Computational Mathematics with SageMath
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Lecture – 20
Applications of derivatives

Welcome to the 20th lecture on Computational Mathematics with SageMath. In this lecture, we shall look at some applications of derivative. In the last lecture, we looked at finding derivatives including implicit derivatives. We also looked at Taylor's theorem and Lagrange mean value theorem and its geometric meaning.

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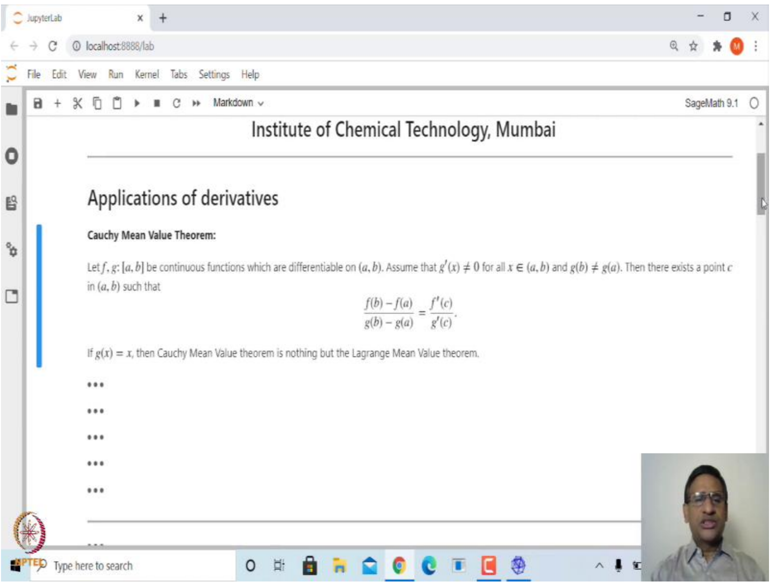

I had asked in that lecture, to look at the Cauchy mean value theorem and also look at what is its geometric meaning. So, before we look at some applications, let us look at Cauchy mean value theorem. Cauchy mean value theorem actually says that, if you have two functions f and g defined on closed and bounded interval [a, b] which is continuous on this interval and are differentiable in open interval (a, b), and let us also assume that derivative of g does not vanish in the open interval (a, b) and the end point value of g are not equal. In this case one can find a point c in the open interval (a, b) such that f(b) minus f(a) upon g(b) minus g(a) is equal to f dash c upon g dash c. If you notice g dash c is in the denominator and that is why we need that condition g dash c nonzero.

Also in this case, if you look at g(x) equal to x then what will you get on the left-hand side denominator it is nothing, but b minus a, on the right-hand side it will be f dash c.

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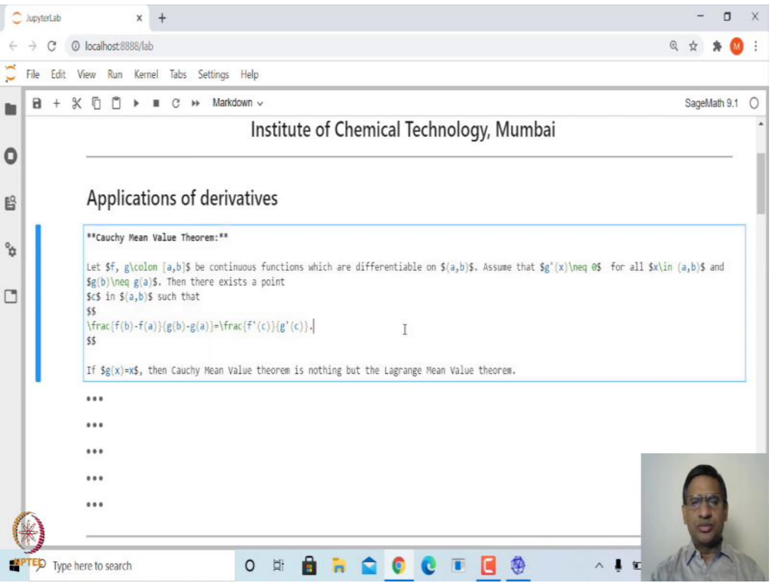

That is nothing, but Lagrange mean value theorem. So, you can think of Lagrange mean value theorem as a particular case of Cauchy mean value theorem.

Now, what is geometric meaning of this? Let us look at a problem and let us try to verify this Cauchy Mean Value theorem for that problem.

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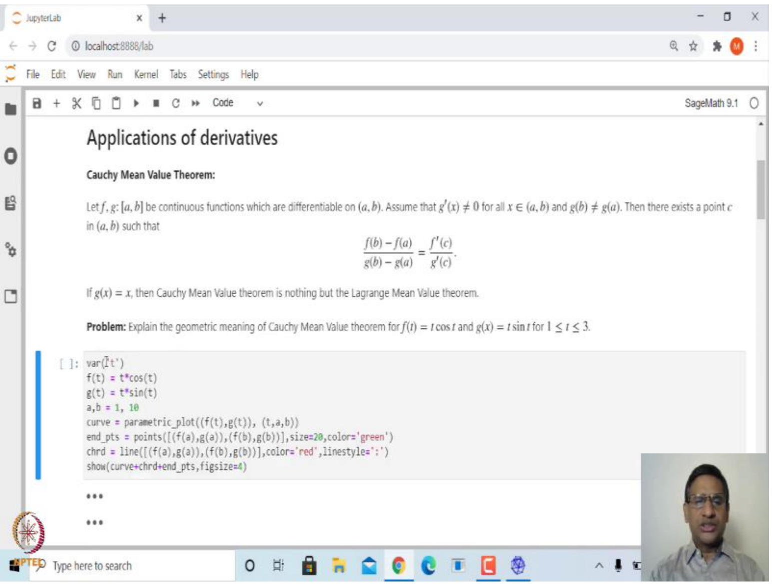

Then, we will see what the geometric meaning is. Consider two functions f(t) and g(t), f(t) is t cos t and g(t) is t sin t, with t varying between 0 and let us say 3.

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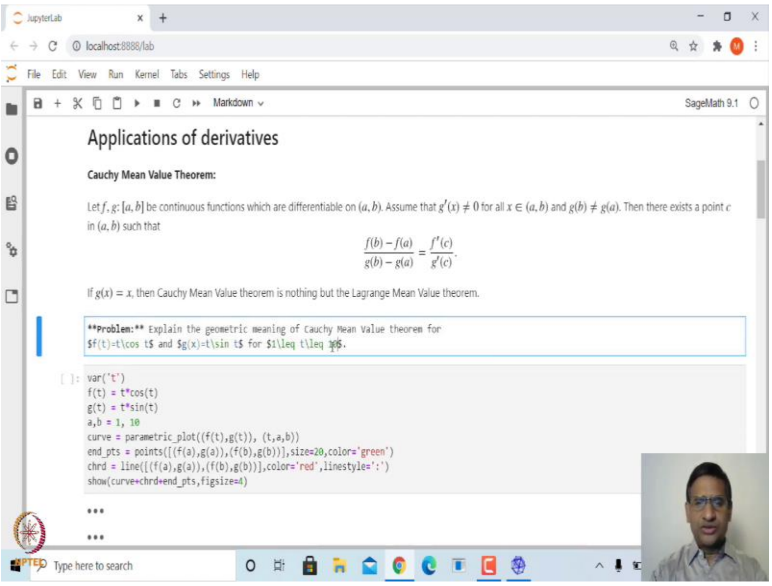

In this case let us take t, between 1 and 10, we can increase the range.

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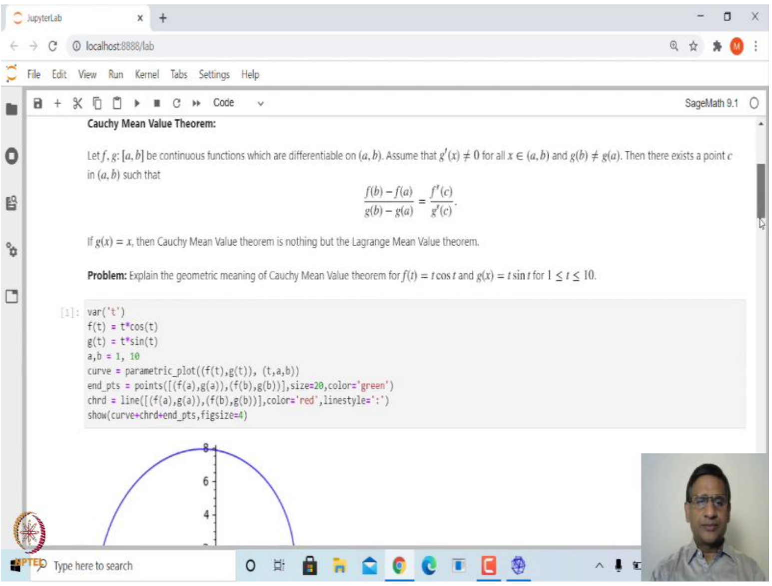

Now, let us define t as a variable f(t), g(t). a, and b are the end points, that is, 1 and 10. Let us plot the curve and the chord joining, f (a), g(a) and f(b), g(b).

