Computational Mathematics with SageMath
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Lecture – 20
Calculus of one variable with SageMath Part 1

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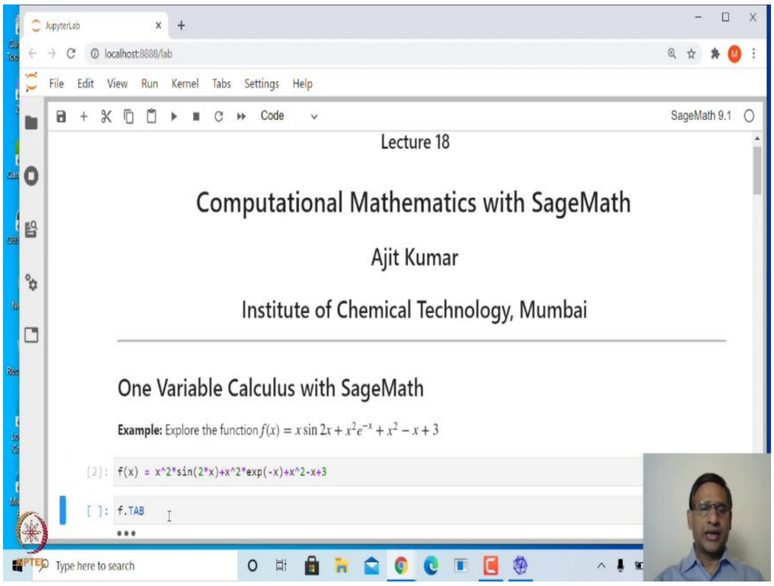

Welcome to the 20th lecture on Computational Mathematics with SageMath. In this lecture, we will explore some of the concepts from one variable calculus using SageMath. So, let us get started. So, suppose you have a function $f(x) = x\, sin(2x) + x^2\, e^{-x} + x^2 - x + 3$. This is a fairly complicated function.

Now, suppose you want to apply concept of calculus to this function f then what we can do is, we can define this function $f(x) = x^2 \sin(2x) + x^2\, e^{-x} + x^2 - x + 3$.

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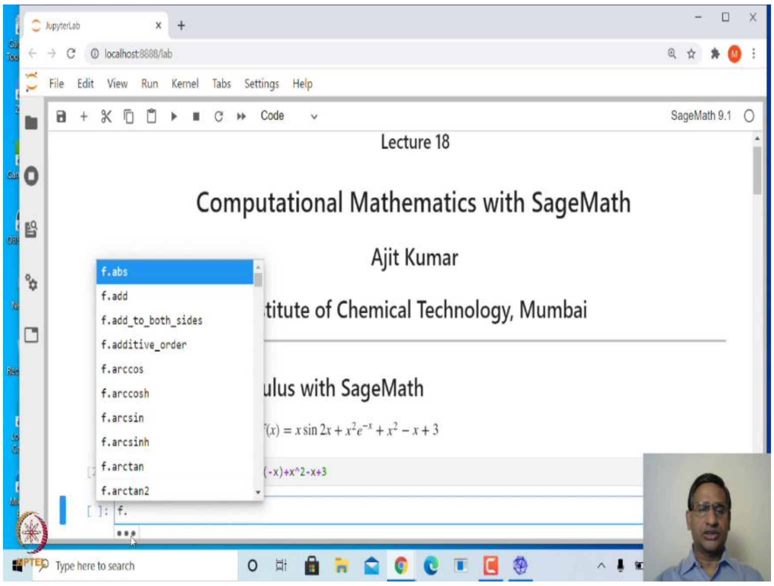

Now, you can apply f dot, and then tab f dot tab, when you apply, you will see some of the standard concepts in calculus. So, for example, the first concept that you generally come across is going to be the limit of a function.

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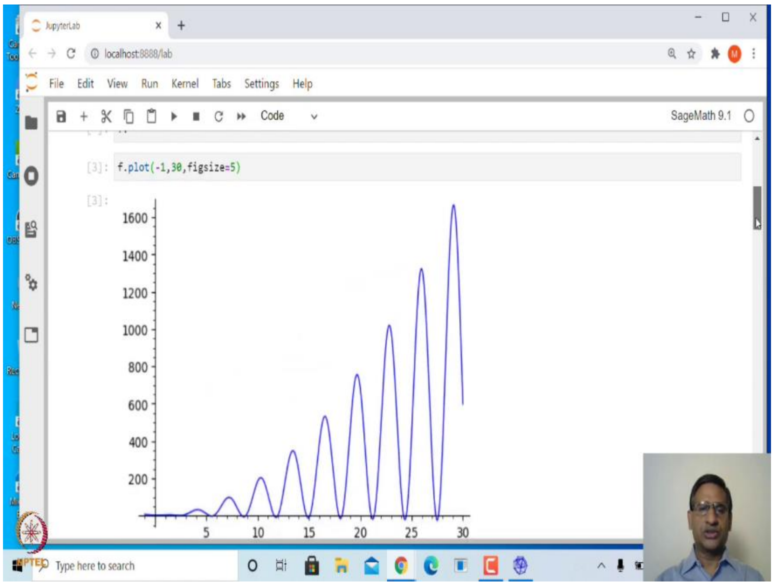

So, there is an inbuilt function to find limit of a function, there is an inbuilt function to find derivative of a function, similarly integral of a function. So, first let us plot graph of this function, because when you know the graph of a function many concepts regarding f will become clear. So, let us first plot its graph between -1 and 30. This is how the graph looks like.

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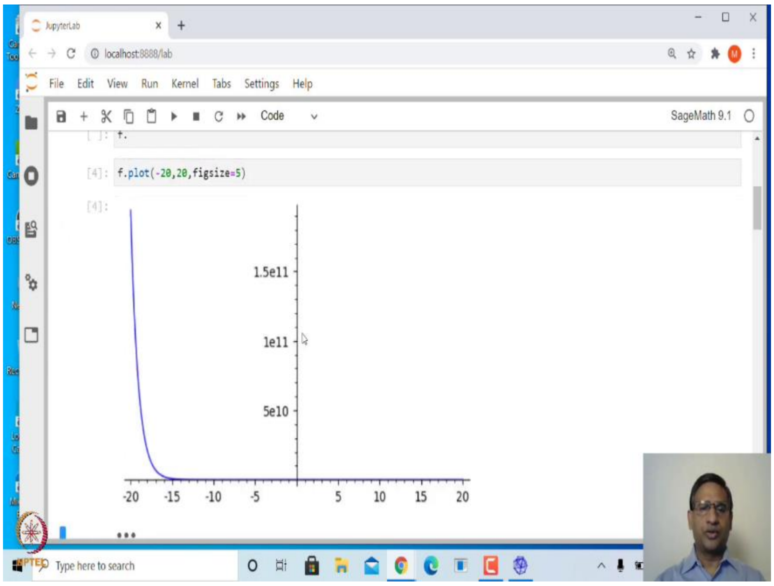

So, let me, let us plot this between -20 and 20. So, when you plot the graph, what happens is on the left hand sides because of the present of e to the power -x when, x is negative e to the power -x will go to infinity.

And therefore, this side you see this is the function, for example, around this point the maximum value is I think one point into 1.5 into 10 to the power 11 ok. So, that is what is happening; whereas, on the right hand side, though the function is increasing, but the increment is not that very high.

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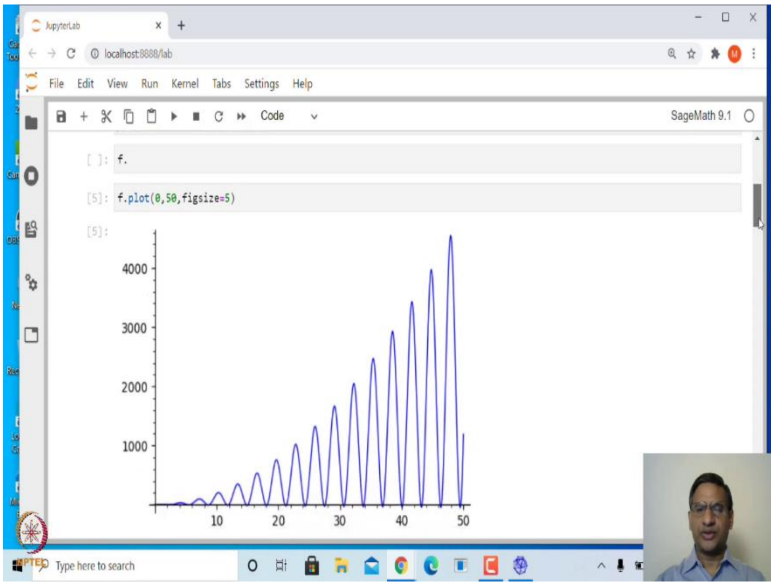

So, if I plot between 0 and let us say 50, then what you see is this function is increasing.

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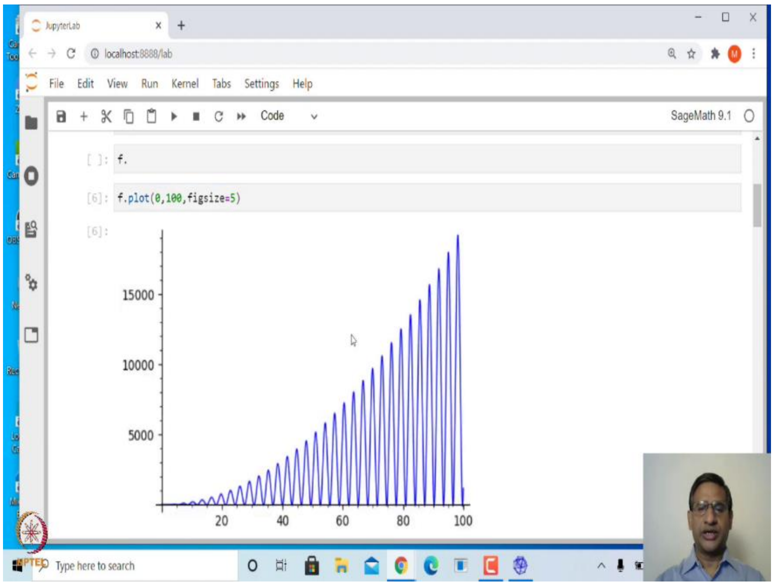

And let us say, if I say instead of 50, let us put 100, and then this is what you get. So, again this function is increasing, but not with that high speed, right.

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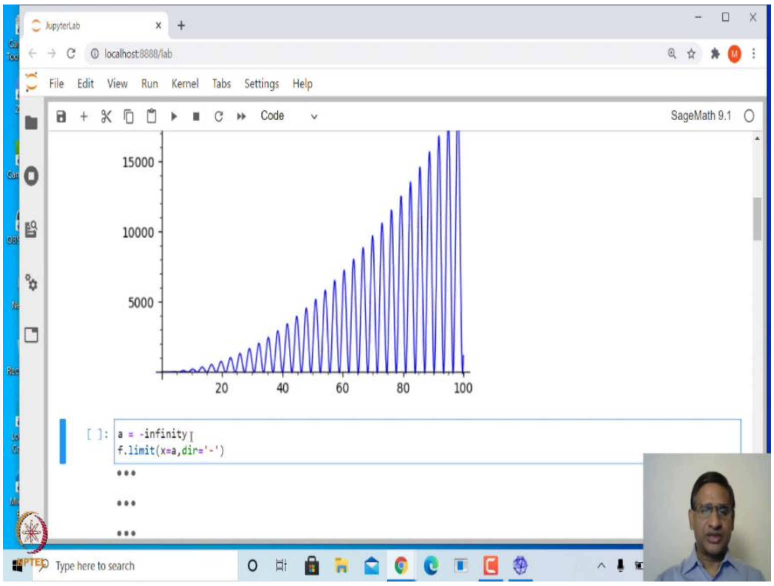

So, now let us look at suppose we want to find what happens to the function as x goes to - infinity. So, we can find limit of f using f dot limit, and you can mention the point at which you want to find a limit. So, x equal to a in this case and we are taking a to be minus infinity.

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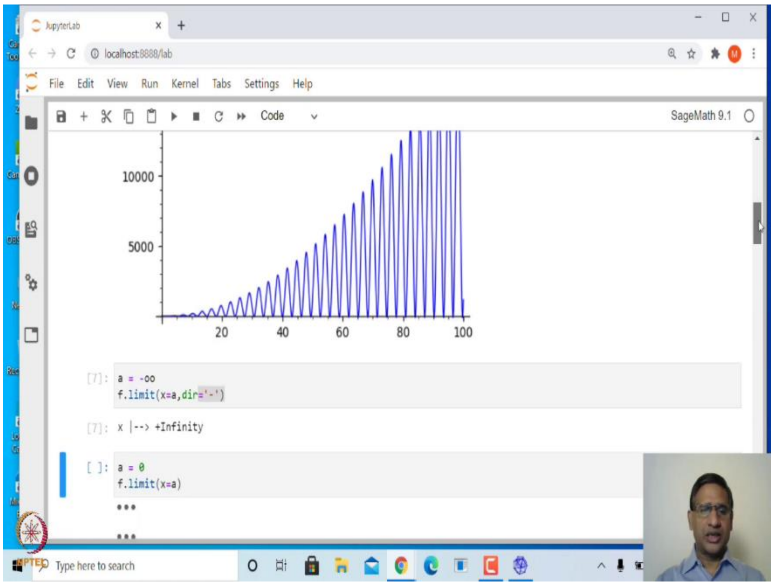

Instead of minus infinity, you could also write minus small o, small o that also is infinity. And here dir is equal to inside single quote minus means it is left hand limit ok. So, this is the left hand limit of f at x equal to minus infinity. And it takes little time, but let us see, yeah. So, this goes to infinity that is what we observed from the graph. So, graph already tells me that this is what should happen.

