

Functional Analysis
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Lecture No. 43

Dual of L^1

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$(L^p(\mu))^* = L^q(\mu) \quad \frac{1}{p} + \frac{1}{q} = 1 \quad 1 < p < \infty$

Thm. (Riesz Representation Thm.) Let $\Omega \subset \mathbb{R}^d$ be an open set. The dual of $L^1(\Omega)$ is isometrically isomorphic to $L^\infty(\Omega)$.

Pf. Step 1 $\exists \omega \in L^2(\Omega)$ s.t. $\forall K \subset \Omega$, compact, $\omega(x) \geq \epsilon_K > 0$
 $\forall x \in K, \quad E_n = \{x \in \Omega \mid n \leq |x| \leq n+1\} \quad \Omega = \bigcup_{n=0}^{\infty} E_n$
 $\omega(x) = \alpha_n > 0$ on E_n . Choose α_n s.t.
 $\sum_{n=0}^{\infty} \alpha_n^2 |E_n| < +\infty$.

Then ω has the required properties.

Step 2. Let $\varphi \in (L^1(\Omega))^*$. $f \mapsto \varphi(\omega f) \quad f \in L^2(\Omega)$.
 $|\varphi(\omega f)| \leq \|\varphi\| \|\omega f\|_1 \leq \|\varphi\| \|\omega\|_2 \|f\|_2$.

We saw earlier that for any measure space (X, μ) , we have that $(L^p(\mu))^*$ was $L^{p^*}(\mu)$ where $\frac{1}{p} + \frac{1}{p^*} = 1$, for $1 < p < \infty$. So, at that time we said that we are not considering the case of L^1 which requires some measure theoretic arguments, that's all. So now, in the case of $L^1(\Omega)$, we will prove this.

Theorem (Riesz Representation Theorem): Let $\Omega \subset \mathbb{R}^N$ be an open set. The dual of $L^1(\Omega)$ is isometrically isomorphic to $L^\infty(\Omega)$. So, that means you can essentially identify L^1 as L^∞ . The proof, we will do in several stages.

Step 1. $\exists \omega \in L^2(\Omega)$ such that for every $K \subset \Omega$, compact $\omega(x) \geq \epsilon_K > 0 \forall x \in K$. So how do we do that? consider E_n as usual, like we did earlier. So, you take an annular

$$E_n = \{x \in \Omega \mid n \leq |x| \leq n+1\}. \text{ So, } \Omega = \bigcup_{n=0}^{\infty} E_n.$$

Then you set $w(x) = \alpha_n > 0$ on E_n , and choose α_n such that $\sum_{n=0}^{\infty} \alpha_n^2 |E_n| < +\infty$. So, you choose your α_n , so you know what the E_n 's are, you know their measure, so you then choose something, for instance α_n can be $\frac{1}{n|E_n|}$. So, that could be a thing which will give you something which is finite. Then w has the required properties. So, because of condition $\sum_{n=0}^{\infty} \alpha_n^2 |E_n| < +\infty$, it is in L^2 and then any compact set would be in some $|x| \leq n$ and therefore, it will be, have a finite α_n 's for each of them and then you take the minimum of them, that will be a strictly positive number and therefore, you have, that it will be positive on any compact set.

Step 2. Let $\phi \in (L^1(\Omega))^*$. Consider the mapping $f \rightarrow \phi(wf)$, $f \in L^2(\Omega)$. So, what is $|\phi(wf)|$? $|\phi(wf)| \leq \|\phi\| \|wf\|_1 \leq \|\phi\| \|w\|_2 \|f\|_2$ by the Holder inequality. So, this defines a continuous linear functional on L^2 .