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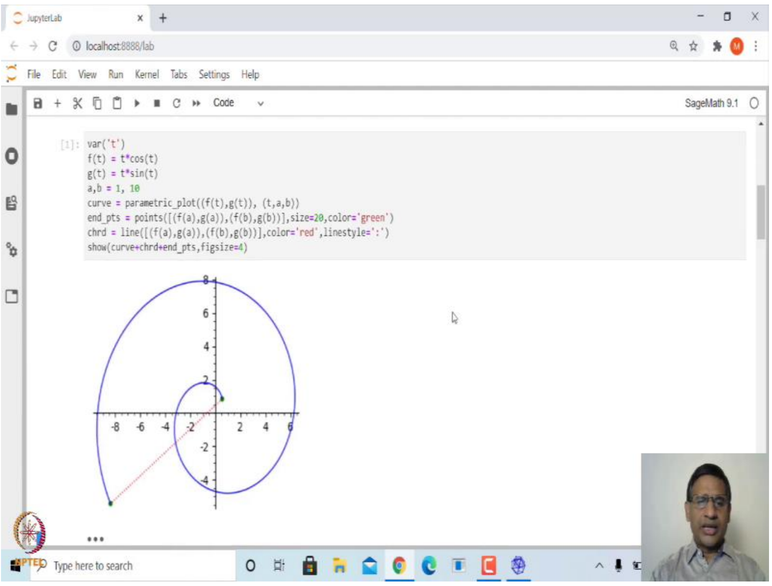

We are taking a curve, which is parametrically defined, and x coordinate is t cos(t), y coordinate is t sin(t). So, this point will be f(a) comma g(a) and this point will be f(b) comma g(b). You are looking at this chord joining these two end points. That is the chord. So, this Cauchy mean value theorem, what it says is that, there exists a point strictly between a and b at which the tangent to this curve is parallel to this chord. That is what it means.

You can imagine geometrically, there could be one point here at which this the tangent to this curve will be parallel to this chord. There will be another point somewhere here at which the tangent will be parallel to this chord. That is the geometric meaning of Cauchy mean value theorem. Let us try to verify that.

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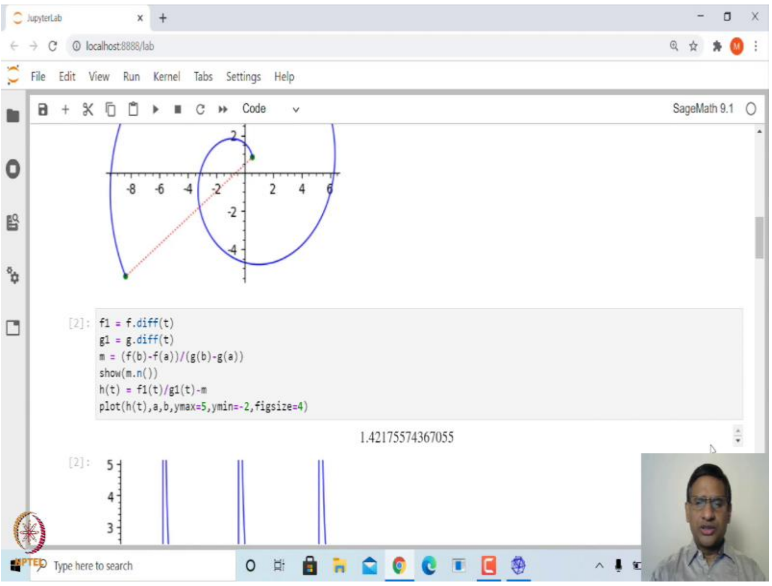

So, in order to verify, let us first define the derivative of f and g. Derivative of f we will define in f1 and derivative of g, we will call as g1. Then, let us take m as the left-hand side in the Cauchy mean value theorem that is, f(b) minus f(a), divided by g(b) minus g(a).

One can think of this as, 1 upon m will be the slope of this this chord. Then, what we are looking at? We are looking at finding c such that f dash upon g dash at c equals to this m.

So, we can define a function h(t) equals to a derivative of f(t) divided by derivative of g(t) minus m. Then, to find such a c, all we need to do is find 0's of h(t). Let us plot the graph of h(t). So, this is the slope one upon the slope of the chord.

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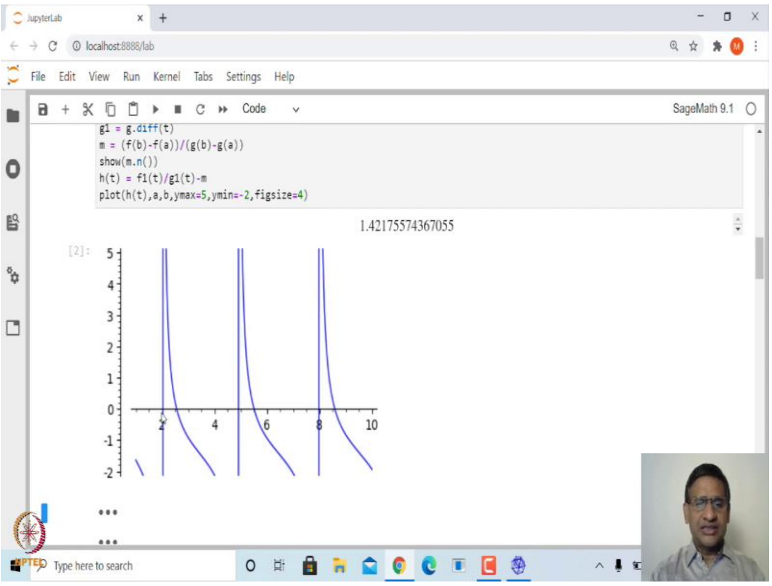

Let us plot the graph of h(t). When you plot this graph it, clearly says that the h(t) will have 0 between 2 and 4, one between 1 and 4 and one between 4 and 6, one between 8 and 10.

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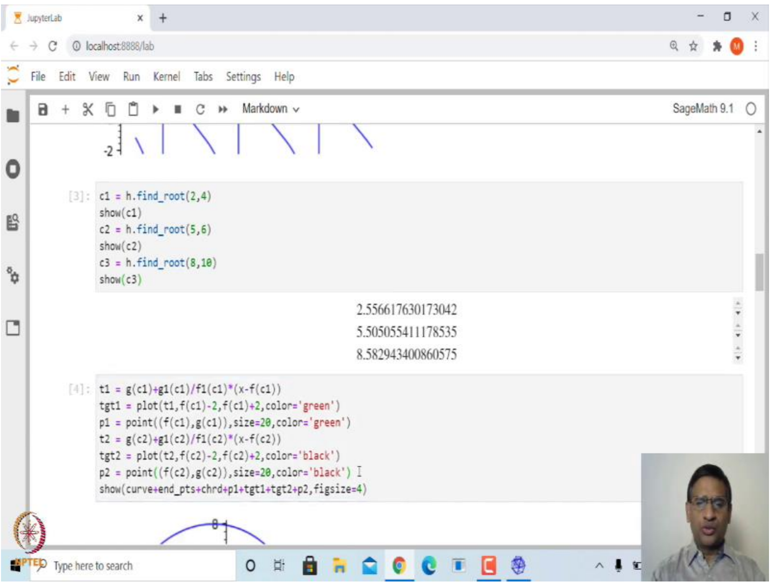

Let us find these zeros using find underscore root command. So, h dot find underscore root between 2 and 4 that is c1, h dot find root between 5 and 6, that is, c2 and h dot find underscore root between 8 and 10, that I have stored in c3. Let us look at what are the values of c1, c2, c3. So, c1 c2 and c3 are these values and that you can see that all these values lie strictly between a and b.

Let us try to plot graph of the tangent to these curves and then see whether that is parallel to the chord. Here we are defining the tangent at the point f(c1), g(c1) and this is the plotting that tangent in green color. Similarly, this is the point p1 the that is (f(c1), g(c1)).

Similarly, define the second tangent and plot that second tangent. Then plot also the second point and add all these things to the given curve, which we have plotted earlier along with the chord. That is what we are doing. Third point I am not adding here, one can add third point as well.

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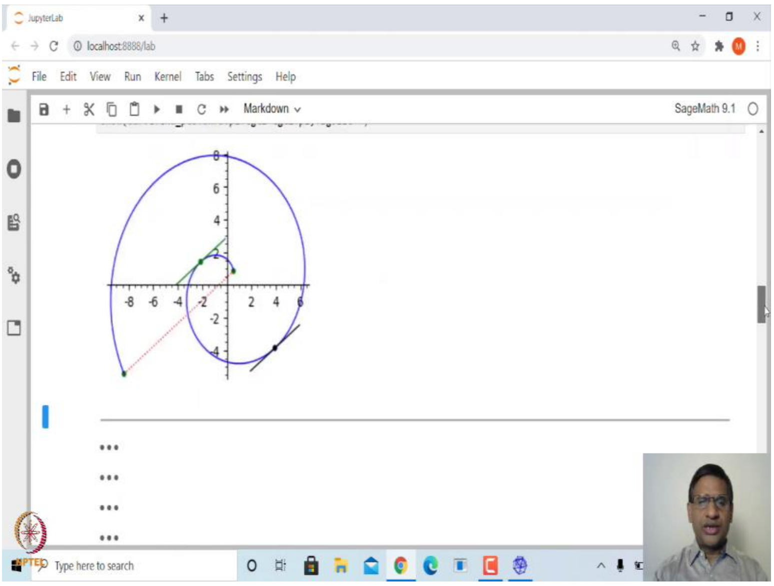

Let us look at these two points. If you run these, you can see here, there are these two points, green one is the c1, actually (f(c1), g(c1)) and this black one is, (f(c2), g(c2)) and the tangent at these point are parallel to this given chord. Here the slope of this tangent will be actually 1 upon m, which we have defined. That is same as saying g1, g dash at c1 divided by f dash at c1. So, that is how it is plotted. You can add the third point, and you can expect third tangent to be somewhere here.