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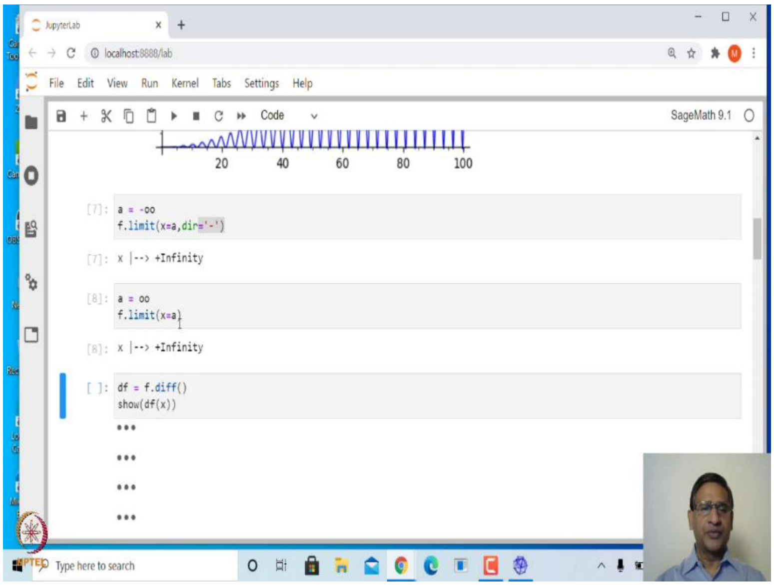

Whereas, if I take a = + infinity that is oo, and then try to find out what is the limit, then it also goes to infinity though we saw that increment on the positive side is not very high, but it goes to infinity.

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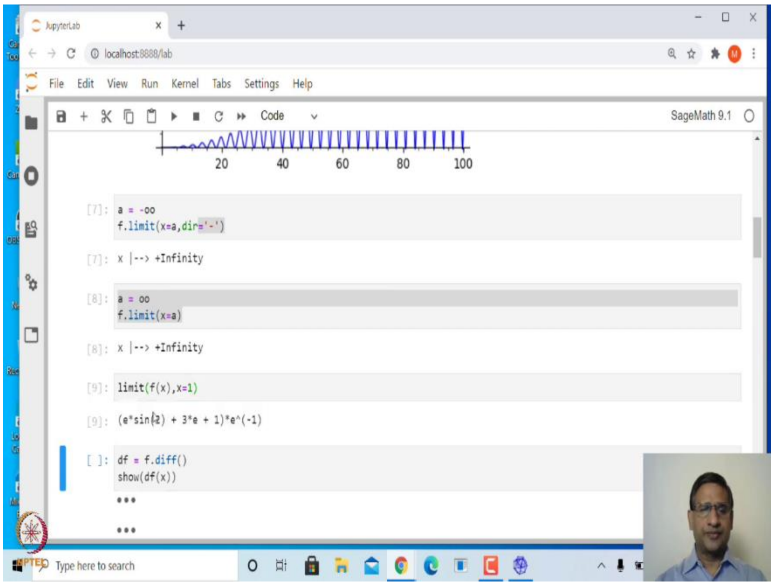

You can also find a limit at any point. So, for example, if I want to find out limit of f at, so let me just use limit of $f(x)$, at x is equal to let us say 1, then this is the value.

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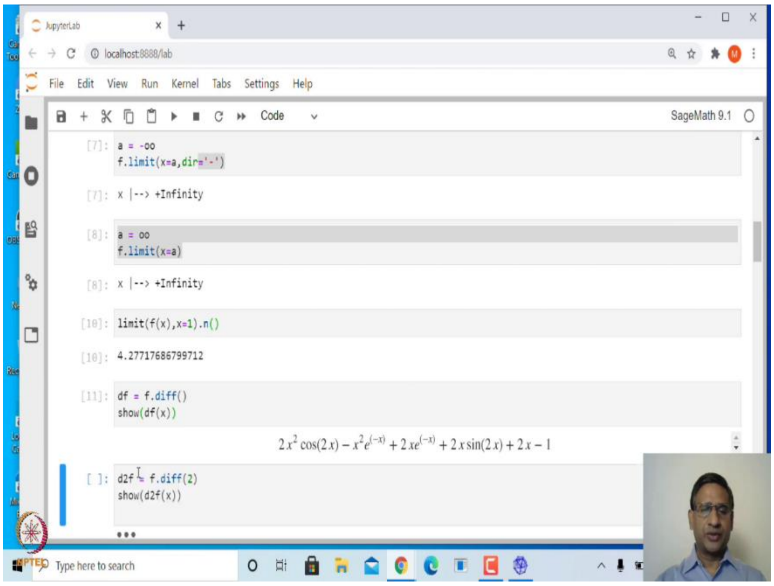

Of course, you can find you can say dot n to get the numerical value. Similarly, you can find the derivative of the function using f dot diff. And let us store this in df and ask it to show, what is the derivative. So, this is the derivative of the function first derivative. If you want to find, let us say second derivative in the bracket you can mention 2. If I say want third derivative, I have to say here 3, and so on.

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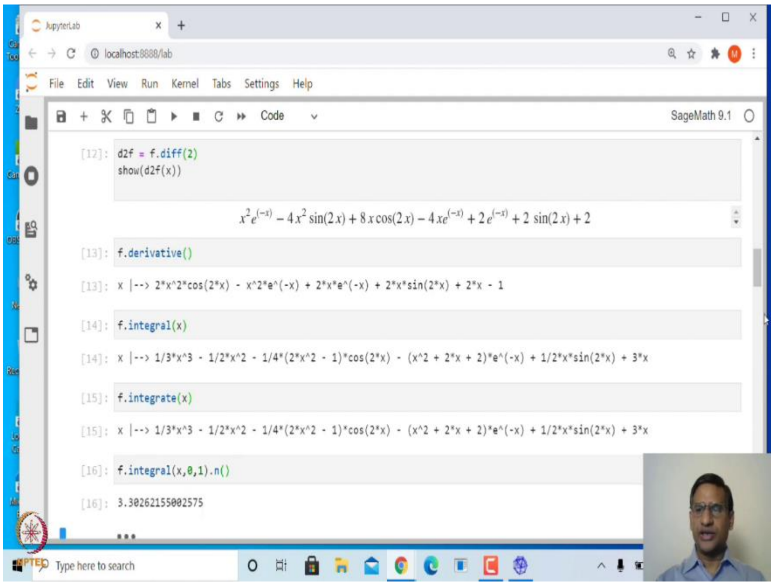

So, let us see what is the second derivative that is the second derivative. Similarly, you can also find derivative using f dot derivative function and it works exactly in a similar way, or you can also we can also find integral of the function. So, f dot integral, and then you need to mention the variable with respect to which you want to integrate so that is an indefinite integral.

Similarly, you can find this integral also using integrate function, so that will also give you the same answer. On you can if you want to find definite integral let us say integral between 0 to 1, then you can say f dot integral with respect to x and comma 0 to 1, 0 comma 1 that is the limit, right. So, this will give you definite integral.

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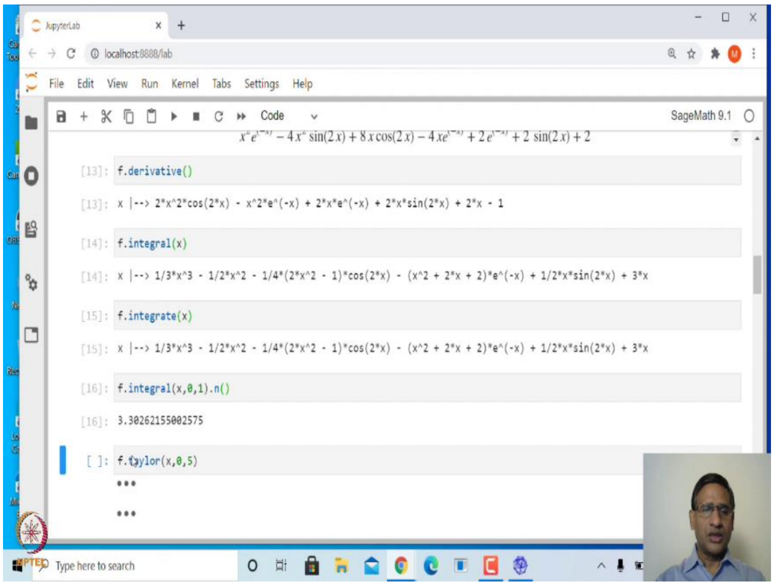

And similarly, you can also find Taylor's polynomial of whatever degree you want with respect to whichever point, so that is also there. Of course, all these concepts we will do it in detail. But what I am trying to tell you is that once you define a function and then apply f dot tab, then you have option to find limit, to find derivative, to find integral, to find Taylor's expansion, all these things exist and we will see in more detail.

Another thing you should keep in mind as I have been telling this repeatedly that for example, we want to find limit derivative integral using a function limit diff or integral then this function will also work not only for function of one variable, but also function for more than one variable so, the same function ok.

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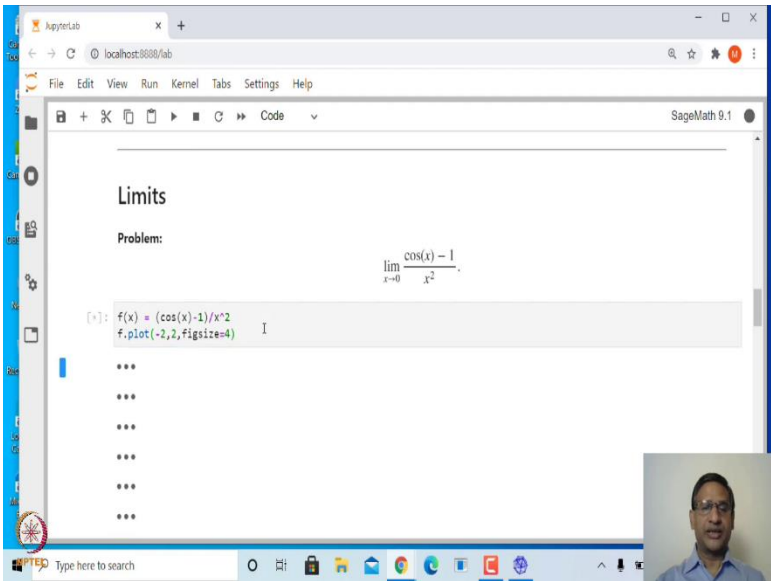

Now, let us look at some more examples of limit. So, first let us say we want to find out what is limit of $cos(x) - 1$ divided by x^2. So, what do we do? First, we define this function $f(x)$, and I have said earlier x is by default a variable in Sage.

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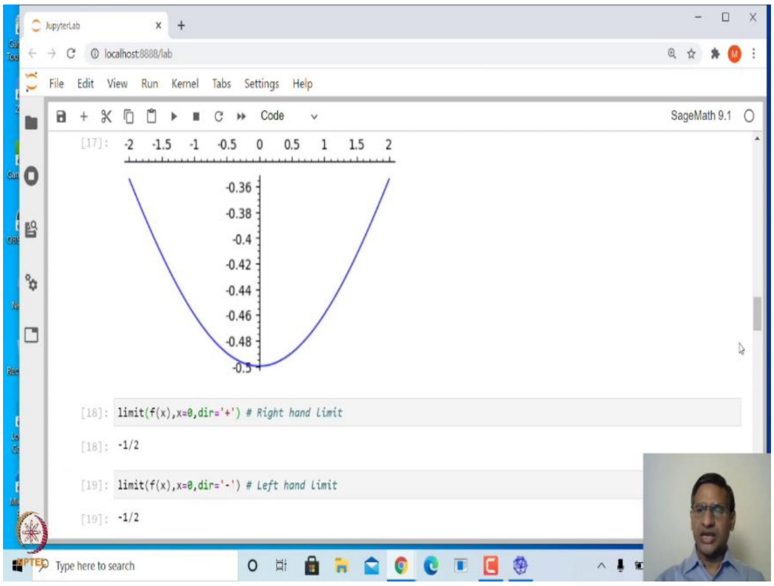

So, and then let us plot its graph between -2 and 2. So, the function the denominator is x square which is 0 when x goes to 0, and that is where it if at all the limit has a problem, it will be at 0. But, by looking at the graph, it looks like this function has a limit at x equal to 0. Of course, this function is not defined at x equal to 0, but we can find its limit.

So, let us look at what will be its limit. So, if I find limit of f(x) at x equal to 0, the right plus will give you right hand limit, similarly this is equal to -1/2 and that is what you can observe from the graph. Similarly, the left hand limit will be -1/2.

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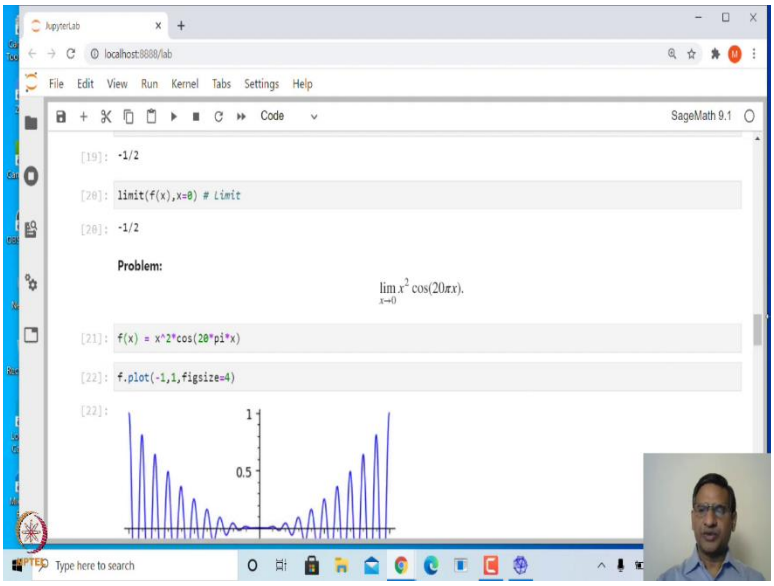

And if we want to find the limit of this function, I do not need to mention the direction, I have to only say x equal to 0, so that is the limit and this is equal to half. So, in this case, for this function left hand limit is equal to right hand limit which is equal to the limit of the function, right.