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$\forall x \in \mathbb{R}, E_n = \{x \in \mathbb{R} \mid n \leq |x| < n+1\}$ $\mathbb{R} = \bigcup_{n=0}^{\infty} E_n$
 $w(x) = \alpha_n > 0$ on E_n Choose α_n st.
 $\sum_{n=0}^{\infty} \alpha_n^2 |E_n| < +\infty$
 Then w has the required properties.
Step 2. Let $\phi \in (L^1(\mathbb{R}))^*$ $f \mapsto \phi(wf)$ $f \in L^2(\mathbb{R})$.
 $|\phi(wf)| \leq \|\phi\| \|wf\|_1 \leq \|\phi\| \|w\|_2 \|f\|_2$
 $\phi = 2 = \phi^* \quad (L^1)^* = L^\infty \quad \exists v \in L^2(\mathbb{R})$ st.
 $\phi(wf) = \int_{\mathbb{R}} f v dx \quad \forall f \in L^2(\mathbb{R})$

If you have $p = 2 = p^*$, then $(L^2)^* = L^2$. Therefore, $\exists v \in L^2(\Omega)$ such that $\phi(wf) = \int_{\Omega} f v dx \quad \forall f \in L^2(\Omega)$ because of the Riesz Representation Theorem we have already shown for L^2 .

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$$\left| \int_{\Omega} \phi w f dx \right| \leq \|\phi\| \|wf\|_1 \leq (\|\phi\| \|w\|_2 \|f\|_2) \quad \checkmark$$

Step 3. Set $u(x) = \frac{v(x)}{w(x)}$ $w \neq 0$ u well-def. m.e.

Claim $u \in L^\infty$ & $\|u\|_\infty \leq \|\phi\|$.

Let $C > \|\phi\|$. $A = \{x \in \Omega \mid |u(x)| > C\}$.

To show A has meas. zero. Assume the contrary.

$\exists B \subset A$ with finite positive measure.

$$f(x) = \begin{cases} +1 & x \in B \quad u(x) > 0 \\ -1 & x \in B \quad u(x) < 0 \\ 0 & \text{if } x \notin B. \end{cases}$$

$f \in L^2(\Omega)$ ($|f| = 1$ on B has fin. meas.).

$$\int_{\Omega} u w f dx \leq \|\phi\| \int_{\Omega} |w f| dx$$


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We have $\left| \int_{\Omega} f v dx \right| \leq \|\phi\| \|wf\|_1 \leq \|\phi\| \|w\|_2 \|f\|_2$. We have already seen this.

Step 3. Set $u(x) = \frac{v(x)}{w(x)}$. $w \neq 0$, we saw in the first step, $w(x) > 0$ in every E_n and therefore $w(x)$ never vanishes and therefore u is well defined and measurable. Claim, $u \in L^\infty$ and $\|u\|_\infty \leq \|\phi\|$. Let $C > \|\phi\|$. Then take $A = \{x \in \Omega \mid |u(x)| > C\}$. To show, A has measure 0. Then essentially you will have $|u(x)| \leq C$ and therefore $\|u\|_\infty \leq C \forall C > \|\phi\|$, and therefore $\|u\|_\infty \leq \|\phi\|$, that is the argument. We have to show that. Assume the contrary. $\exists B \subset A$ with finite positive measure. If A has a finite measure, then we can of course consider subset of A which still has positive measure and which is also finite.

Now consider $f(x) = +1$ if $x \in B, u(x) > 0$

and $f(x) = -1$ if $x \in B, u(x) < 0$

$f(x) = 0$ if $x \in \Omega \setminus B, u(x) > 0$

So, f is a bounded function. See, $|f| = 1$ on a set of positive measure and therefore $f \in L^2(\Omega)$ ($|f| = 1, |f| = 0$ on $\Omega \setminus B$ & B has finite measure). So, we can use this f in

previous inequality. Here, $v = uw$. So $\int_{\Omega} u w f dx \leq \|\phi\| \int_{\Omega} |w f| dx$.

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Let $\phi \in L^1(\mu)$. $\mu = \mu|_B + \mu|_{B^c}$.

To show A has mean zero. Assume the contrary.