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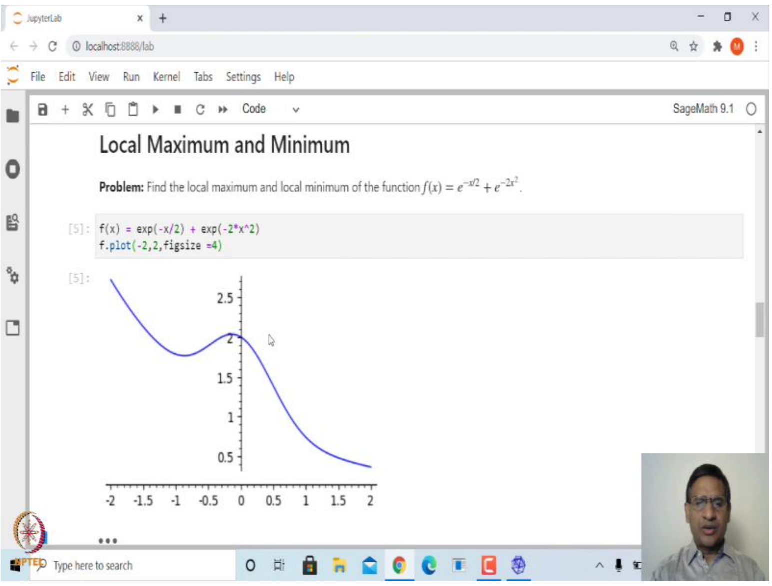

So, this is verification of Cauchy mean value theorem and its geometric meaning. Now, let us look at how to find local maximum and local minimum of a function that is an application to derivative. I am sure, all of you know how to find local maximum local minimum using first and second derivative tests.

Let us take an example. Suppose, you have a function e to the power minus x by 2 plus e to the power minus 2 x square and suppose, for this we want to find local maximum and local minimum, let us say either in some interval or entire line.

Let us first plot graph of this function. Graph of this function, this is how it looks like and what is happening is this?

If you go to minus infinity this will keep on increasing. So, you cannot have any local maximum, local minimum and on the right-hand side again it will keep on decreasing and it will go to 0.

So, it seems that, it has two critical points, where the tangent is parallel to x axis. One of them is here, other one is here. And there will be two point of inflection,

where the curve changes the concavity or convexity. So, in case we want to find out local maximum and local minimum. Of course, you can find derivative, equate it to 0, and you find those critical points. Then find second derivative and check whether the second derivative is positive or negative at critical points.

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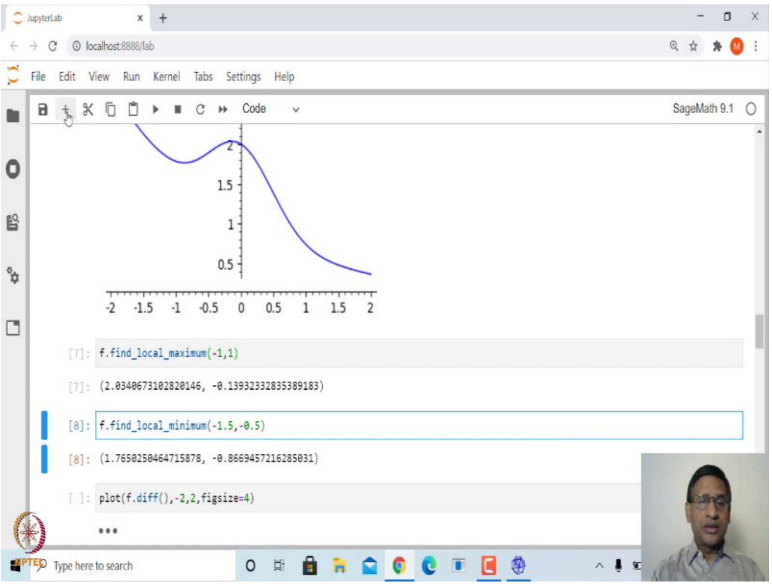

Let us use the inbuilt function first. The Sage has an inbuilt function called find underscore local maximum. In this case, we see that there is a local maximum between minus 1 and 1.

So if I execute this it will give me two values, the first value is the value of the function, that is, the maximum value, and this is the point at which the local maximum occurs. This points is obtained numerically, that is why you can save these numbers in decimal.

Similarly, if I want to find local minimum, you have mention the interval. In this case I have mentioned minus 1.5 to 0.5 and if you find this, you will see that the local minimum occurs at -0.86694 and so on and this is the minimum value of the function.

So, Sage has inbuilt function to find local maximum and local minimum. Of course, in case you mention some interval in which it does not have a local maximum or local minimum.

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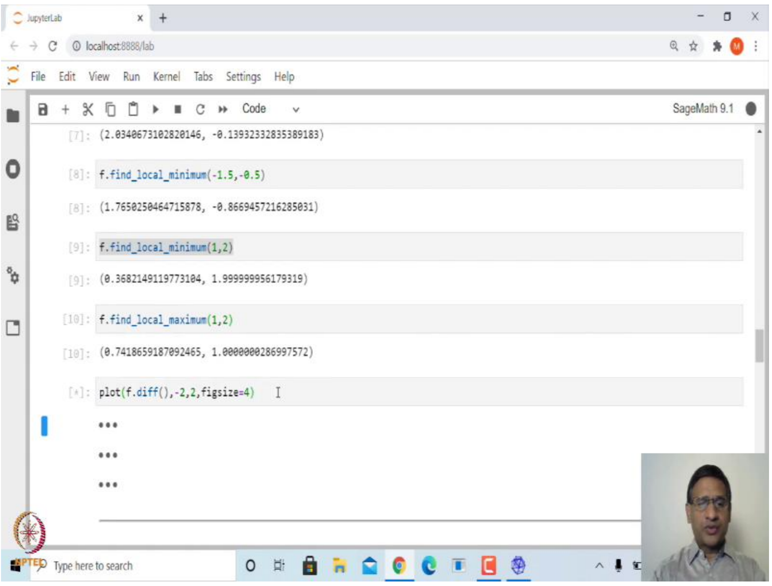

But it will find maximum and minimum of the function in that interval. For example, if I look at find local minimum between, let us say, 1 and 2.

By looking at the graph you can see that the minimum will be will lie at 2 and the maximum will be at 1. That is what you will see numerically. So, for example, the minimum value is at 1.999 which is very close to 2 and this is the minimum value.

Similarly, if I ask for what is the maximum value then, you will see the maximum value lies at 1. So, let us say, f dot find local maximum between 1 and 2. This will give you that the maximum lies at 1.00 and this is the maximum value.

So, this will always find maximum or minimum, though it may not be local maximum or local minimum, it may not even be the point.

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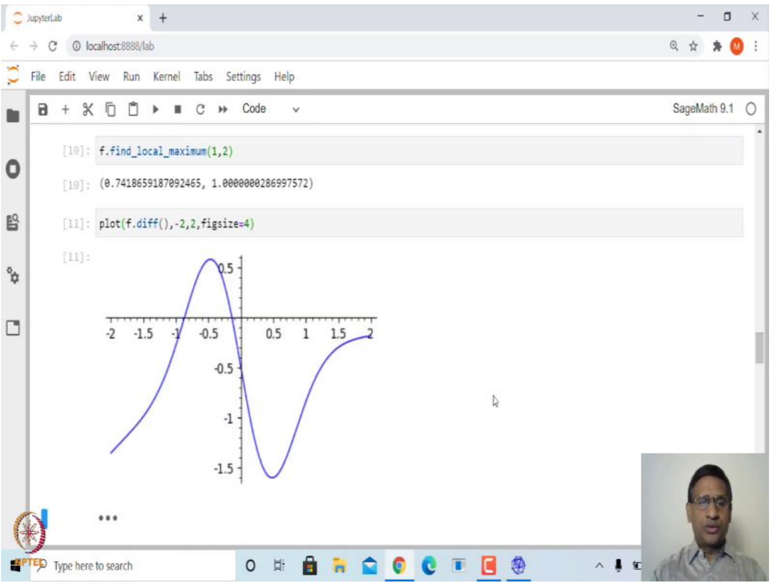

Now, if you try to find the derivative of f equate it to 0, then in this case you have to find the roots numerically.