So, of course, if we define, so if I define function value f(x) is equal to half at x equal to 0, then this function will also be a continuous function because left hand limit is equal to right hand limit is equal to value of the function, that means, function will be continuous.

Similarly, let us look at another problem. So, if we want to find limit of $x^2\cos(2 * pi * x)$. Now, to find the limit of this, again let us plot graph of this function. So, first let us define what is $f(x)$ and then let us plot its graph between -1 and 1.

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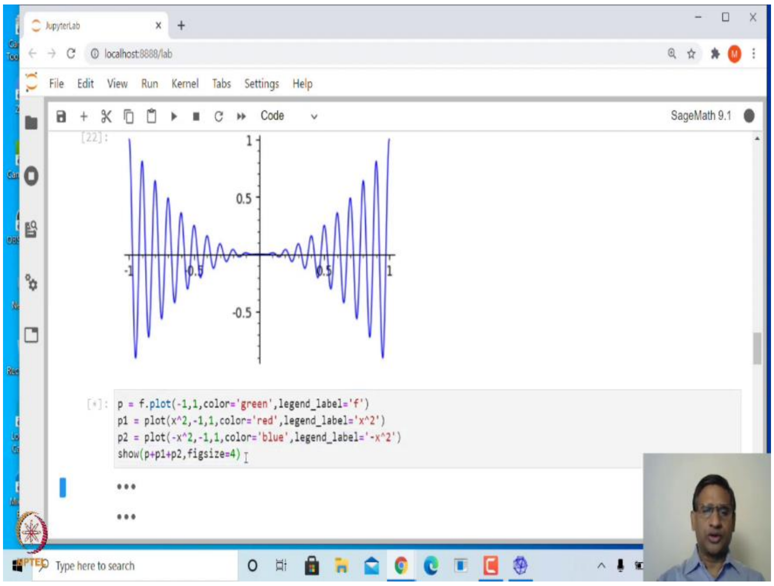

So, this is how the graph looks like. So, you may observe here, for example, x square, this cos always lie between -1 and 1. So, this function x square into cos x will always lie between -x square and + x square, and that is what you can see from the graph. But, let us plot the graph of $y = x^2$ and $y = -x^2$, and add it to the graph of this, and then see what happens.

So, if p, I am storing the plot of the function, p1 is the graph of x^2 with red color, and we are also giving the legend x square; and p2 is the graph of the function $-x^2$ and with blue color and let us ask it to plot.

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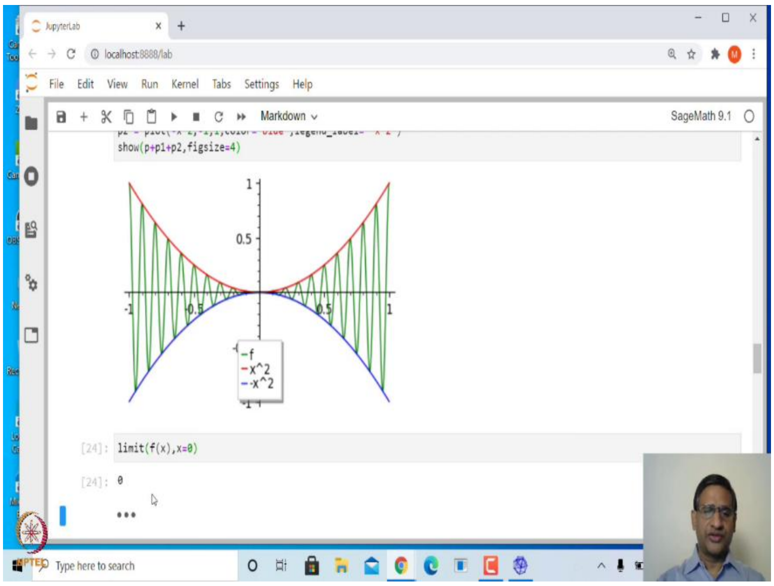

So, when you plot this, you see that, this graph of the function $f(x)$ is sandwiched between $y = x^2$ and $y = -x^2$. Now, both this function $y = x^2$ and $y = -x^2$, the limit of x^2 and $-x^2$ at x equal to 0 is 0.

And since the function is sandwiched between these two, the sandwich theorem tells you that the limit of the function $f(x)$ will also be equal to 0 at x equal to 0. So that is a demonstration of sandwich theorem, and which tells you that in case $f(x)$ is a function which is sandwich between two function let us say g1 and g2.

And the limit of g1 and g2 at some point x equal to a is let us say, it exist and equal to a, then limit of the function at x equal to a will also be a, that is what it means, right. So, now we can compute what is the limit of the function at x equal to a, this is equal to 0.

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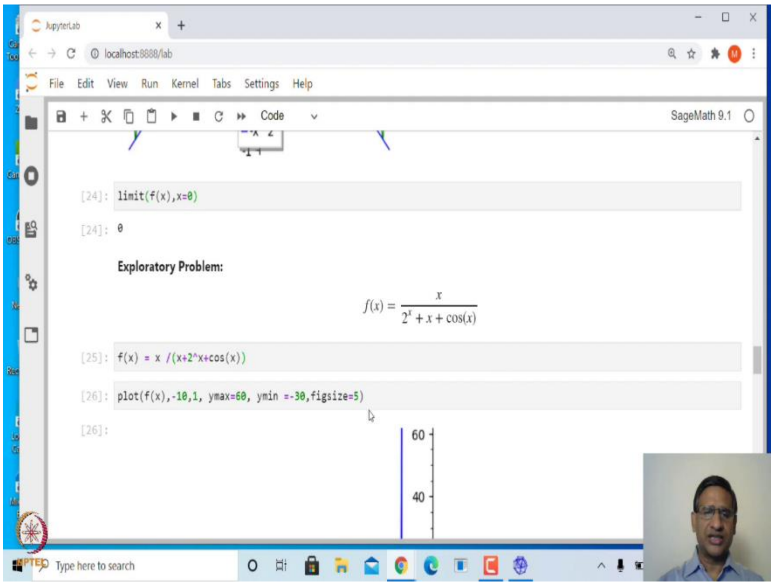

Now, let me also look at another problem which is $f(x) = \frac{x}{2^x + x + \cos(x)}$. So, this let us explore what happens to the limit of this function at various points. So, as usual before we explore the limit, we will plot graph of this function.

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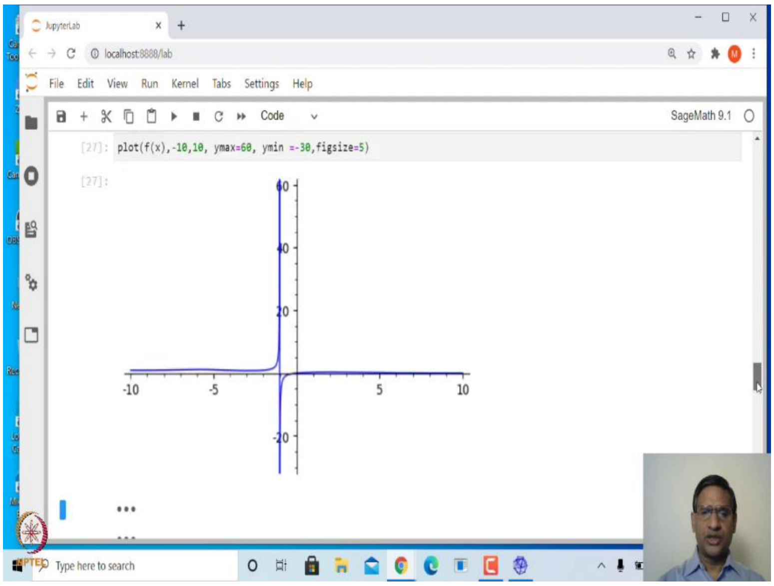

So, when you plot graph of this function, let us first plot graph of this function between -10 and 10, so that is how the graph looks like, ok. So, what looks like is that when x is increasing, then it seems to be touching the x-axis, when x is decreasing along negative side, again it is very close to x-axis. Whereas, at some point here to the left of origin the function seems to be going to infinity from the left and going from to minus infinity from the right that is what we see ok.

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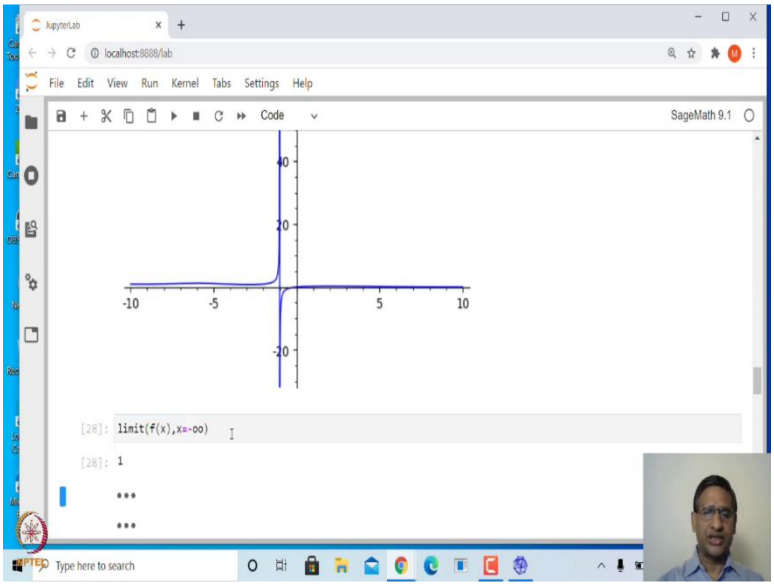

So, let us try to look at what happens to the function when x goes to -infinity. So, limit of $f(x)$ at x equal to -infinity, this is equal to -1 actually. So, it is not touching x axis, but it is becoming close to 1 y equal to 1.

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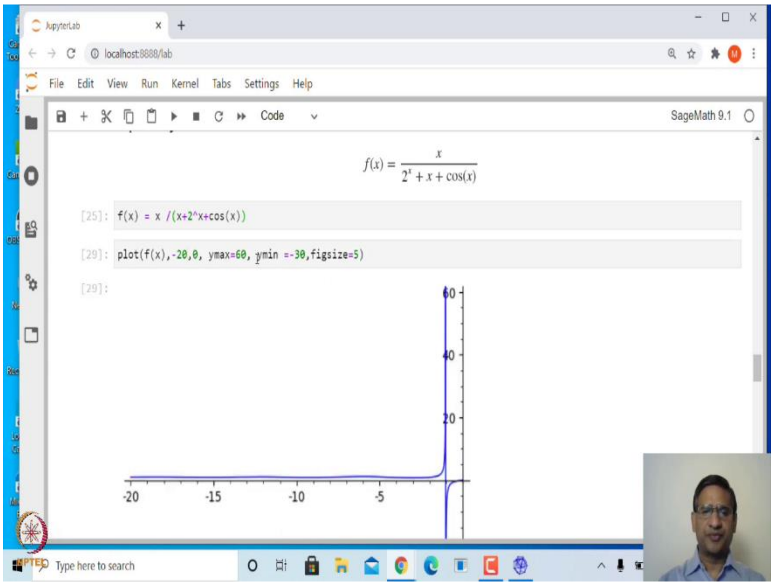

So, if you want, we can even plot this graph between let us say -20 and let us say 0, and then you will see that this is let me restrict the maximum value of y to let us say 2 and minimum value to let us say -3.

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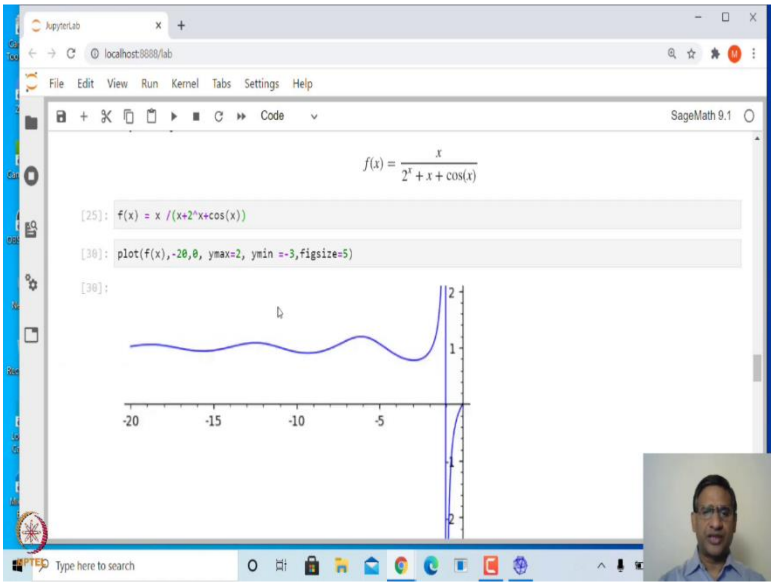

And then you will see what happens to this, right.

