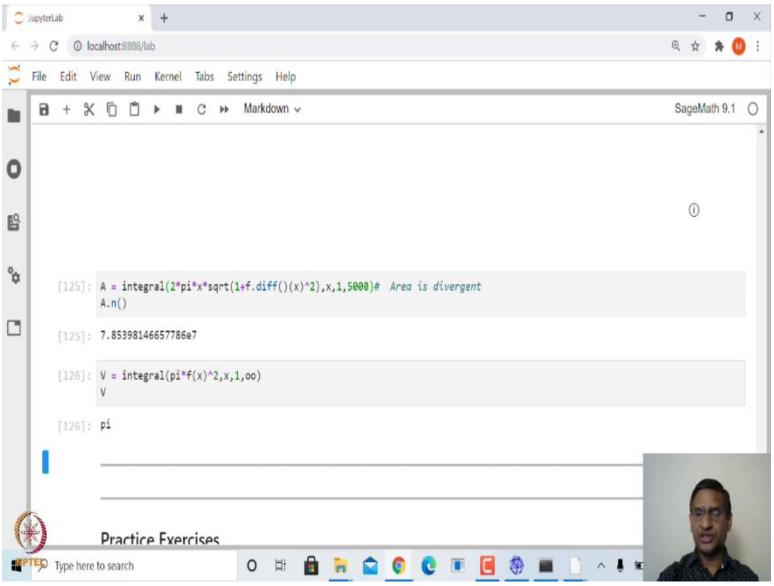

And if I go up to 5000, it is you can see here this is becoming very large. The surface area is increasing. Whereas, what happens to the volume? If I look at the volume of this Gabriel horn, we have to calculate the integral of this $f(x)^2\pi f(x)$ from 1 to ∞, this is π. The volume is finite, whereas, the surface area is infinite, that is the interesting thing about this Gabriel horn surface.

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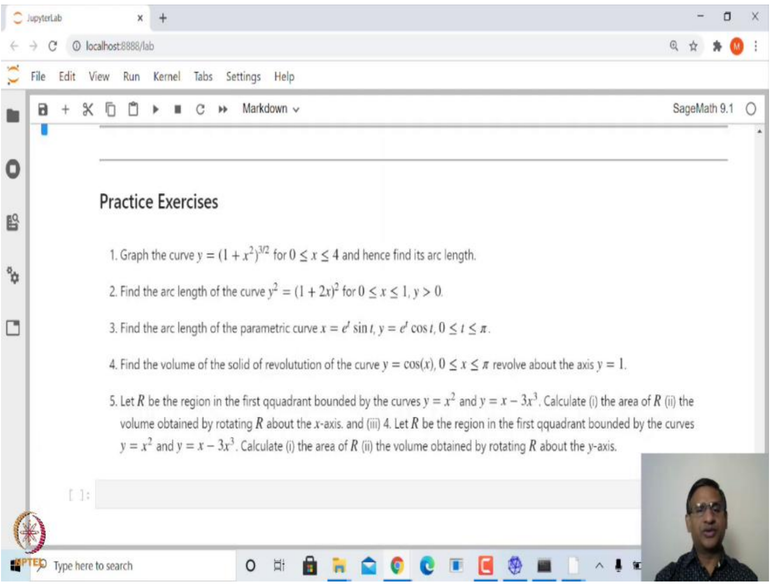

Let me leave you with exercises. The 1st one is to graph the surface $y = (1 + x^2)^{\frac{3}{2}}$ between 0 and 4, and find its arc length. The 2nd problem is again to find the arc length of this function defined implicitly $y^2 = (1 + 2x)^2$ between 0 and 1, y positive. 3rd one is to find the arc length of the curve defined by parametric curve $x = e^t sin(t)$, $y = e^t cos(t)$ t varying between 0 and pi.

The 4th one is to find the volume of the lid of a revolution of $y = cos(x)$ x between 0 to π revolving about the y-axis. And the 5th one is again you look at a region R which is bounded by two curves $y = x^2$and $y = x - 3x^3$. Calculate the area of R, and then calculate the volume obtained by rotating R about the x-axis.

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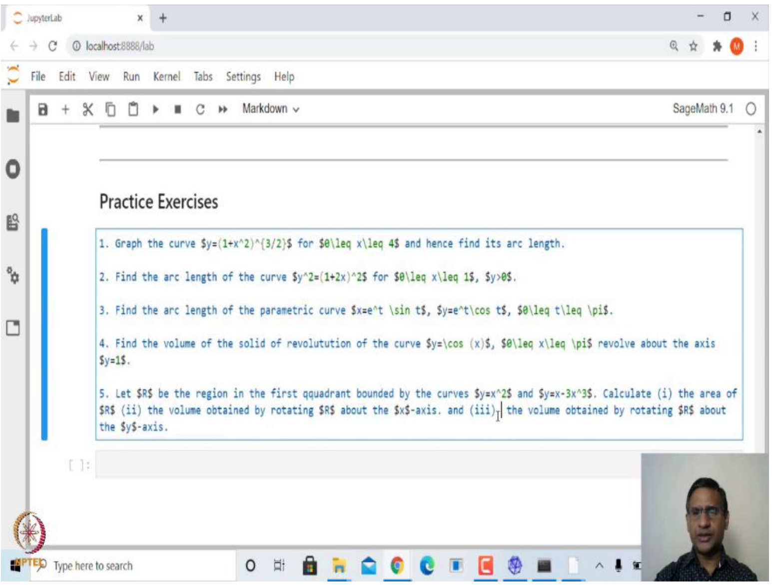

The volume is obtained by rotating R about the y-axis. These are the few practice problems. I will see you in the next lecture. Where we will look at maybe one or two more applications, then we will move on to multivariable calculus.

Thank you very much.