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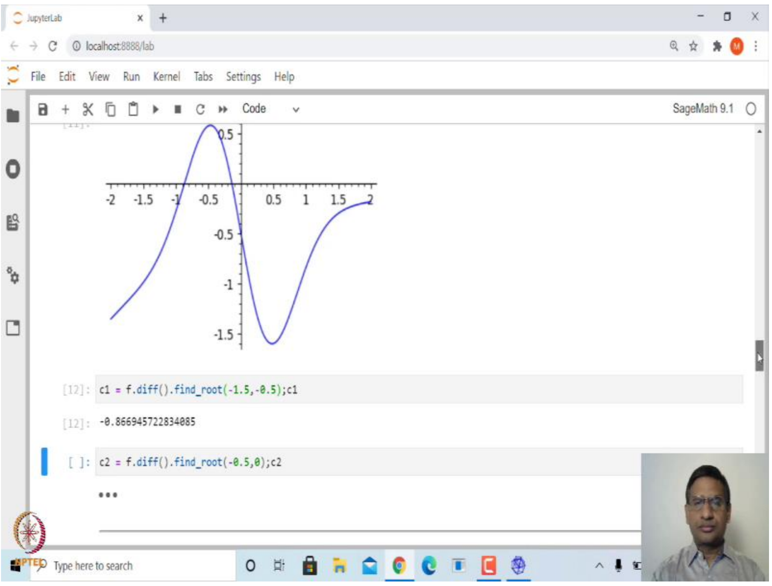

Now, if you try to find the derivative of f equate it to 0, then in this case you have to find the roots numerically. So, this is the first critical point where the derivative is 0, that is negative of 0.866 at this is the point of local maximum.

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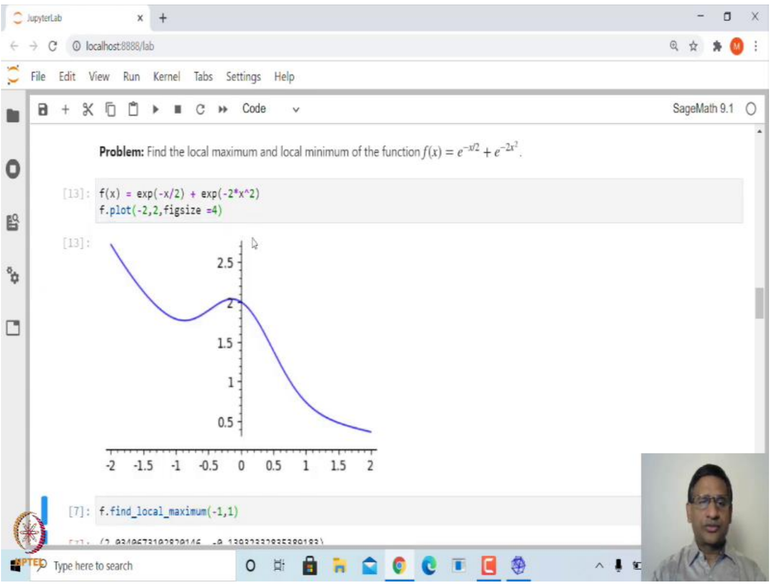

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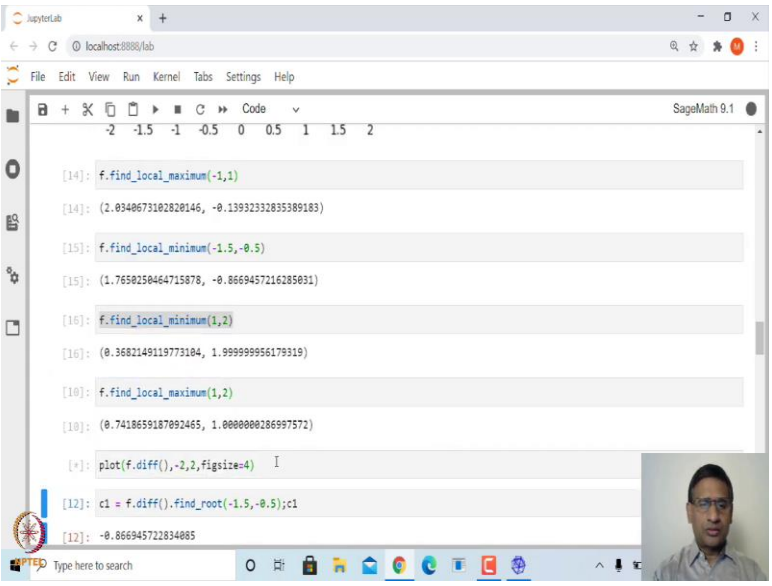

So, this is f of x. Just let me run once again, yeah.

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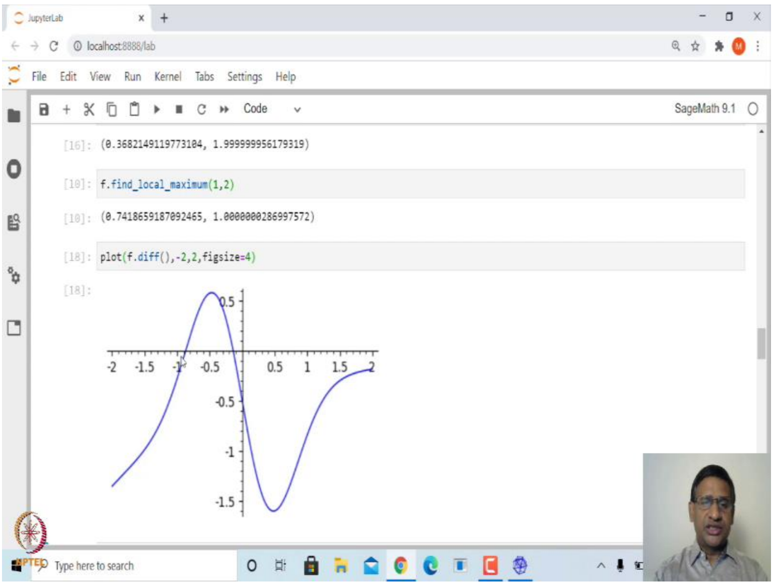

If you try to plot graph the derivative of this function, you see that it has a one zero between minus 2 and -0.5 and one between minus 0.5 and 0.5. So, these are the two critical points.

And to the left of this point the function derivative is negative; that means, function will be decreasing between this and this, the function the derivative is positive.

So, function is increasing again between this. And after that this is all negative so, function is decreasing. That is what you can also explain from this.

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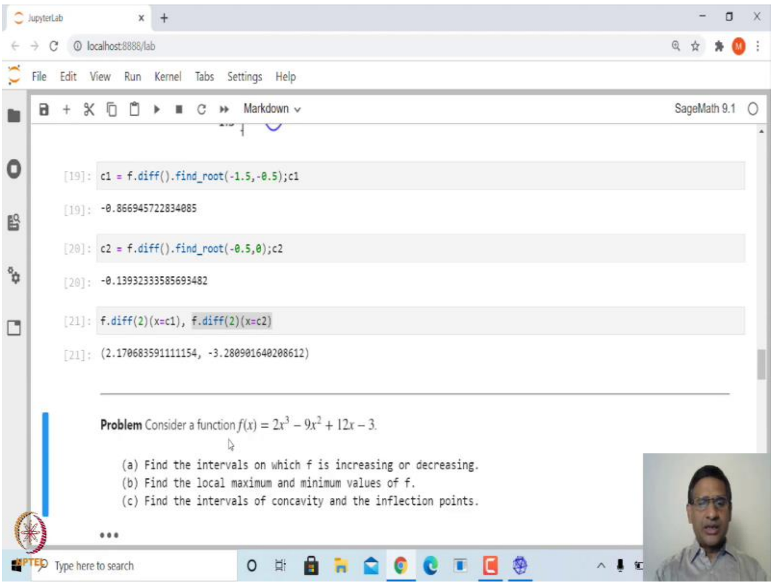

If you find the critical points the first critical point is this and then second critical point between minus 0.5 and 0, that the second critical point which was local maximum, Let us find the second derivative at these points. You can see here that second derivative at the first point is positive, so this point is local minimum. The second the second derivative at this point is negative so, this point c2 will a point of local maximum. That is what you see from the graph.

This is how you can find local maximum local minimum. In case, you have to find absolute maximum or minimum, you also need to evaluate the function value at end points along with local maximum and local minimum and then, compare all these. Then, you can find global maximum.

Next let us look at one more problem. Consider the function f(x) equal to 2 x cube minus 9 x square plus 12 x minus 3 and try to find interval on which this function is increasing or decreasing.

Already, we saw this in previous example where we found the local maximum local minimum of f. Now we find the interval of concavity and the inflection point.