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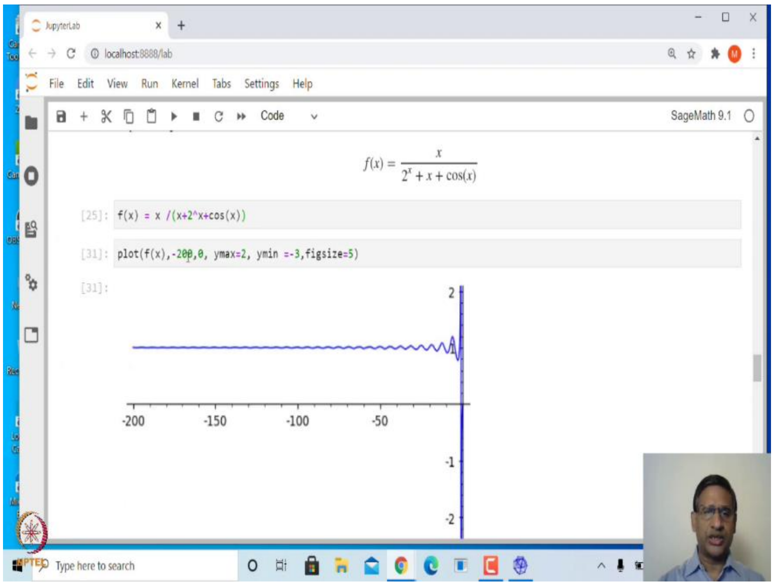

So, as if you plot a horizontal line $y = x$, you will see that as x goes to let us say -200, this will become very close to 1, right so that is what is happening. So, this is how you can possibly get to know about the function behavior by plotting graph in various domain and then you can verify this using inbuilt function limit, right.

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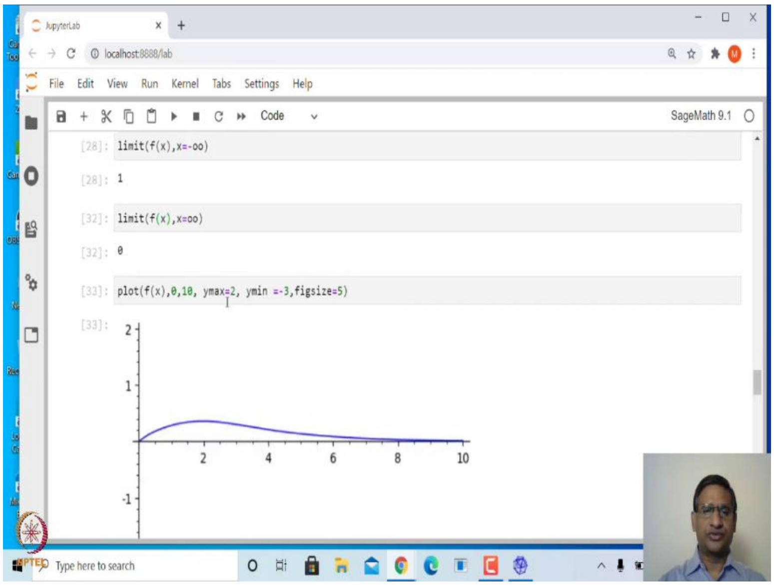

Similarly, if you look at what happens with the function when x goes to infinity, this goes to 0. So, again you can try to plot its graph, if you plot its graph let us say, let me plot graph of this from let us say 0 to 100, and then that is what you can see here.

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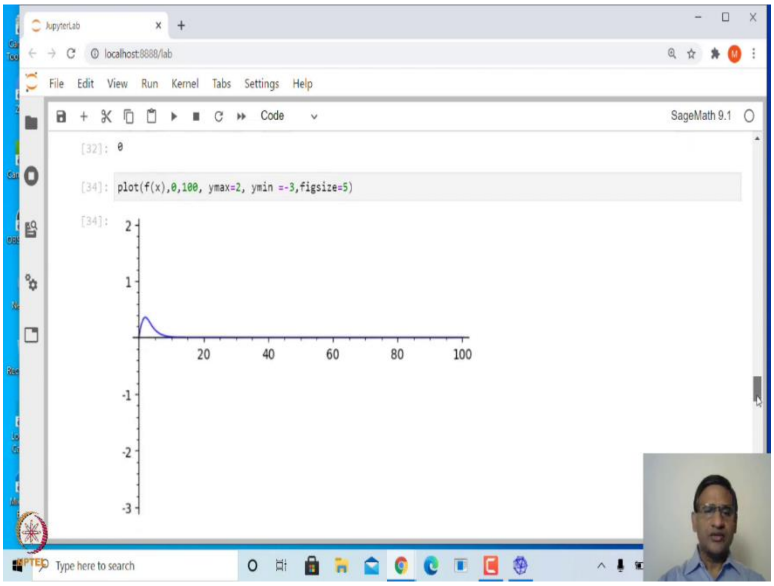

This is becoming very this is 0 to 100; this will become very close to the origin, right.

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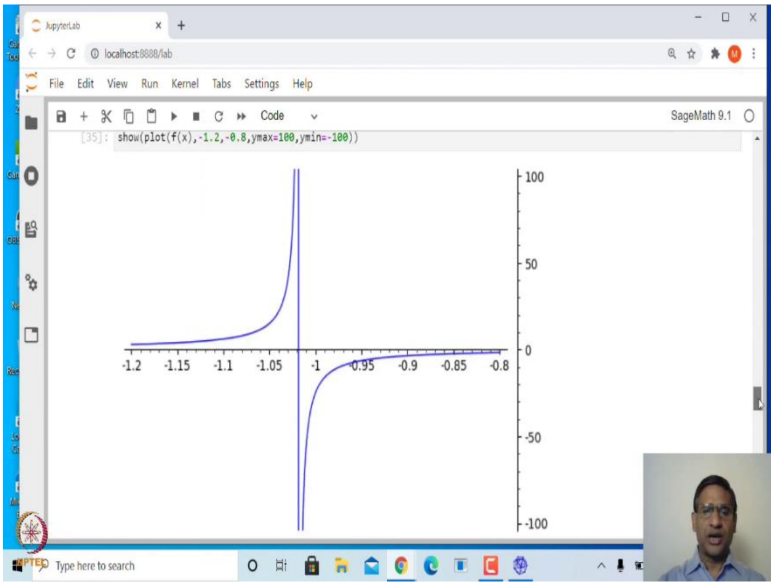

So, whereas, we have already seen that in case we look at the function its graph near this -1, then what is happening between -1.05 and -1, there is a point at which this function does not have limit; from the left hand side, it goes to + infinity; from the right hand side, it goes to -infinity.

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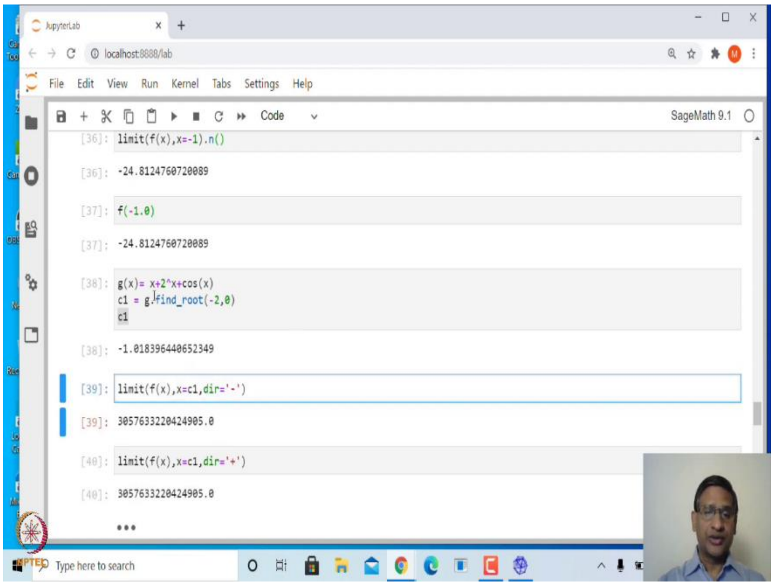

So, how do we find, what is that point at which this behavior is happening? So, that is actually exactly a point where this denominator is becoming 0. So, let us find out what is the point at which denominator is 0. So, how do we do that? So, we can first of all at -1, the function does not have problem. So, if I look at the limit at -1, this limit is -24.8124 and so on, ok. But that is the same as value of the function. So, this function is continuous at -1, right.

So, how do we find out? So, first let us define the denominator, which is $x + 2^x + \cos(x)$ in $g(x)$, and we can find 0 of this function using find underscore root, this finds a numerical root, but you need to give the interval in which you want to find a root. So, we already saw that between minus let us say between -2 and 0, there is only one 0 for this. So, let us find that and this is -1.01833 and so on, right.

So, let us find out what is the limit at this point of the function. The left hand limit, the left hand limit is some very large positive quantity, whereas if you find the right hand limit, that again is the same. So, actually what is doing is since this value c1 which we have found is numerical approximation, it is not the exact value. So, this lies very close to the actual root, but it is to the left of actual root, that is why both at this point the function actually has a limit and this is equal to quite large quantity.

So, this also tells you that when you have found this numerical approximation that is not exact value and that is where left hand limit, right hand limit are becoming same. So, one has to be also careful when you want to find this limit using numerical approximation. It may not give you proper answer. So, you when you are using software to do some computation, you should also be careful just do not believe everything what you get, right.

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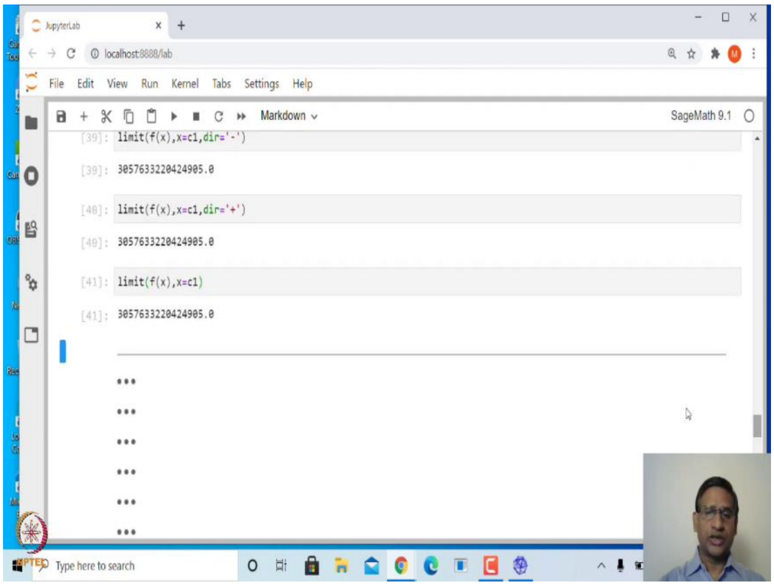

Similarly, the limit of the function is also the same at this point ok.

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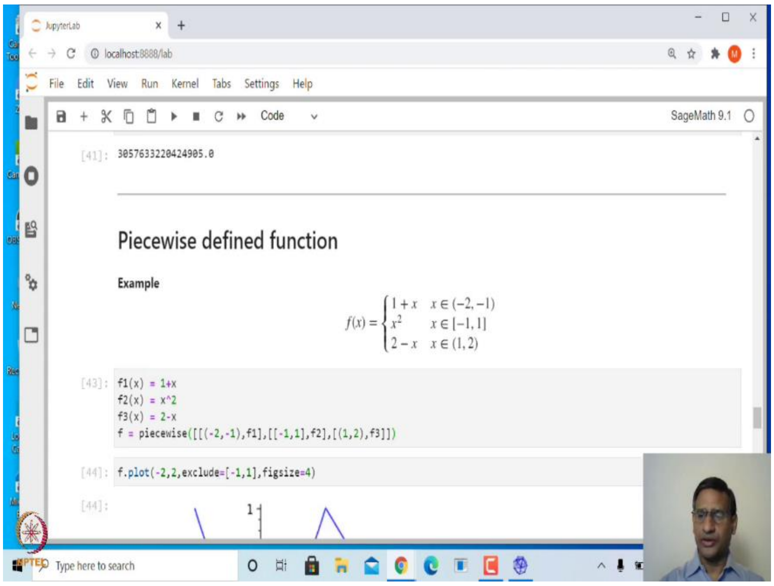

Let us look at, if you have a function which is defined in pieces – piecewise defined function, we have already seen how to define such function and plot its graph. So, suppose you have a function $f(x)$ which is defined is equal to $1 + x$ in the -2 to -1; and x^2 between -1 and 1; and $2 - x$ between 1 and 2 in open interval.

So, we define this function. First, we define this f1, f2, f3, each of this component. And then we use piecewise defined function, mention the interval, mention the what is the function in that interval and things like that. And then let us plot its graph. When we plot its graph, since this function is defined between -2 and 2, we have to plot between -2 and 2 else where it will not work, right.

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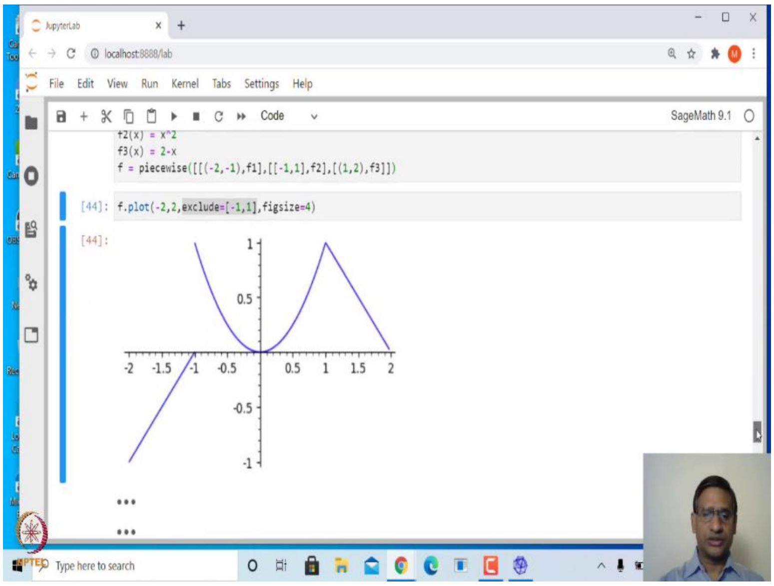

So, that is the graph of this function. And you can see here we have excluded the points between points -1 and 1, because that is where it the two functions are getting clubbed, right. So, at -1, you see that this function has does not have a limit, whereas at 1 this function has a limit. So, you can obtain. But again, if you want to find out if you just apply limit function to f, it may not work. In, in this case, you have to apply limit to this f1 and f2.