$\exists B \subset A$ with finite positive measure.

$$f(x) = \begin{cases} +1 & x \in B \text{ and } u(x) > 0 \\ -1 & x \in B \text{ and } u(x) < 0 \\ 0 & \text{if } x \notin B. \end{cases}$$

$f \in L^1(\mu)$ ($\because |f| = 1$ & B has fin. pos. meas.).

$$\int_{\Omega} u f \, d\mu \leq \|\phi\| \int_{\Omega} |u f| \, d\mu$$

$$c \int_B u \, d\mu \leq \int_B |u| w \, d\mu \leq \|\phi\| \int_B u \, d\mu \quad \begin{matrix} B \subset A \\ |u| \geq c \text{ on } A \end{matrix}$$

B has pos. meas., $w > 0 \Rightarrow c \leq \|\phi\| \quad \checkmark$





Step 3. Let $u(x) = \frac{u(x)}{w(x)}$ w/o u well-def. where.

Claim $u \in L^\infty$ & $\|u\|_\infty \leq \|\phi\|$.

Let $C > \|\phi\|$. $A = \{x \in \Omega \mid |u(x)| > C\}$. \checkmark

To show A has mean zero. Assume the contrary.

$\exists B \subset A$ with finite positive measure.

$$f(x) = \begin{cases} +1 & x \in B \text{ and } u(x) > 0 \\ -1 & x \in B \text{ and } u(x) < 0 \\ 0 & \text{if } x \notin B. \end{cases}$$

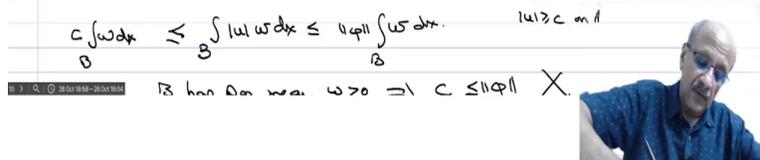
$f \in L^1(\mu)$ ($\because |f| = 1$ & B has fin. pos. meas.).

$$\int_{\Omega} u f \, d\mu \leq \|\phi\| \int_{\Omega} |u f| \, d\mu$$

$$c \int_B u \, d\mu \leq \int_B |u| w \, d\mu \leq \|\phi\| \int_B u \, d\mu \quad \begin{matrix} B \subset A \\ |u| \geq c \text{ on } A \end{matrix}$$

B has pos. meas., $w > 0 \Rightarrow c \leq \|\phi\| \quad \times$





Now, $f = 0$ outside B , so, all the integrals come only to B which has got positive measure and $u f = |u|$. So, $\int_B |u| w \, d\mu \leq \|\phi\| \int_B w \, d\mu$ (as $|f| \leq 1$ & w is non negative)

What do you know about u on B ? $B \subset A$ and $|u| \geq C$ on A . Therefore, you can write,

$$C \int_B w \, dx \leq \int_B |u| w \, dx \leq \|\phi\| \int_B w \, dx, \quad B \text{ has positive measure and } w \text{ is strictly}$$

positive and therefore $\int w$ can be cancelled so this implies $C \leq \|\phi\|$ which is a contradiction because we have taken $C > \|\phi\|$, and therefore, $C \leq \|\phi\|$ becomes a contradiction.

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- $$f = \begin{cases} -1 & x \in B \\ 0 & \text{if } x \notin B \end{cases}$$
- $$f \in L^1(\Omega) \quad (C: |f|=1 \text{ on } B \text{ has fin. pos. meas.})$$
- $$\int_{\Omega} w f \, dx \leq \|\phi\| \int_{\Omega} |w f| \, dx$$
- $$C \int_B w \, dx \leq \int_B |u| w \, dx \leq \|\phi\| \int_B w \, dx \quad (|u| \geq c \text{ on } B)$$
- $$B \text{ has pos. meas.}, w > 0 \Rightarrow C \leq \|\phi\| \quad \times$$
- $$u \in L^\infty, \|u\|_\infty \leq \|\phi\|$$

Therefore, you have that $u \in L^\infty$ and $\|u\|_\infty \leq \|\phi\|$.