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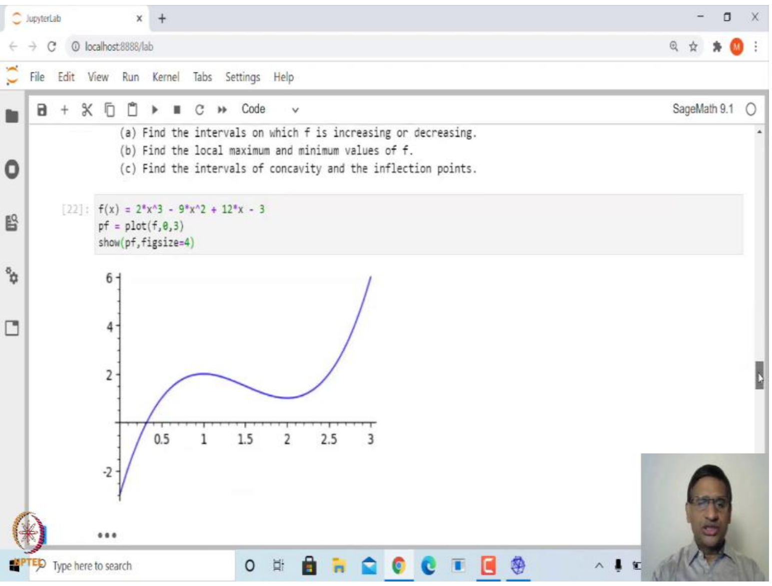

Again steps are very similar to what we have done earlier. I will just run this and then show you. This is the graph of this function and you can see here, there is one point at local maximum, this one local minimum and the function increases from here to here.

then starts decreasing and then starts increasing from here to here and somewhere here, there will be one point of inflection. So, we will try to locate all these things.

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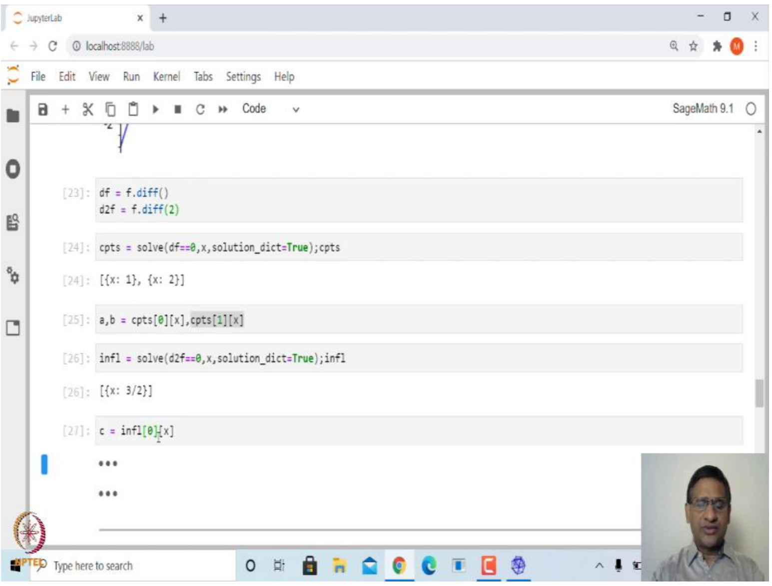

First let us find the first two derivative. Call it f1 and f2, and then, let us find these critical points. So, solve this derivative equal to 0 for x and I am using this option solution dictionary. It will give me solution as a dictionary. It says that there are two critical points x equal to 1 and x equal to 2; that is what you can see from the graph.

Now, let us store these critical points in a and b. That is the advantage of using dictionary, we can very easily extract the values.

Then also find the point of inflection that is the points at which second derivative vanishes. So, solve the second derivative equal to 0 and in this case, there is only one point which is x equal to 3. This point let me call as c. So, let us store this point of inflection in c.

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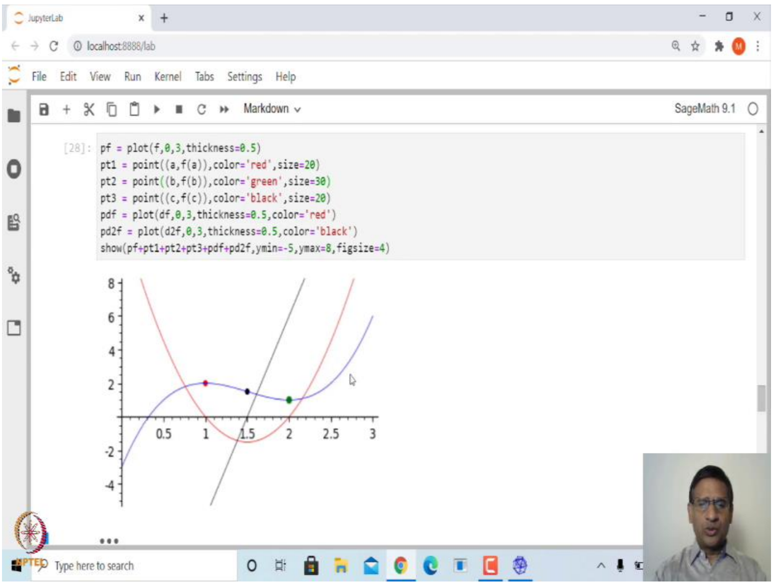

And then, let us plot all these things together. When we plot all these things together and this is how the. We have plotted the curve, that is in blue in color and these are the three points; red one is local maximum, green one is local minimum, and this black one is a point of inflection. This red color curve is the graph of derivative and you can see here it vanishes at 1 and 2 and those are the local maximum local minimum points.

Also this this black line is the second derivative, which you can see here, to the left of 1.5 it is negative and to the right of 1.5 it is positive. So, that is the at which the function is changing its concavity, and also the first derivative is negative from minus infinity to 1. When the function is increasing from 1 to 2, the second first derivative is negative. Function is decreasing and after 2 it is positive therefore, function is increasing. This connects all these notion of finding interval in which function increases, decreases in interval or the point at which the function changes its nature of concavity all these things are explained through this particular example.

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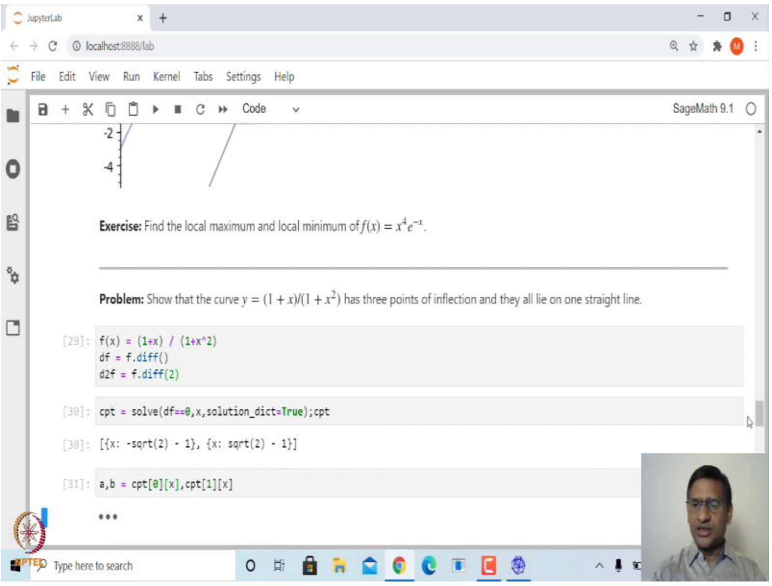

As a exercise you can find local maximum and local minimum of f(x) is equal to x to the power 4 into e to the power minus x. Let us look at another example. In this example, we want to show that in this curve y equal to 1 plus x divided by 1 plus x square has three points of inflection and they all lie on a single straight line. That is quite easy actually.

You need to find the critical points, these critical points you join the line and then, see whether they lie on the same line. Let me run this hm. Define f(x), define derivative and then, find out the critical points; these are the two critical points in this case and let us store these critical points in a and b.

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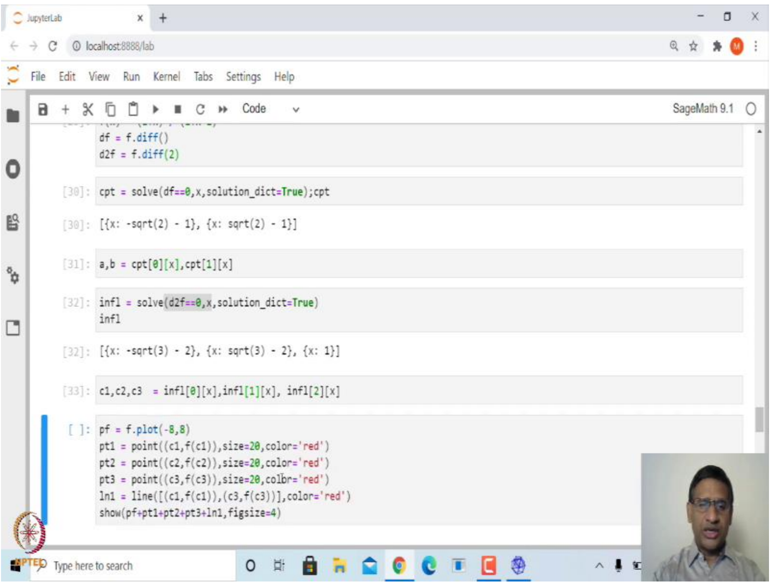

And let us find out the point of inflection, that is equate second derivative equal to 0. So, you can see here there are three critical three points, one is at minus 2 minus square root 3, other one is square root 3 minus 2 and the third one is at 1. So, again let us just store them in c1, c2, c3 and then, plot graph of the function along these inflection point and then, join the line joining c1, f(c1) and c3, f(c3).