For example, if you want to find limit at x equal to -1 i.e., left hand limit at x equal to -1, you have to find limit of f1, left hand limit of f1. Similarly, you have to find right hand limit of f2 and then see whether they are equal ok. So, Sage will not give you limit of f, but you have to find using the limit of each of this component right, so that is what you need to do ok.

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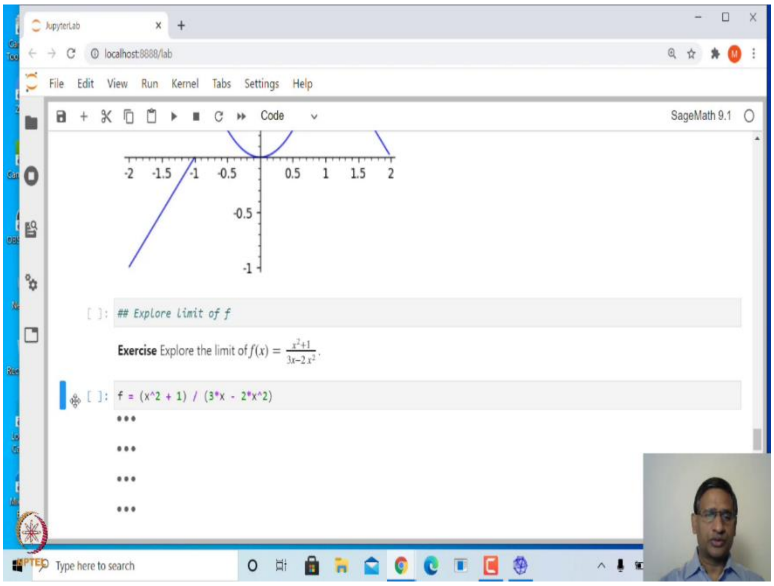

So, suppose so you should try to find these limits at 1 and -1 ok, graphically it is quite clear, right. Now, let us look at another example, let me just give you this as an exercise you explore the limit of the function $f(x)$ equal to $x^2 + 1$ divided by $3x - 2x^2$

So, the point at which the denominator is 0 that will be 2.0 and 3 by 2 at those points you should see what is the limit, and you should try to explore ok. So, let me not get into these things. Let me move to the next example. This is very easy. I am sure all of you can get it quite easily.

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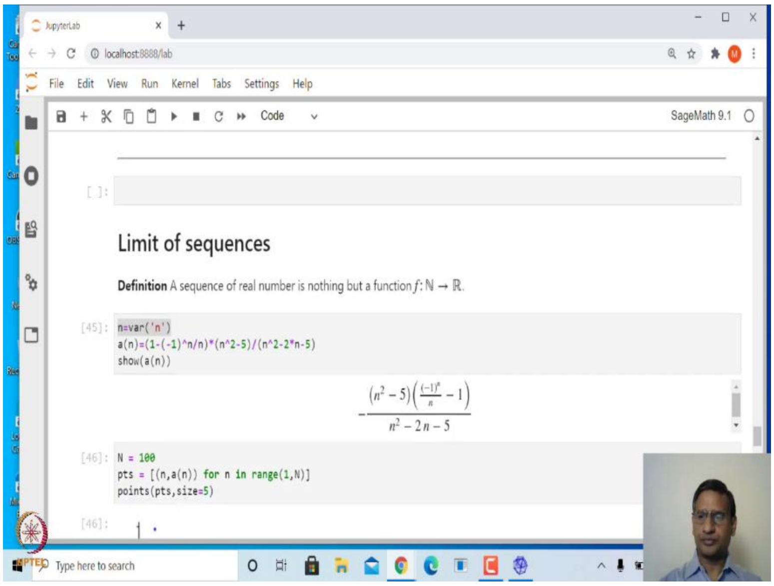

Let us look at how to find limit of a sequence? Actually, one can define limit of a function using limit of a sequence. So, actually natural way is to first study sequences and its limit and then go to limit, continuity and all that etc. That is generally the approach I prefer but, since I believe many of you may not have done course in real analysis, if you have learned calculus most of the time, they first teach you limit, and then the sequence ok.

So, let us say we want to look at finding limit of a sequence of real numbers. So, first of all what is sequence of real number? It is nothing but a function from natural number to R, any function from natural number to R is called a sequence. And the Nth term the image of f .. image of let us say N under f, which is we can denote it by f N generally is called Nth term of this sequence usually you denote it by x n, right.

And so let us look at an example. So, this is a function, this is a sequence f n which I have denoted here by a(n), I have defined in a(n). You need to declare n as a variable. So, this is a sequence looks fairly complicated. But, since a sequence is also a function, it will be a good idea to plot graph of this function. So, when I say graph of this function, you need to plot the n comma f(n). So, in this case n comma a(n).

So, if you plot, for example, let us say plot first 100 terms of the sequence, then we can say n = 100, and points is n comma a n for n in range 1 to n. You should not say range n because at 0 it is not defined here. So, this is and then you can use option called points to plot this, these points.

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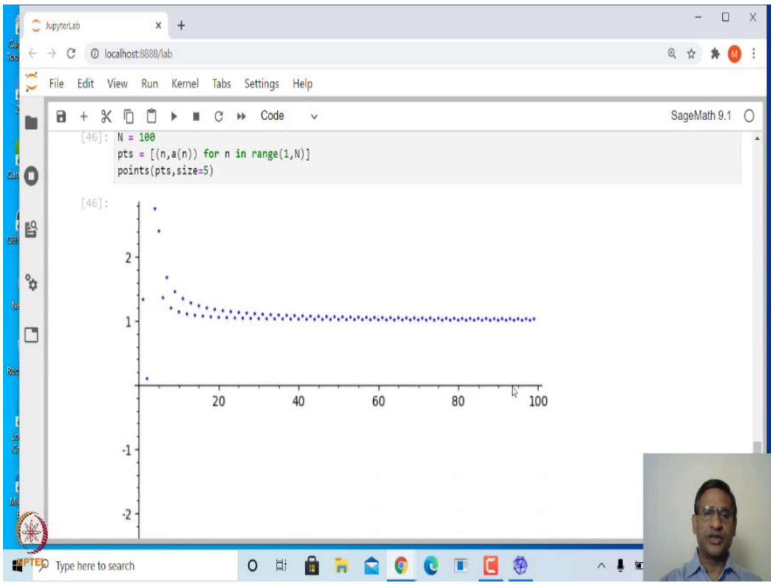

So that is how the terms of the sequence first 100 terms look like. So, you can see here the beginning the it is quite random, but as we increase n, this is becoming somewhat closer to y equal to 1.

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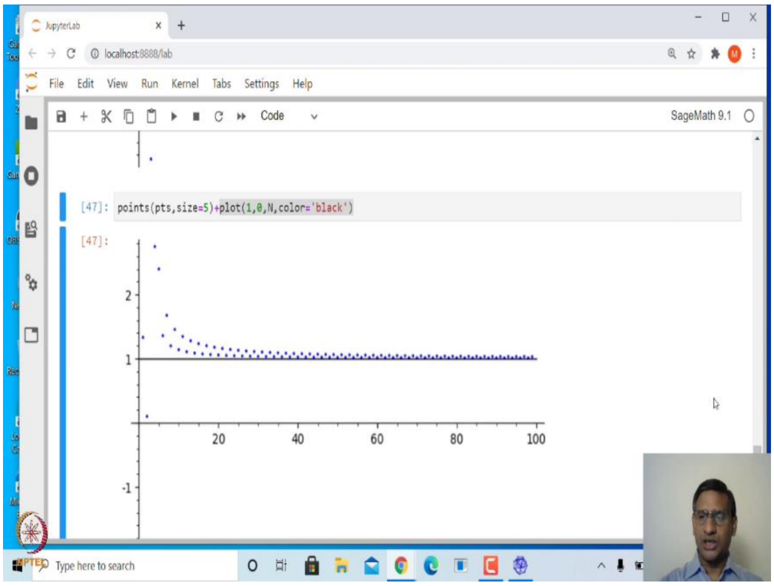

So, for example, if I draw, try to plot or add a horizontal line at y equal to 1, this is what is added in black color. Then you will see that this the terms of the sequence are becoming very close to each other. How close to y equal to 1? That again is it depends how close you want, but in case, in this case you can make as close as possible. So, suppose we draw a horizontal line y equal to, so this is y equal to let me say the limit a and y a plus epsilon to a minus epsilon.

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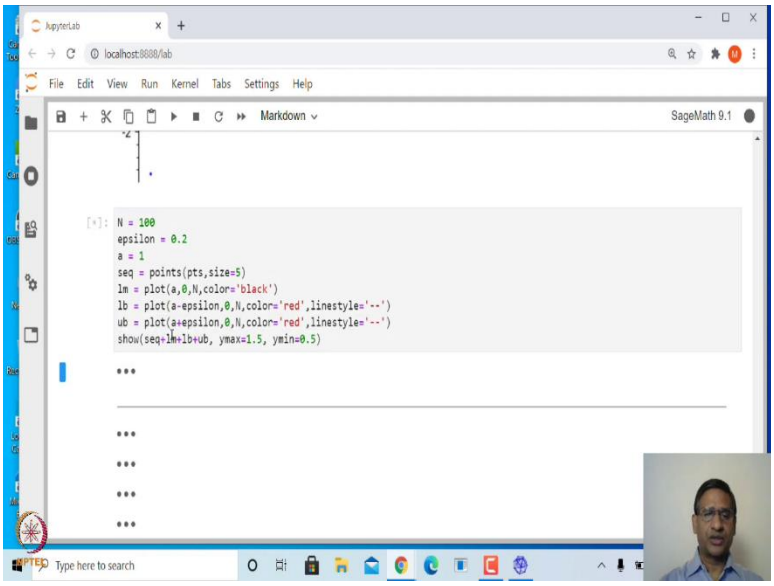

So, a horizontal band around y equal to 1 and then see what happens to the sequence so that is what is done here. So, you take n = 100, and epsilon is equal to some 0.1 band, let us to begin with let us say 0.2 band, and the point is at a and then let us plot the line – this is the line. Then the upper line that is 1 minus epsilon is lower line, and upper line is 1 plus epsilon.

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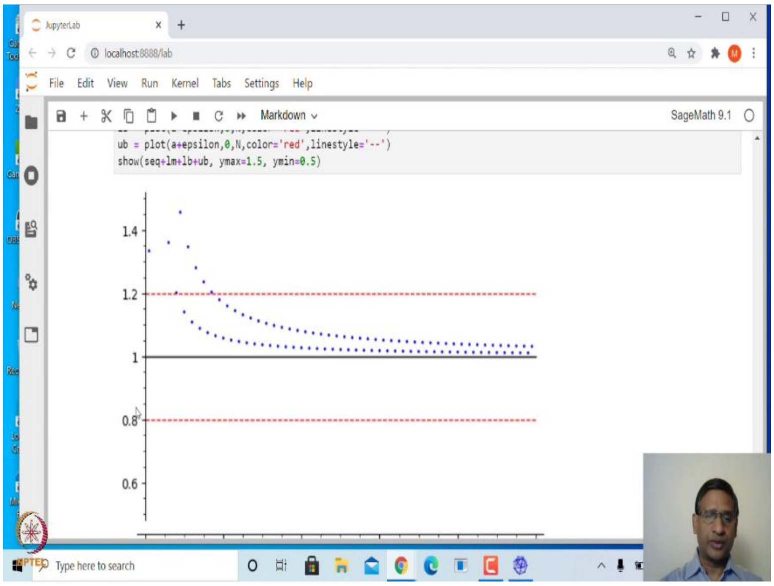

And let us, so what happens? In this case, you can see here this is the band which is at one point at y equal to 1, 1.2, and 0.8. So, you can see that after few terms, all the terms of the sequence lie inside this band.

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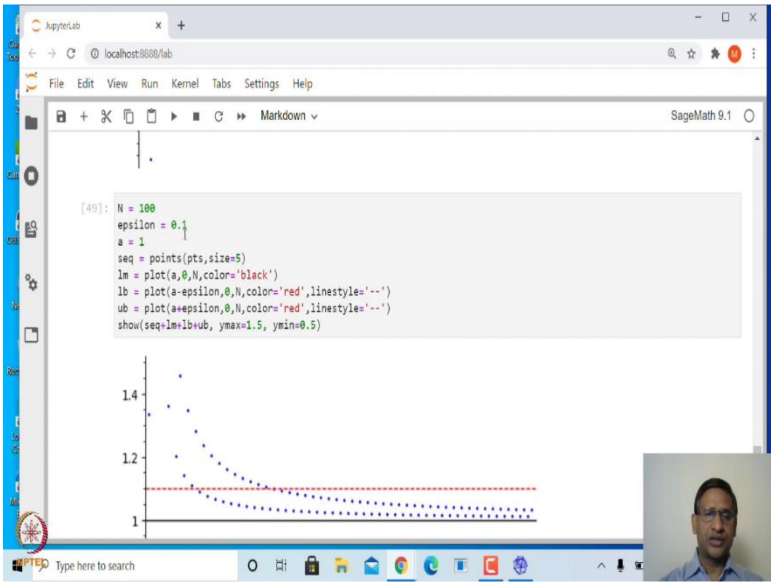

However, if I decrease this bandwidth, so let us say if I say 0.1, then you will see that the point after, which the terms are going inside this band has increased ok.

