Functional Analysis
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Lecture No. 43

Dual of L-1

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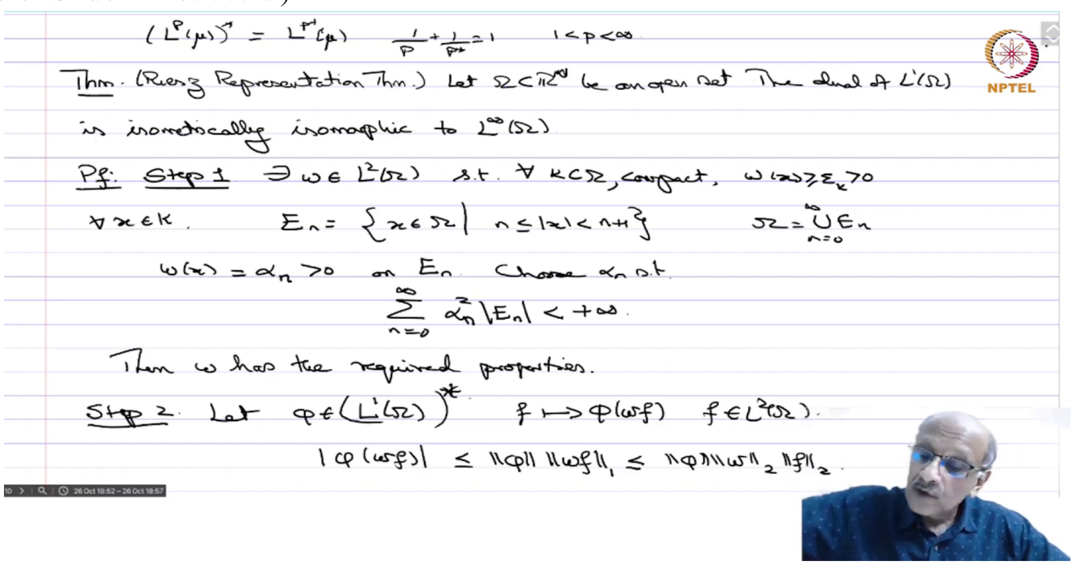

We saw earlier that for any measure space $x \in \mu$, we have that $(L^p(\mu))^*$ was $L^{p^*}(\mu)$ where $\frac{1}{p} + \frac{1}{p^*} = 1$, for $1 < p < \infty$. So, at that time we said that we are not considering the case of L^1 which requires some measure theoretic arguments, that's all. So now, in the case of $L^p(\Omega)$, we will prove this.

Theorem (Riesz Representation Theorem): Let $\Omega \subset \mathbb{R}^N$ be an open set. The dual of $L^1(\Omega)$ is isometrically isomorphic to $L^\infty(\Omega)$. So, that means you can essentially identify L^{1^*} as L^∞. The proof, we will do in several stages.

Step 1. $\exists\ \omega \in L^2(\Omega)$ such that for every $K \subset \Omega$, compact $\omega(x) \geq \epsilon_k > 0\ \forall\ x \in K$. So how do we do that? consider E_n as usual, like we did earlier. So, you take an annular

$E_n = \{x \in \Omega \mid n \leq |x| \leq n + 1\}$. So, $\Omega = \bigcup_{n=0}^{\infty} E_n$.

Then you set $w(x) = \alpha_n > 0$ on E_n, and choose α_n such that $\sum_{n=0}^{\infty} \alpha_n^2 |E_n| <+ \infty$. So, you choose your α_n, so you know what the E_ns are, you know their measure, so you then choose something, for instance α_n can be $\frac{1}{n\,|E_n|}$. So, that could be a thing which will give you something which is finite. Then w has the required properties. So, because of condition $\sum_{n=0}^{\infty} \alpha_n^2 |E_n| <+ \infty$, it is in L^2 and then any compact set would be in some $|x| \le n$ and therefore, it will be, have a finite α_n's for each of them and then you take the minimum of them, that will be a strictly positive number and therefore, you have, that it will be positive on any compact set.