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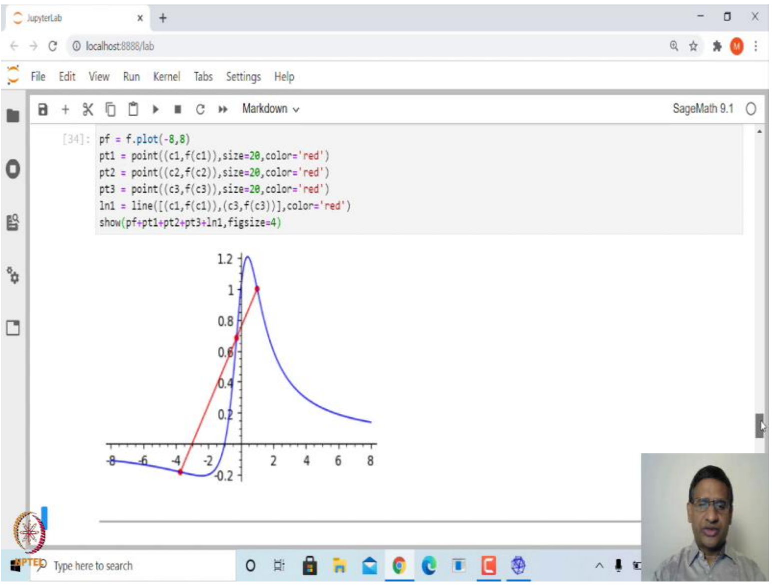

And then, see whether the second point also lies within this. So, that is what you can see here. So, this is your c1, f(c1); this is c3, f(c3) and this is c2, f(c2) and that all these three points are along the same curve. You as a exercise, you should also try to find out point of local maximum local minimum in this case as well. That you take as a practice exercise.

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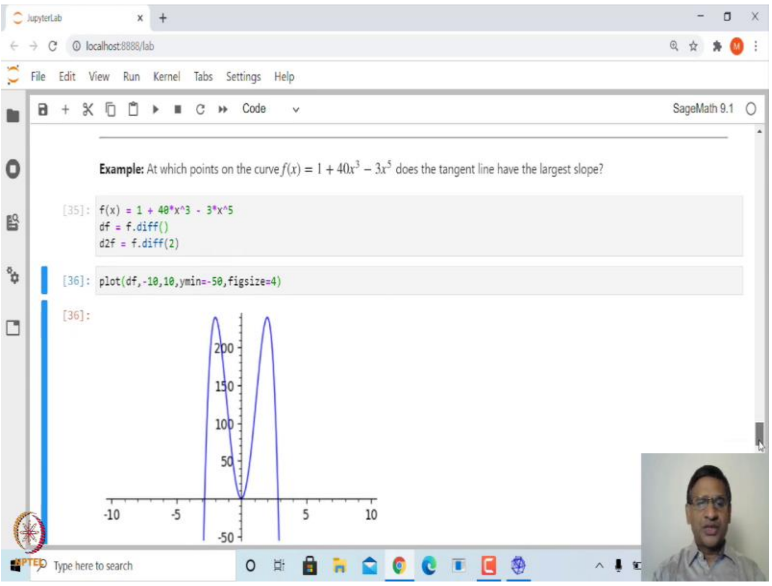

Next, let us look at, for example, this this function f(x) equal to 10 plus 40 x cube minus 3 x power 5, and look at the tangent and check at what point this tangent have highest slope. How do we do that? First let us define the function, find the derivative, first derivative, second derivative. Let us plot the graph of this function.

Plot the graph of this df, and this is the derivative. So, the point at which it has highest slope is the point at which the function the derivative has is maximum. And these are the two points you can see here. This is one point, this is another point. You can simply find the derivative, and find the local maximum and local minimum of derivative of f.

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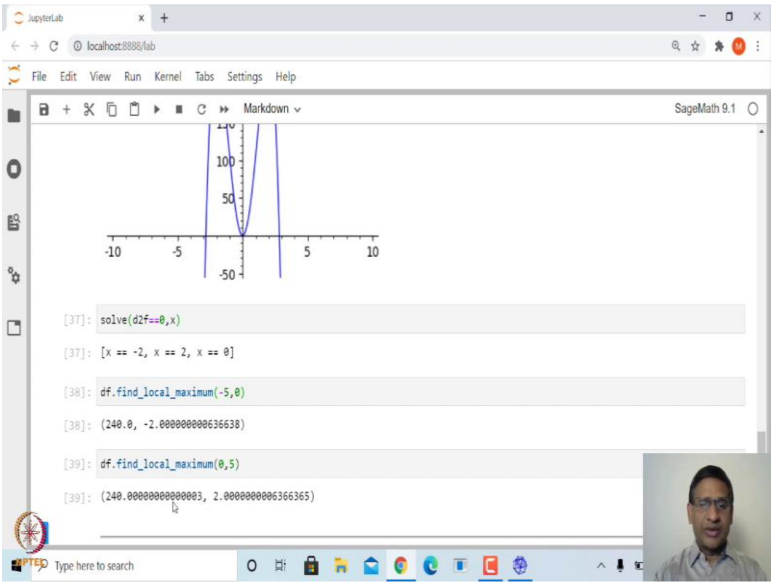

So, you have to find out second derivative equate to 0, that will be critical points of derivative of f and these are the three points. but at 0, it has local minimum.

So, if I try to find the local maximum between, let us say minus 5 and 0. That is the point, close to minus 2 it has local maximum. You can check the second derivative,

it will be positive, similarly second derivative will be negative that is why point of local maximum. Similarly, you can find another local maximum at 2.00. So, these are the two points at which this function, tangent has maximum slope.

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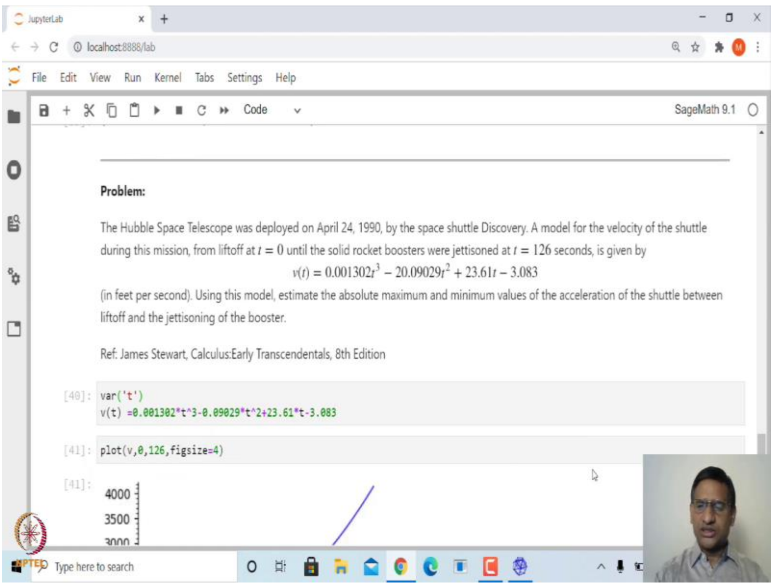

So, let us look at one problem as an application this problem. This taken from James Stewart book on Calculus Early Transcendent 8th Edition and this the problem is as follows:

The Hubble Space Telescope was deployed on April 24, 1990 by the space shuttle Discovery. A model for the velocity of the shuttle during this mission, from liftoff at equal to 0 until the solid rocket boosters were jet jettisoned at timed 126 seconds, is given by this particular equation. And this is in feet per second the velocity. Using this model, estimate the absolute maximum and absolute minimum values of the acceleration of the shuttle between the lift off point and jettisoning of the booster. This is quite easy actually, but this is an application in in Physics in rocket launching. If you want to find the acceleration. This is a velocity, acceleration is nothing but rate of change of velocity.

So, all we need to do is find the derivative of v and find the local maximum and local minimum of that. Let us define this function v. Let us plot its graph.

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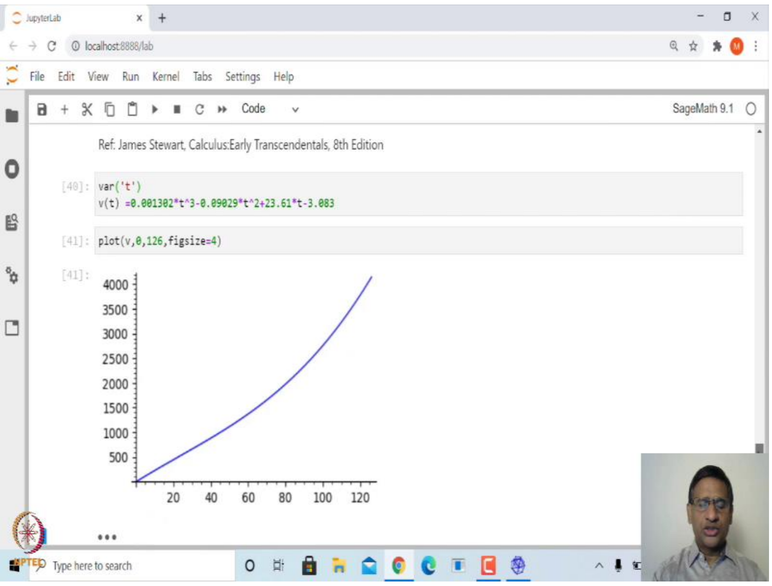

So, the graph of the velocity looks like this. You can see here this is not a straight line it is going like this.