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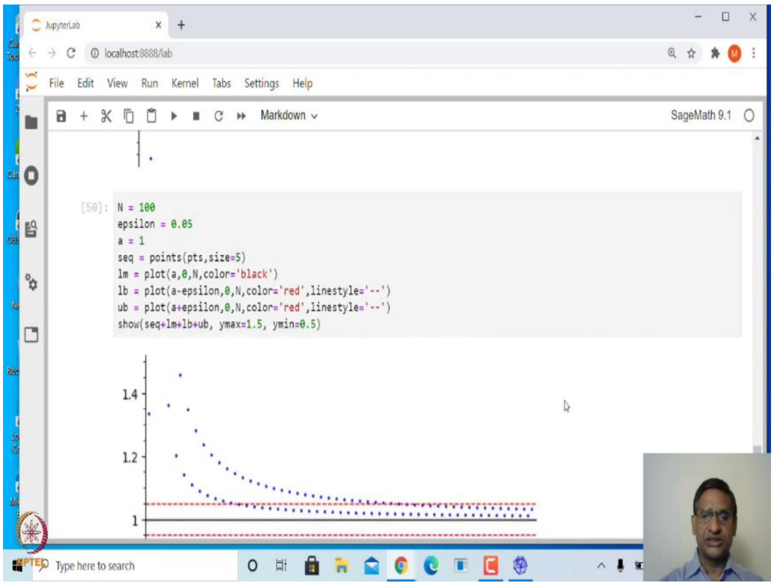

If I increase this to further let us say 0.5, then you will see that.

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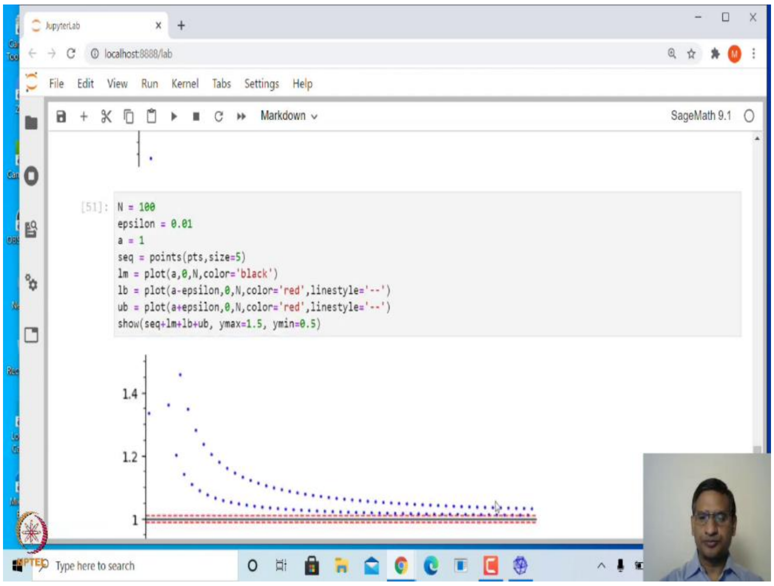

Again, if I increase this further, if I say 0.1, then it is not even going inside, right. So, I need to maybe increase this.

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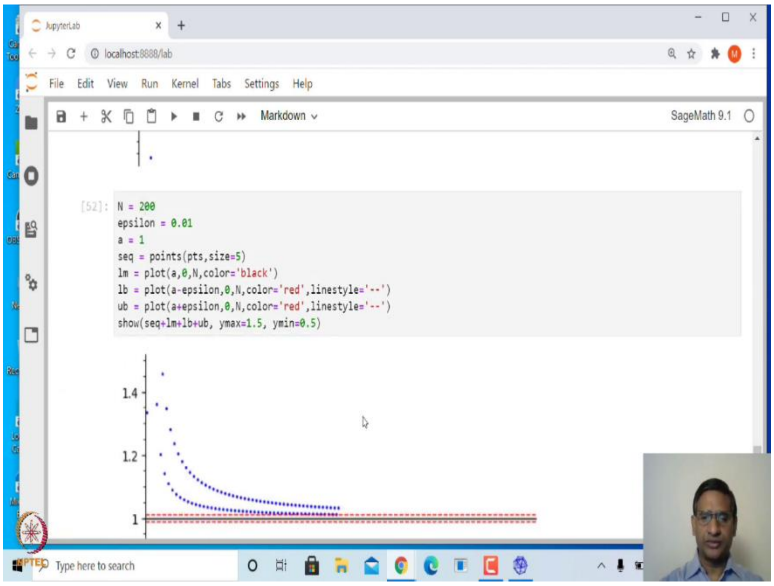

For example, if I say 200, the terms 200 terms, this also I have to say n is 200 and the terms of the sequence. So, you need to also increase the points here which I did.

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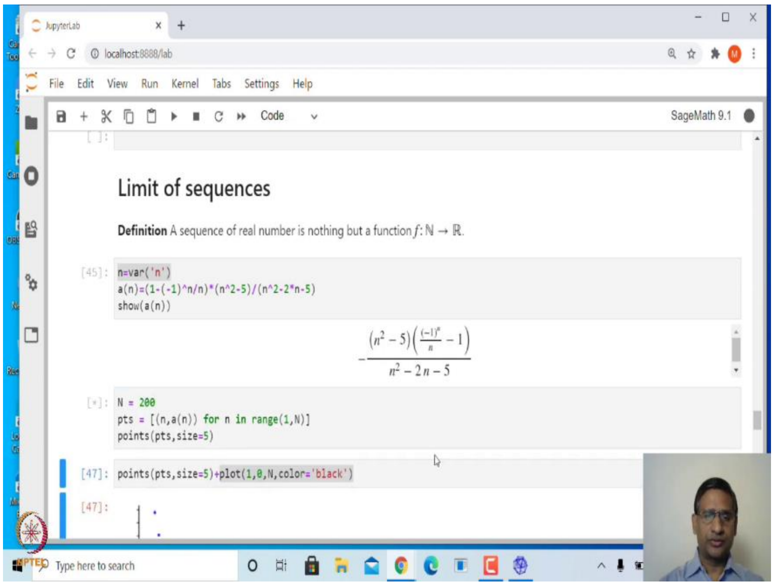

So, let me say here 200. So, these are the 200 points which are generated. Now, let me plot this graph.

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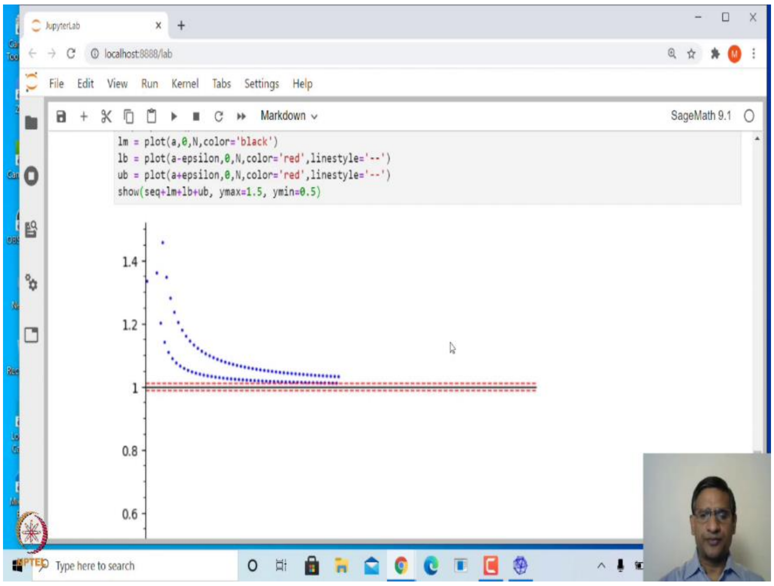

So, when I plot this graph, yeah so, still this 200, I think this.

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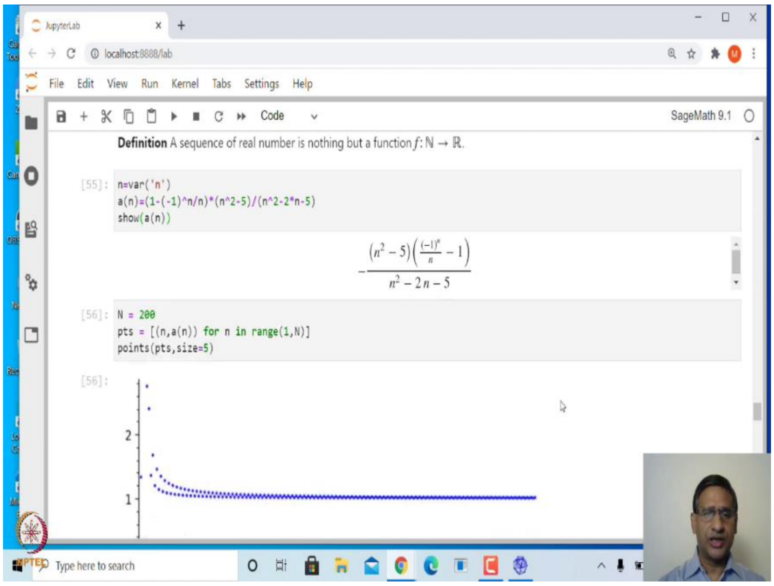

Let me I think, this something has happened. So, let me just re-run this. So, 200 points and now, let me go to this particular band and then try to plot ok.

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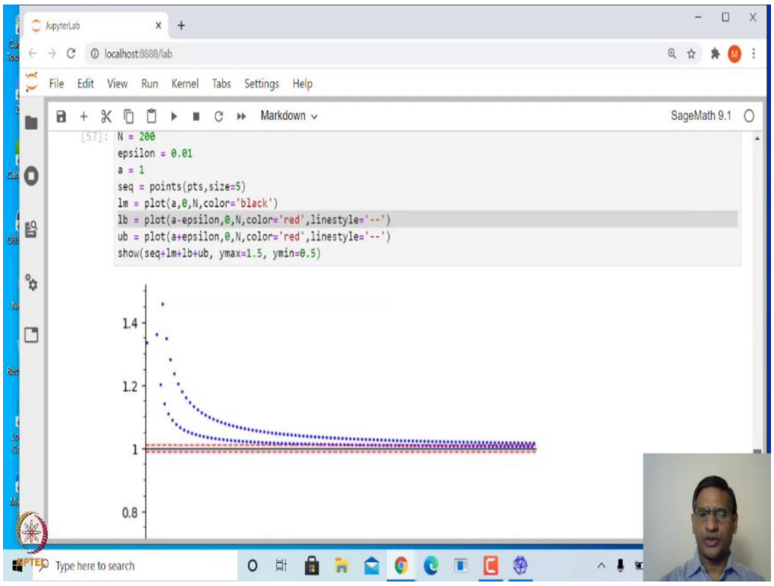

So, yeah even after this it is not going inside so, it may even require more number of iterates and so maybe let me, ok. So, you can play with this N and in fact, this leads to a definition of limit of a sequence.

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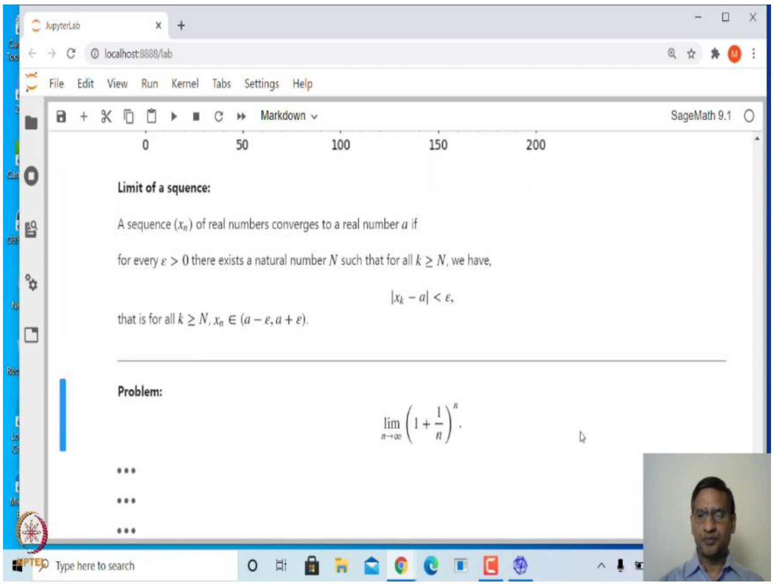

And what is the definition? So, it says that if you have a sequence $x(n)$ of real numbers, and we say that it converges to a real number a, if you take any epsilon positive in particular you take any band about a, then you can find a natural number after, which all the terms has a property that x_k for k bigger than equal to n mod of $(x_k - a)$ will be less than epsilon is same as saying x_k will lie in the band, a - epsilon to a + epsilon that is what was demonstrated. So that is the definition of limit of a sequence.

Now, suppose, we look at this sequence $\left(1 + \frac{1}{n}\right)^{1/n}$, and try to find out what happens to this sequence whether this sequence converges. And if it converges, where does it converge?

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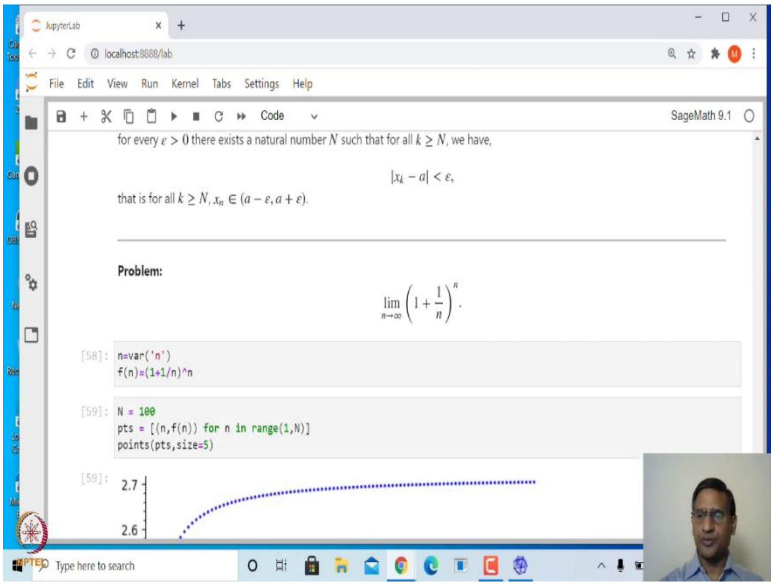

So, let us first again define this sequence as $f(n)$. And let us plot some first 100 points of the sequence.