Step 4: $\forall f \in L^2$

$$\phi(wf) = \int_{\Omega} f u v = \int_{\Omega} f u w dx$$

$g \in C_c(\Omega)$. $f = \frac{g}{w} \in L^2$ $K = \text{supp } g \text{ ct.}$
 $w > \epsilon_K$.
 $\frac{g}{w} \leq \frac{1}{\epsilon_K} g \in L^2$

$$\phi(g) = \int_{\Omega} g u dx$$

$C_c(\Omega)$ dense in $L^1(\Omega)$. By Lebesgue-Riesz cont. w.r.t g in L^1 norm.

$$\Rightarrow \forall g \in L^1(\Omega) \quad \phi(g) = \int_{\Omega} g u dx$$

$$|\phi(g)| \leq \|g\|_1 \|u\|_{\infty}$$

$$\Rightarrow \|\phi\| \leq \|u\|_{\infty}$$



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 $w > \epsilon_K$.
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$$\phi(g) = \int_{\Omega} g u dx$$

$C_c(\Omega)$ dense in $L^1(\Omega)$. By Lebesgue-Riesz cont. w.r.t g in L^1 norm.

$$\Rightarrow \forall g \in L^1(\Omega) \quad \phi(g) = \int_{\Omega} g u dx$$

$$|\phi(g)| \leq \|g\|_1 \|u\|_{\infty}$$

$$\Rightarrow \|\phi\| \leq \|u\|_{\infty} \Rightarrow \|\phi\| = \|u\|_{\infty}$$



Step 4: $\forall f \in L^2$, we have $\phi(wf) = \int_{\Omega} v f dx$. But what is v ? $v = \int f u w dx$ because u is defined as v/w . so v is nothing but uw .

So now, take $g \in C_c(\Omega)$. Now, then if you look at $f = g/w$, now, g is nonzero only on a compact set $\subset \Omega$ and on any compact set $w > 0$. So, $\frac{1}{w}$ is bounded above. $k = \text{supp } g$ compact so $w > \epsilon_k$, so $\frac{g}{w} \leq \frac{1}{\epsilon_k} g \in L^2$ and therefore $\frac{g}{w} \in L^2$. So, we can substitute this in our equation and therefore $\phi(g) = \int g u dx$ for every $g \in C_c(\Omega)$. We have that $C_c(\Omega)$ is dense in $L^1(\Omega)$ and both L.H.S. and R.H.S. continuous with respect to g in L^1 norm. This implies, $\forall g \in L^1(\Omega)$, $\phi(g) = \int g u dx$. $|\phi(g)| \leq \|g\|_1 \|u\|_\infty$ (by the Holder inequality) $\Rightarrow \|\phi\| \leq \|u\|_\infty$. We already saw in the previous step, that $\|u\|_\infty \leq \|\phi\|$ and therefore $\|u\|_\infty = \|\phi\|$.

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- $\phi(g) = \int_{\Omega} g u dx$
- $w > \epsilon_k$
 $\frac{g}{w} \leq \frac{1}{\epsilon_k} g \in L^2$
- $C_c(\Omega)$ dense in $L^1(\Omega)$. Both L.H.S and R.H.S cont w.r.t g in L^1 norm.
- $\Rightarrow \forall g \in L^1(\Omega)$ $\phi(g) = \int_{\Omega} g u dx$.
- $|\phi(g)| \leq \|g\|_1 \|u\|_\infty$
- $\Rightarrow \|\phi\| \leq \|u\|_\infty \Rightarrow \|\phi\| = \|u\|_\infty$.
- $\forall \phi \in (L^1(\Omega))^*$, $\exists u \in L^\infty(\Omega)$ s.t. $\|u\|_\infty = \|\phi\|$, $\phi(g) = \int g u dx \forall g \in L^1$

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So, $\forall \phi \in (L^1(\Omega))^*$, $\exists u \in L^\infty(\Omega)$ such that $\|u\|_\infty = \|\phi\|$ and

$$\phi(g) = \int_{\Omega} g u \forall g \in L^1(\Omega).$$

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Step 5. Such u is unique

Assume u_1, u_2 $\|u_1\| = \|u_2\| = \|\varphi\|$

$\forall g \in L^1(\Omega)$ $\int_{\Omega} u_1 g dx = \int_{\Omega} u_2 g dx = \varphi(g)$






Now, this is a well defined thing.