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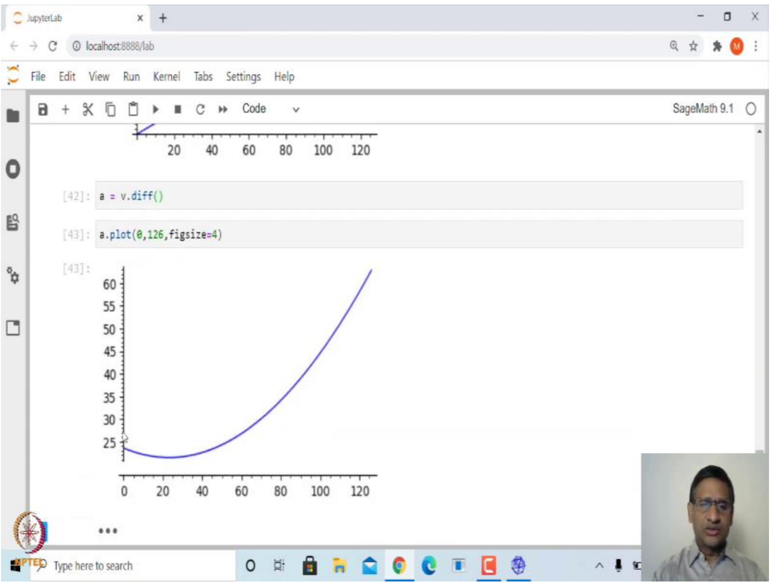

Let us now find out what will be the derivative of the velocity that I have stored in a that is, acceleration. And if you try to plot graph of this acceleration this is how it looks like. You can see here, somewhere is a local minimum, somewhere here then, it keeps on increasing till 126 second.

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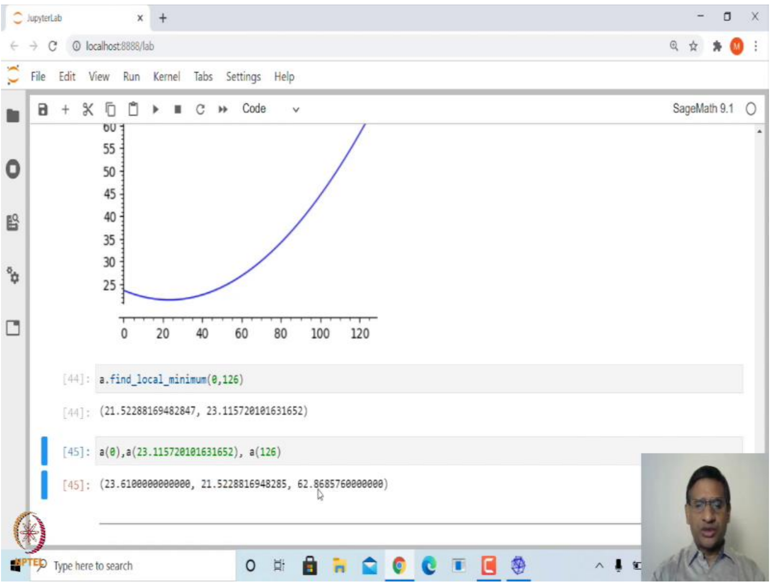

And if you try to find the local minimum of acceleration, it is about 23.1157 second. We need to compare the acceleration. At 0 the acceleration is 20, at 23.61 about 21 and here it is 62. So, the acceleration is minimum at 23.1157, at 126 second and the maximum acceleration is this. So, that is a very easy problem.

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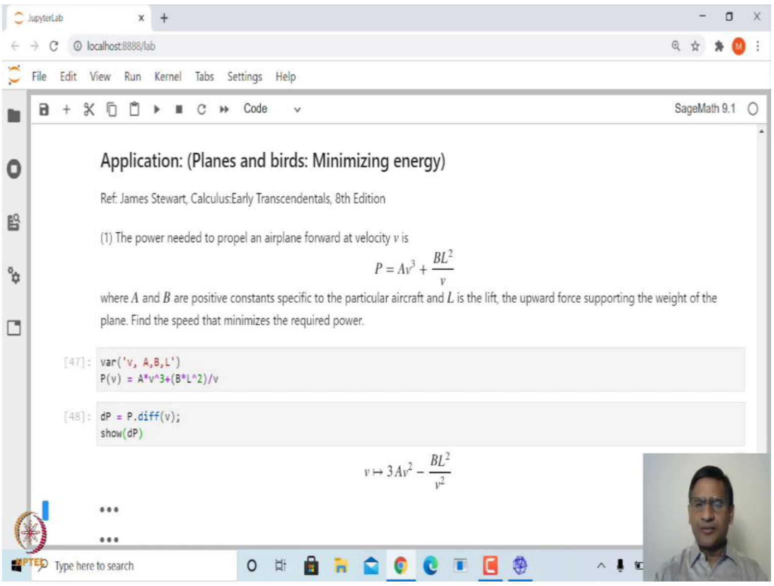

Now, let us look at one more problem as an application. This is again taken from this James Stewart calculus book. This is regarding the planes and birds minimizing energy when they fly. The first problem is power needed to propel an airplane forward at velocity v is given by this.

So, the power is a function of v, where a and b are positive constants specific to the particular aircraft, L is the lift and upward force supporting the weight of the plane. You have to find the speed that minimizes the required power. You have to minimize P for v. Let us define this. Declare all these variables A, B, L and v and define this power function as a function of P and then, let us find the derivative. So, the derivative of this looks like this.

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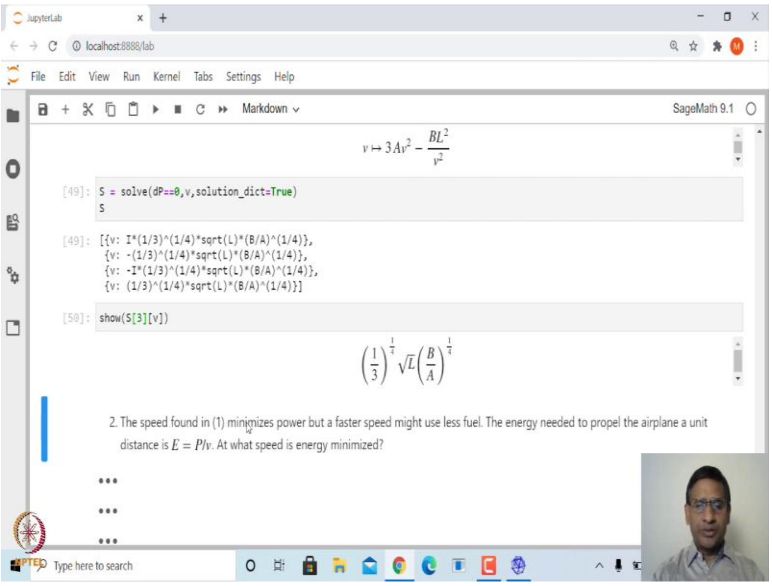

Then, we find the derivative equate to 0 and solve for v. In this case, you can see here that there are four roots, but then three of them are imaginary. Only the fourth root is real. That is the point at which this power is minimum, that is the velocity at which this power is minimum. Let us store this and let me show you. So, this is the point, the velocity at which the power is minimum.

Next, let us look at the speed found in problem 1, that minimizes the power. But a faster speed might use less fuel. The energy needed to propel the airplane a unit distance is given by E, the energy is given by P, power upon the velocity. Then at what speed, is the energy minimum.

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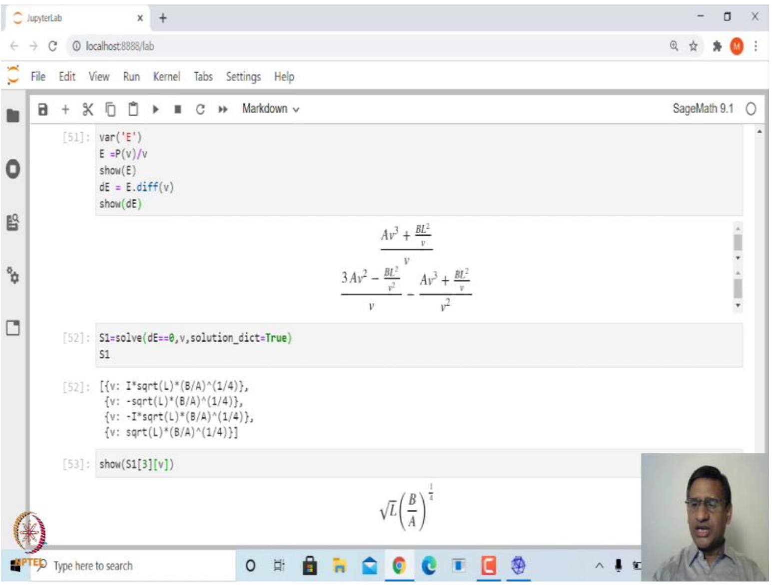

Let us define E as P v upon v and then find it's derivative. Equate it to 0 and then solve for velocity v, and then, again it has a single real root. That is the velocity at which this energy is minimum. So, this is the velocity at which the energy required is minimum.