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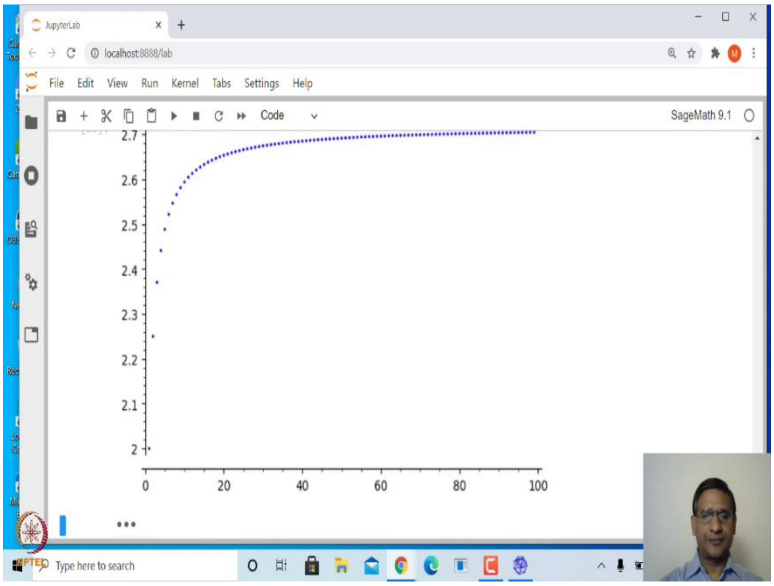

So, you can see that this sequence is actually increasing, and it is kind of converging to some number which is close to 2.7.

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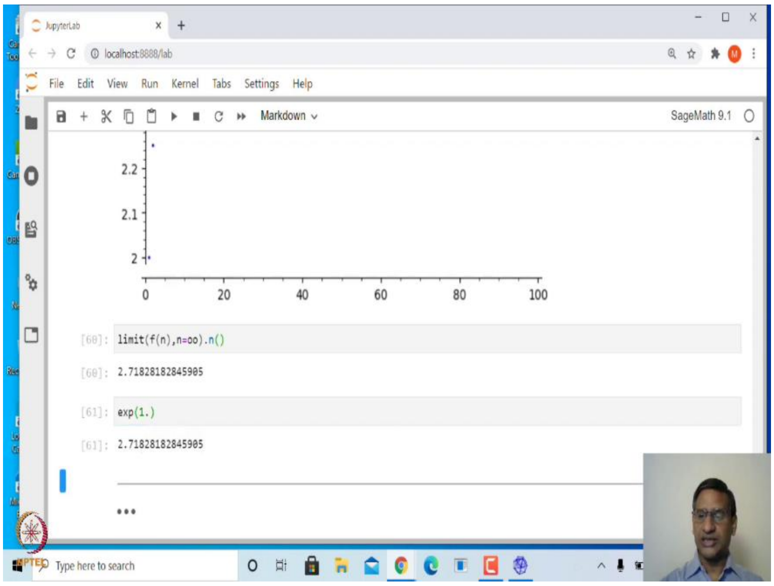

And so let us look at find the limit. So, again you can find limit of a sequence by the same function limit using which we found limit of a function. So, limit of the function the sequence $f(n)$ at n equal to infinity. And let us take its numerical value. So, that is 2.718281, if you recall this number is nothing but Euler number which is exponential of 1.

Similarly, so you can find out what is exponential of 1 and you can see that these two are the same. So that is the limit of this sequence. This is a very important and very famous sequence which and this again it is actually a sequence of rational number which converges to e which is irrational ok.

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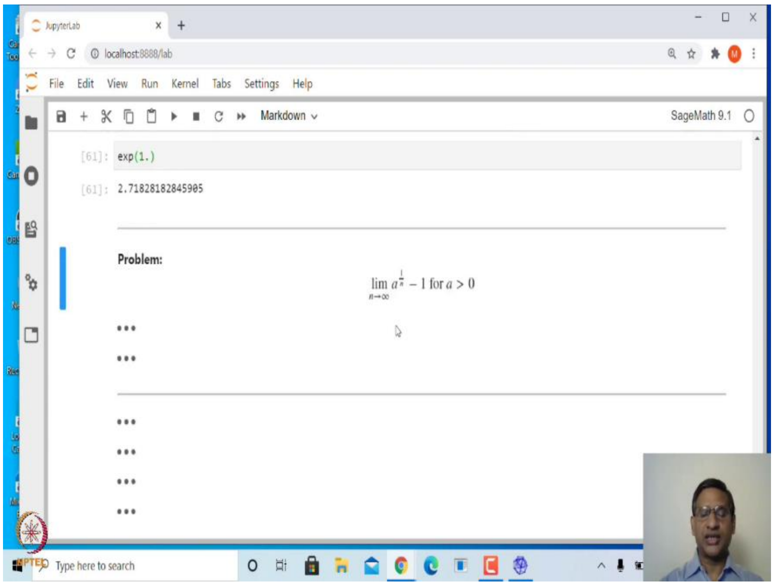

Let us take another example.

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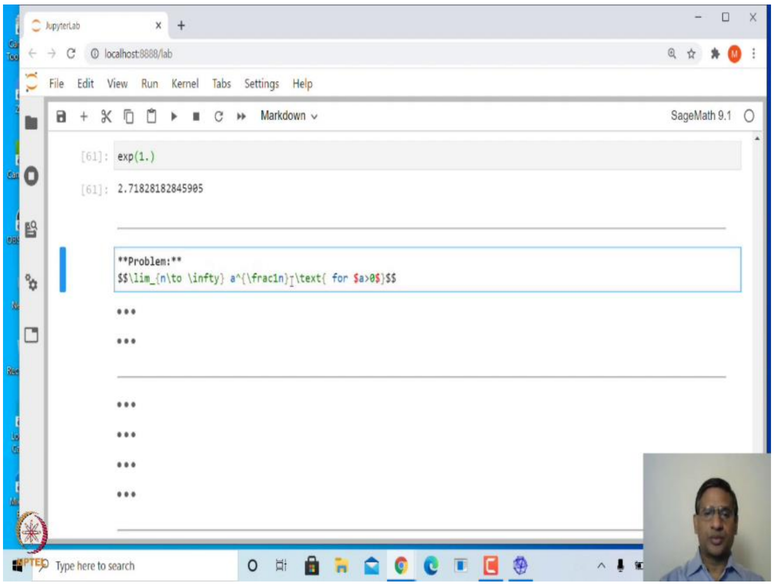

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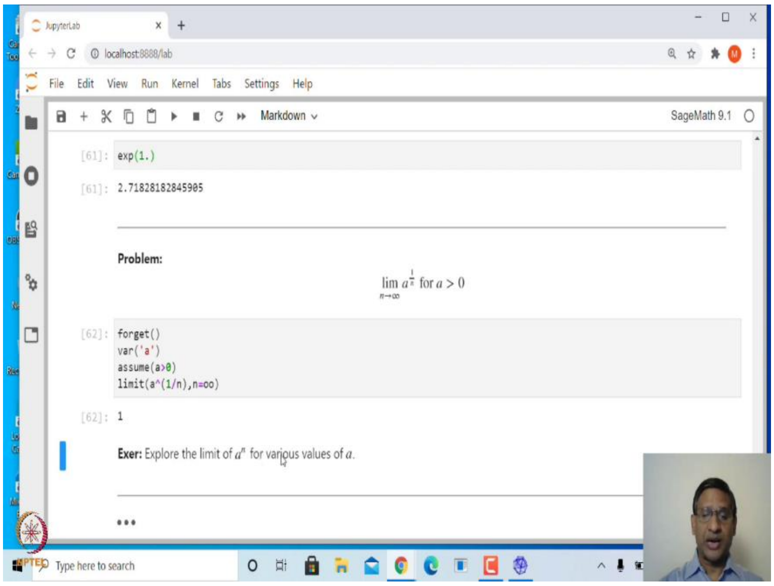

Suppose you want to explore a sequence let us say a^n, a to the power n for a positive; $a^{\frac{1}{n}}$ for a positive. So, how do we do that? One way is you can take a particular value of a positive value of a, and look at that to the power 1 by n, and then plot the graph, and then find out that limit.

But here, you can define a as a variable and assume that a to be positive. Here forget means in case we have already defined a equal to something which we did, we want a to forget that variable, and then you can find the limit of a to the power 1 by n as n equal to infinity and that answer is 1. So, a to the power 1 by n goes to 1 for any positive a ok.

And you can explore what happens if a is negative, or you can explore what happens to a^n, when for various values of a so that is another thing, you can explore. You can take various values of a and then see what happens to the limit.

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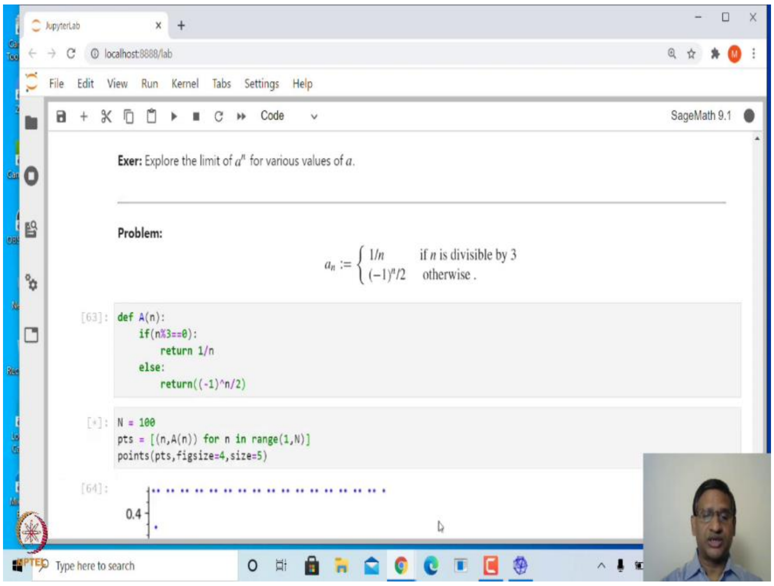

Let us look at this sequence $a(n)$ is equal to 1 by n, when n is divisible by 3, n = $(-1)^n/2$ otherwise. So, this you have to define this using def, and return whenever 3 divides n, you return 1 by n and otherwise return the second component. So, let us plot again maybe first 100 terms of this sequence.

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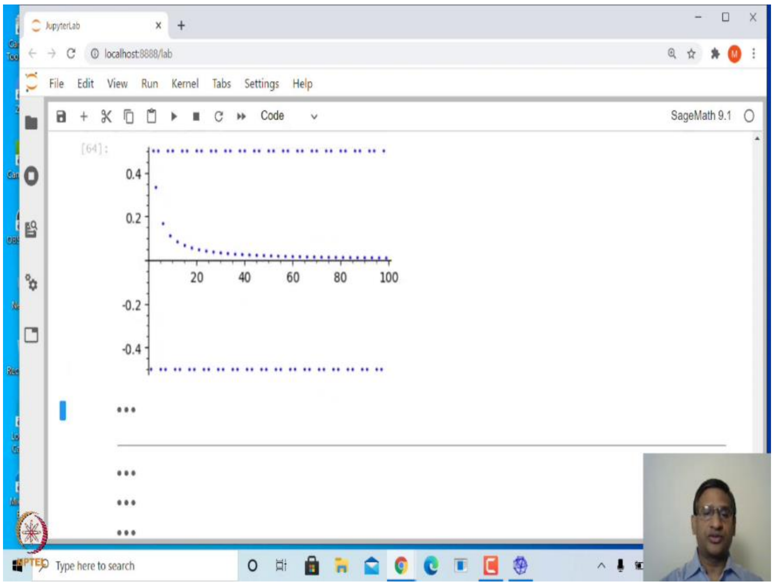

This is how it looks like and you can see here the behavior. Behavior it goes to the plus side something, negative side something, which is quite far apart. And at some at -1 to the power n this goes at whenever it is divisible by n, it is 1 by n. So, this will be close to the origin where other two are oscillating between some point -0.5 and 0.5. So, this sequence is not going to converge that is what this will not converge, right.

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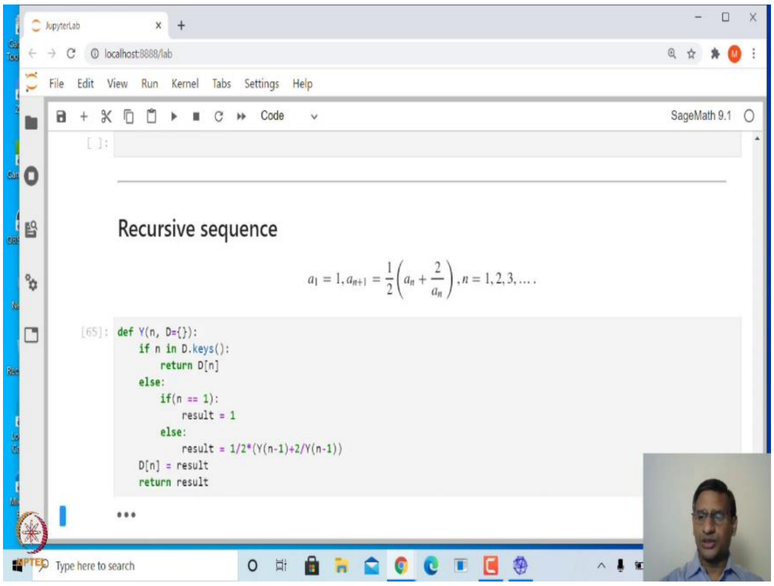

Now, let us look at sometimes you have a sequence which is defined recursively. So, for example, you start with a(1) is equal to 1, and define a_{n+1} is equal to half times a_n + 2 by a_n, n going from 0 to infinity. So, this is if you look at what is the behavior of this sequence, so again let us define this sequence.