Step 5, such u is unique. Assume we have two, assume u_1, u_2 such that

$$\|u_1\| = \|u_2\| = \|\varphi\| \quad \forall g \in L^1(\Omega), \int_{\Omega} u_1 g dx = \int_{\Omega} u_2 g dx = \varphi(g).$$

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Step 5. Such u is unique

Assume u_1, u_2 $\|u_1\| = \|u_2\| = \|\varphi\|$

$\forall g \in L^1(\Omega)$ $\int_{\Omega} u_1 g dx = \int_{\Omega} u_2 g dx = \varphi(g)$

$\Rightarrow \forall g \in L^1(\Omega)$ $\int_{\Omega} (u_1 - u_2) g dx = 0$

$u_1 - u_2 \in L^{\infty}(\Omega) \Rightarrow L^1(\Omega)$

$\int_{\Omega} (u_1 - u_2) g dx = 0 \quad \forall g \in C_c(\Omega) \subset L^1(\Omega)$

$\Rightarrow u_1 = u_2$ a.e. i.e. $u_1 = u_2$ in $L^1(\Omega)$.

$\varphi \mapsto u$ well def.

Step 6. $(L^1(\Omega))^* \rightarrow L^{\infty}(\Omega)$ $\varphi \mapsto u$

$\varphi(f) = \int_{\Omega} u f dx \quad \forall f \in L^1$






So, this implies $\forall g \in L^1(\Omega)$, you have $\int_{\Omega} (u_1 - u_2) g dx = 0$. So, and so this means

what? Now, $(u_1 - u_2)$ is, so $(u_1 - u_2) \in L^{\infty}(\Omega)$, so they are bounded so on any compact

set, they are integrable, therefore they belong to $L^1_{loc}(\Omega)$. Because they are bounded functions on any compact set of finite measure, and therefore, it is integrable on, so this is in L^1_{loc} and $\int_{\Omega} (u_1 - u_2)g \, dx = 0 \, \forall g \in L^1$, therefore in particular, $\forall g \in C_c(\Omega)$, and therefore by our previous proposition, this will imply that $u_1 = u_2$ almost everywhere. That is, $u_1 = u_2$ in $L^1(\Omega)$. So, the mapping $\phi \mapsto u$ is well defined.

Step 6. So, you take, $(L^1(\Omega))^* \rightarrow L^\infty(\Omega)$, so, $\phi \mapsto u$, where

$$\phi(f) = \int_{\Omega} uf \, dx, \forall f \in L^1(\Omega).$$

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$u_1 - u_2 \in L^1(\Omega) \Rightarrow L^1_{loc}(\Omega)$

$\int_{\Omega} (u_1 - u_2)g \, dx = 0 \quad \forall g \in C_c(\Omega) \subset L^1(\Omega)$

$\Rightarrow u_1 = u_2 \text{ a.e. i.e., } u_1 = u_2 \text{ in } L^1(\Omega)$

$\phi \mapsto u$ well def.

Step 6. $(L^1(\Omega))^* \rightarrow L^\infty(\Omega) \quad \phi \mapsto u$

$\phi(f) = \int_{\Omega} uf \, dx \quad \forall f \in L^1(\Omega)$

well-def, isometry onto

$(L^1(\Omega))^* \cong L^\infty(\Omega)$

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This map is well defined isometry and one-one and onto. So, it is one-one because it is an isometry and it is also onto. And therefore, you have that $(L^1(\Omega))^*$ is isometrically isomorphic to $L^\infty(\Omega)$. So, this proves the Riesz Representation Theorem.