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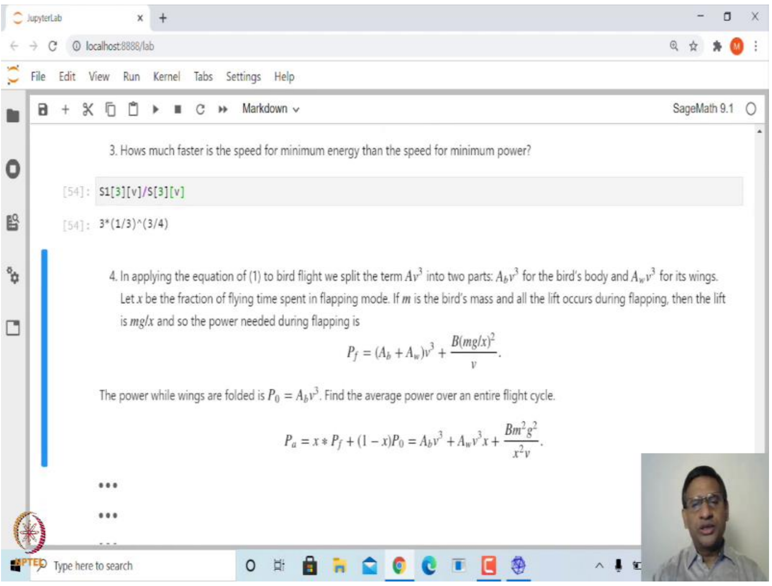

The 3rd problem is how much faster is the speed of the minimum energy than the speed of the minimum power? We have already computed these two things. We will just take the ratio of minimum power divided by minimum energy and it says that the ratio is this 3 into 1 by 3 to the power 3 by 4.

The 4th problem is suppose, this we applied to bird flight and in this case, the term A v cube is split into two parts Ab v cube and Aw v cube for.

So, Ab v cube is the bird's body and Aw v cube is bird's wing. Suppose, x is the fraction of the flying time spent in flapping mode that is in wing flapping and if m is the bird's mass and all the lift occurs during the flapping. That is what that is the assumption then, the lift is given by m g upon x, where g is the acceleration due to gravity.

So, the power needed during this flapping will be given by Pf is equal to Ab plus w into v cube plus B into m g by x, whole square upon v. And then, the power while wings are folded is let us say, P0 which is a b v cube and then, you have to find the average power over an entire flight cycle.

The average power P, will be x times the Pf, that is during the flapping and this 1 minus x0 during this folded one. So, this is given by this that suggests a expression.

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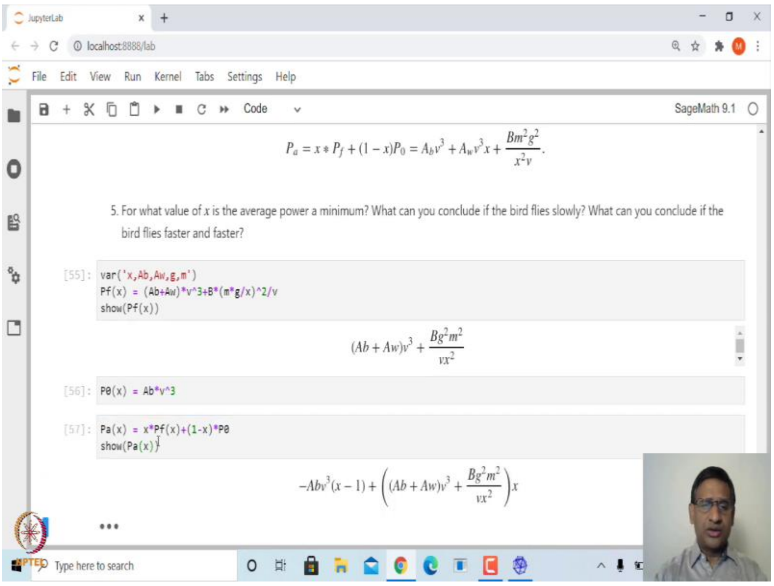

And then, now, the question is for what values of x the average power is minimum and. Let us find out. Again let us denote this average power, this Pf is equal to this. This is Pf, which we have defined earlier and next we defined P0. So, that is, Ab v cube and then, let us take the average power as Pa, is equal to x into Pf plus 1 minus x into P0. So, that is the average power.

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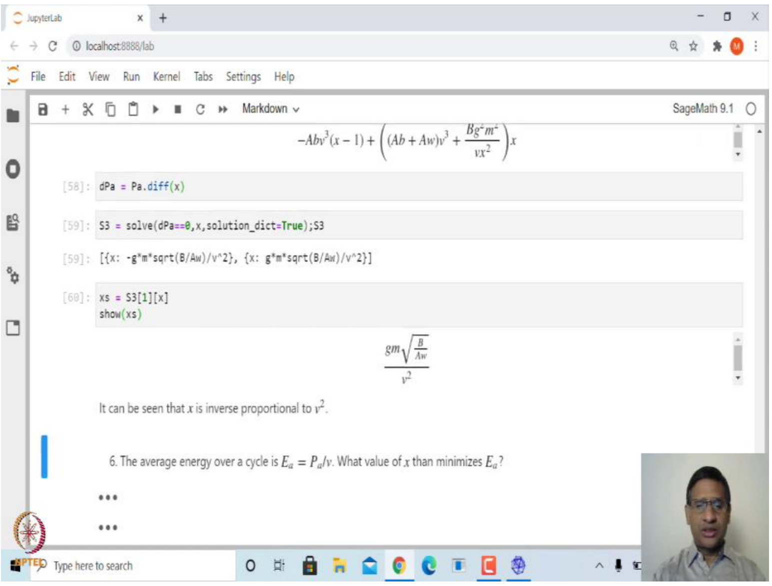

And then, let us find the derivative of average power dPa and solve the derivative equal to 0 for x. Again, it has two roots, but one is negative, which is not possible.

The positive root is given by this. That is the x at which this average power is minimum. You can see here, this is proportional to 1 upon v square.

So, as the velocity increases, this x will decrease. That is the observation. Next, let us look at the average energy over a cycle Ea is given by the Pa upon v. Pa is the average power, then for what value of x, this is minimized.

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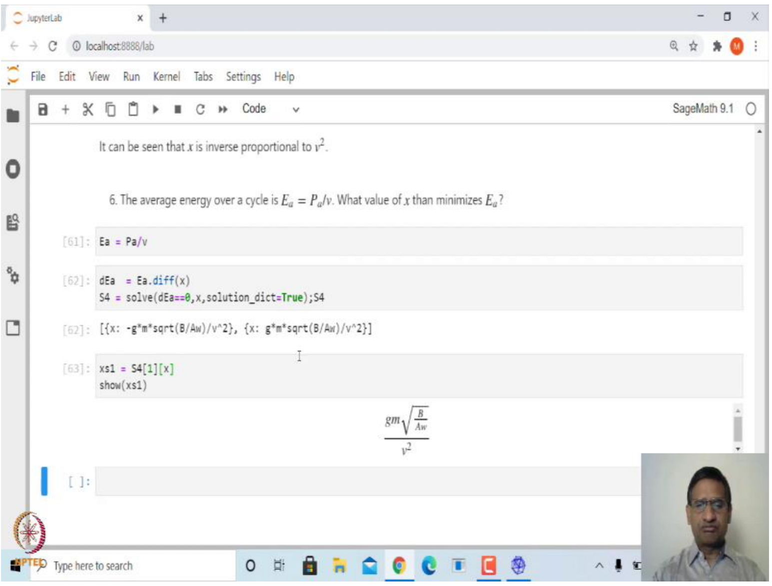

Again that we can just define Ea and then, find the derivative of Ea, equate it to 0 and then, solve for this again. It gives me two values, but only one is negative one is the positive. That is the point which is equal to this. This is exactly same as what you got for minimum value of average power in terms of x.

So these are the some applications of derivative and you can look at more applications. Solving these problems of minimum, maximum etc, basically, is very easy in in SageMath.

Whenever you are unable to find in the closed form of the critical points, you can use this numerical local maximum local minimum that is at find underscore local maximum local minimum. Similarly, to compute the critical points, you need to use find root function, but first you need to plot the graph of this function.

Let me stop here. Thank you very much. Next time we will look at finding integrals in SageMath.