Again, when you want to define this, this is since it is recursively defined, you may have to define something like this. This is what is called dynamic programming approach where initially you are defining it as an empty dictionary. And in that you keep on adding that, D[n] is equal to the result and then you return the last term which is …

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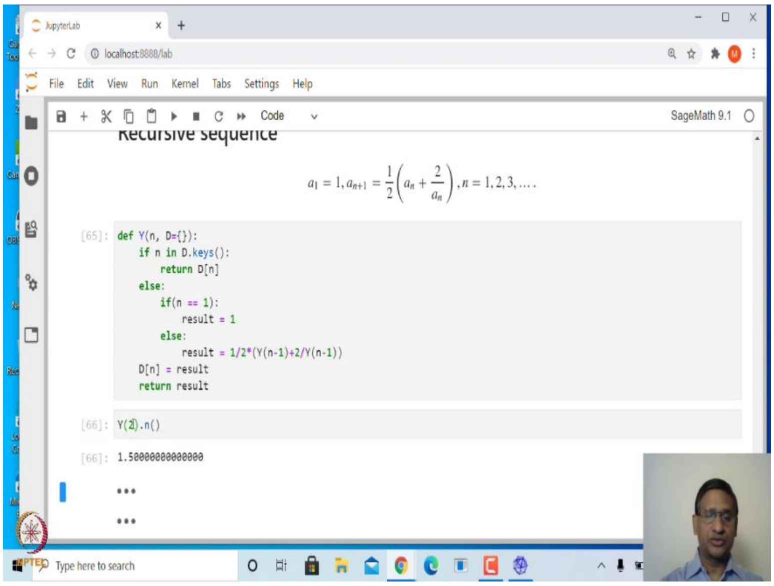

So, that is the now, if I want to find out what is the term, let us say y, let us say what is second term so, second term is 1.5.

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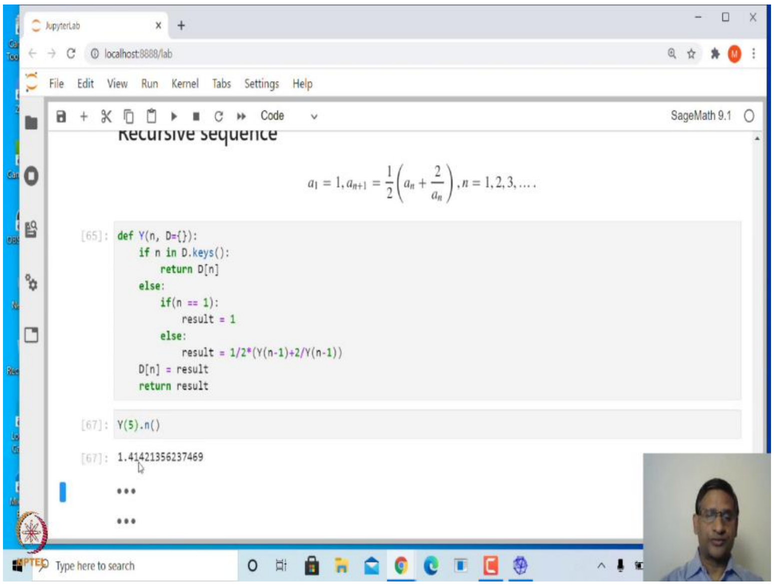

What is the 5th term? The 5th term is 1.4142.

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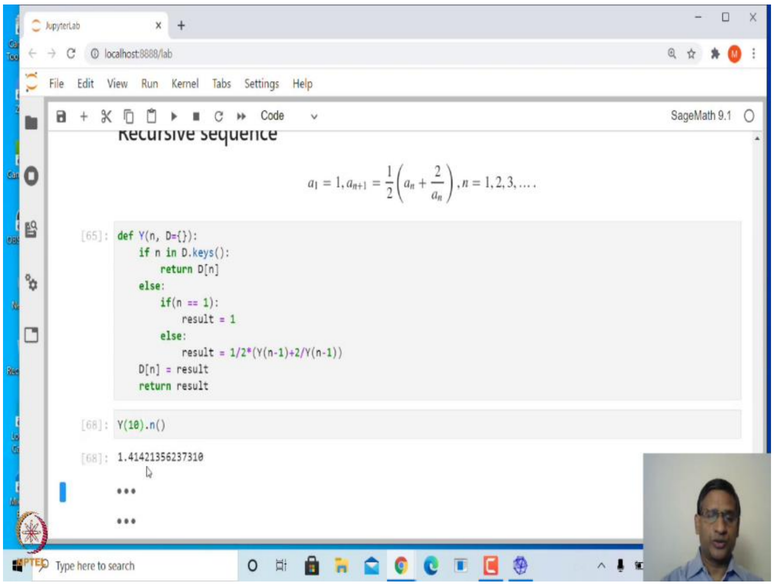

If I say what is the 10th term? This is again 1.42, so, you and you may look at recall what kind of number it is, so it is close to actually root 2.

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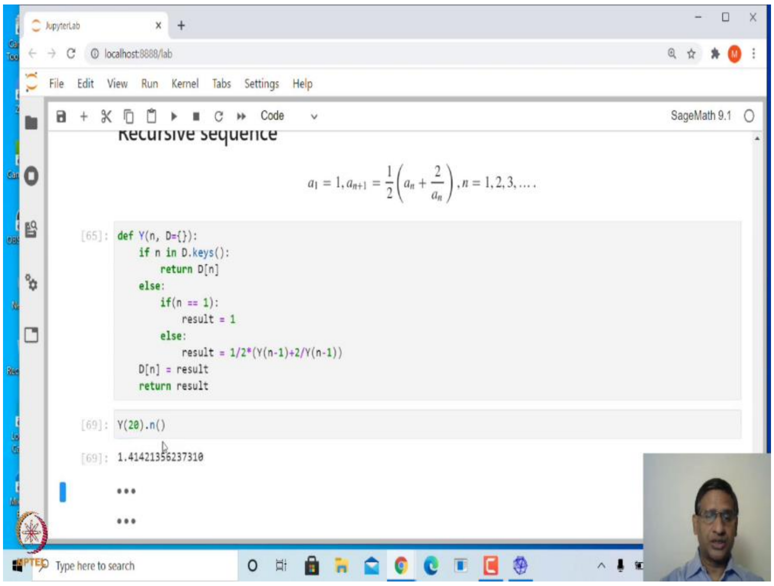

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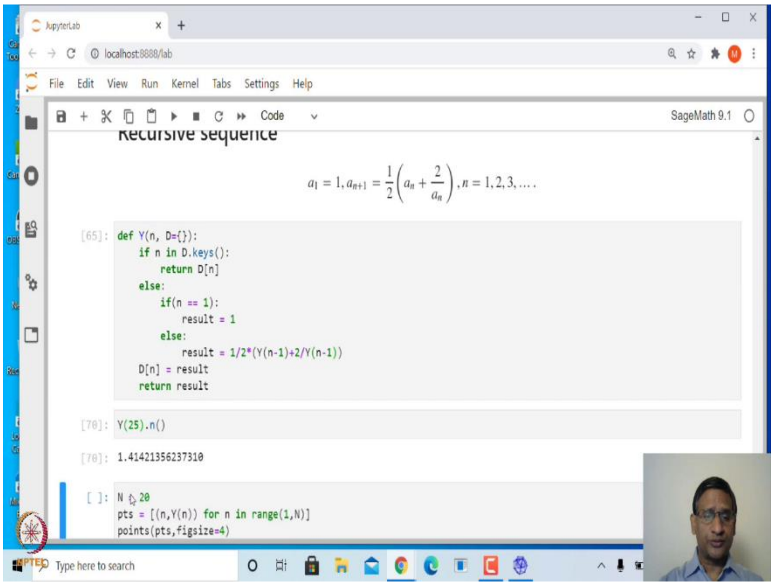

So, and if I say 20, then this if I say 25, it may take more time, it will take time, now it is still running you can see here the star. So, if you take higher value of n, it will take quite some time because this computation it has to do all this recursive, but it will come. But, if you take so if you take very large number, it will take a lot of time. And as a result, you may not be able to find the limit of this sequence, only you can look at the behavior ok. But, so how do we look at it?

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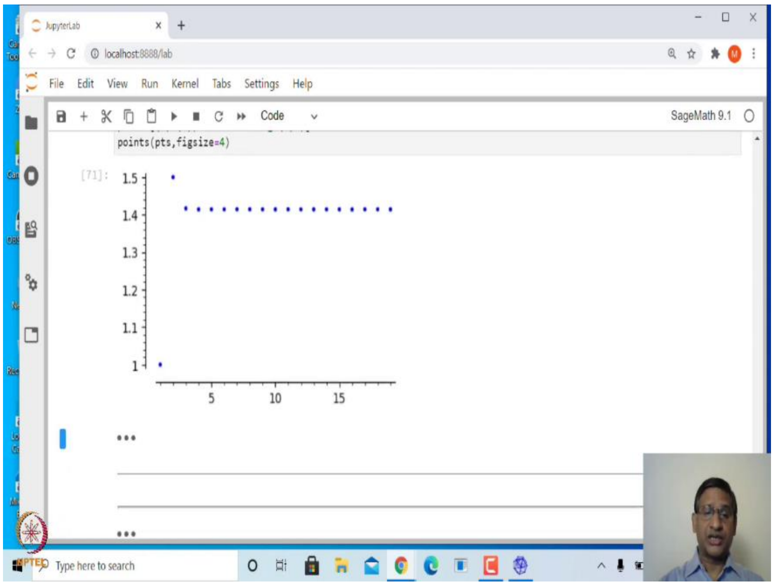

So, suppose if you want to plot let us say first 20 term of the sequence, you can mention see what is happening is that this function seems to be kind of after second term onwards, this seems to be decreasing and it is bounded below. So, therefore, it is going to converge. And if it converges to let us say some real number x, then you can find its limit by substituting.

So, if it converges to x then a_{n+1} will converge to x, a_n will converge to x, 2 upon a_n will converge to 2 by x. So, if you put this in this relation, so you are going to get x, which satisfy this relation x is equal to half times x + 2 by x.

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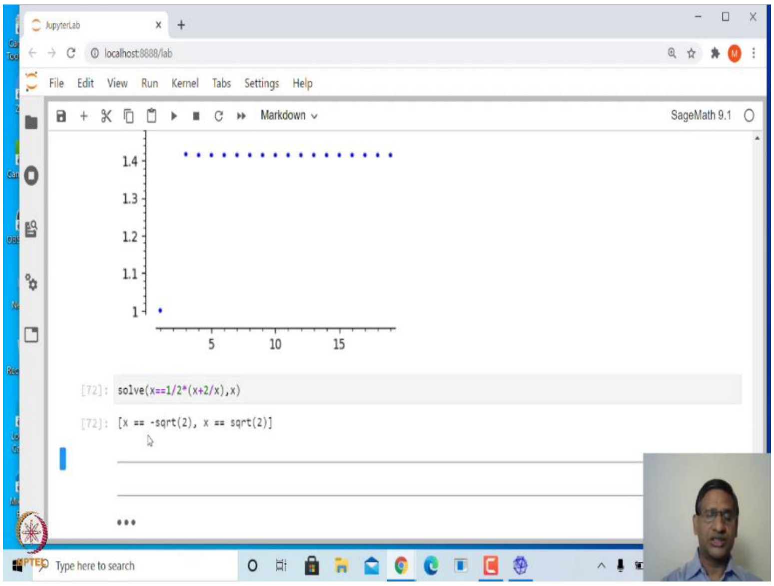

So, let us solve that for x and see what you get. So, x is equal to half into x + 2 by x this is the a_{n+1}, this is a_n, and this is a_n. So, if I try to solve this it says that the roots are square root 2 and minus square root 2.

Limit cannot be negative because the terms of the sequence are all non-negative, in fact, positive, therefore, the limit cannot be negative. Therefore, limit has to be square root 2. So, here again you see that this is a sequence of rational numbers, which converges to an irrational number namely square root 2 ok.

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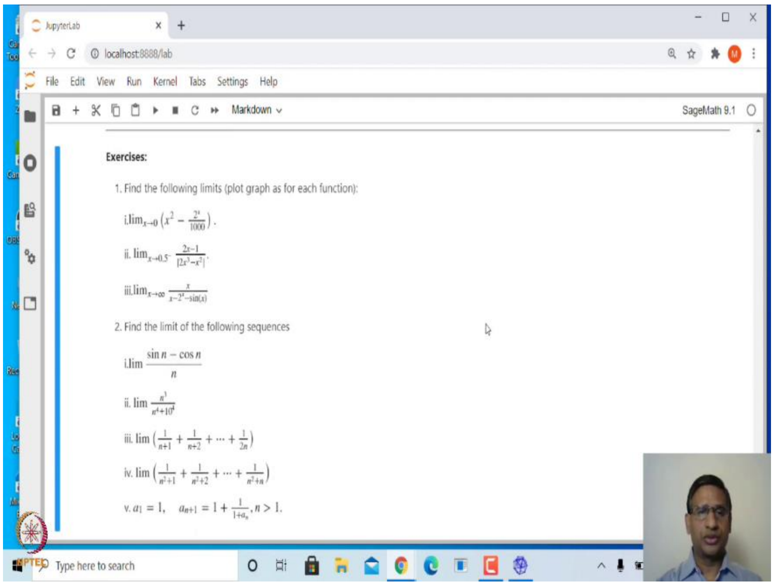

So, let me leave you with some exercises. You should try to find limit of some of these functions and including limit of some of these sequences they are quite easy. Let me make it slightly small, so that you can see entire screen if you want to capture, right. So, these are the, these are the way in which you can explore limit of a sequence and limit of a function.

And once you know the limit of left hand limit, right hand limit of a function, and the function is defined at that point you know the continuity and things like that. But, usually ah you can define continuity of a function using limit of a sequence approach ok. So, let me stop here. In the next class, we will look at more on derivative and some more concepts regarding the derivative of the function.

Thank you very much.