Step 2. Let $\phi \in (L^1(\Omega))^*$. Consider the mapping $f \to \phi(wf)$, $f \in L^2(\Omega)$. So, what is $|\phi(wf)|$? $|\phi(wf)| \le ||\phi||\,||wf||_1 \le ||\phi||\,||w||_2\,||f||_2$ by the Holder inequality. So, this defines a continuous linear functional on L^2.

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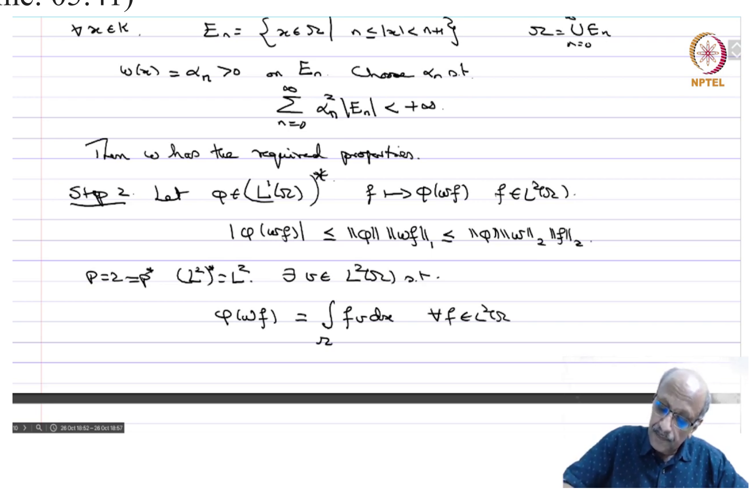
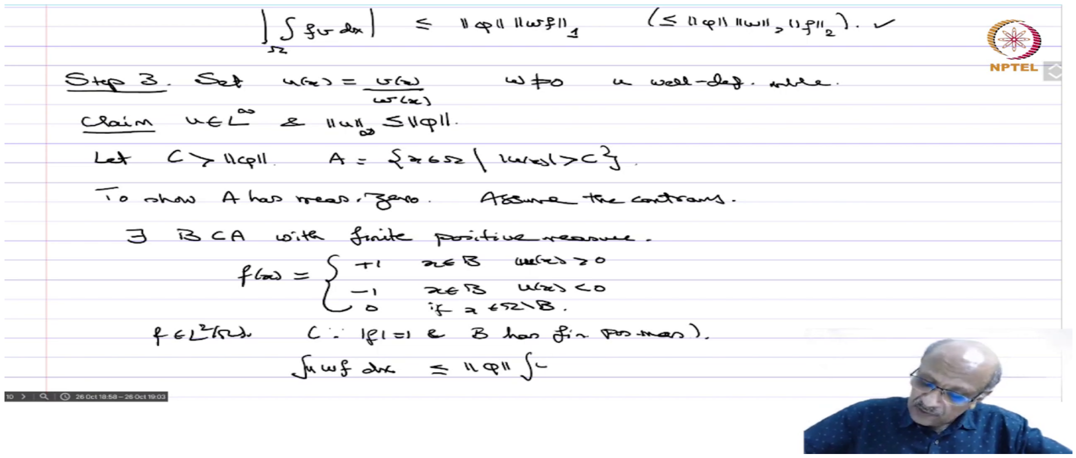


If you have $p = 2 = p^*$, then $(L^2)^* = L^2$. Therefore, $\exists\, v \in L^2(\Omega)$ such that $\phi(wf) = \int_\Omega f\,v\,dx \quad \forall f \in L^2(\Omega)$ because of the Riesz Representation Theorem we have already shown for L^2.



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We have $|\int_\Omega f\,v\,dx| \leq ||\varphi||\,||wf||_1 \leq ||\varphi||\,||w||_2\,||f||_2$. We have already seen this.

<u>Step 3.</u> Set $u(x) = \frac{v(x)}{w(x)}$. $w \neq 0$, we saw in the first step, $w(x) > 0$ in every E_n and therefore $w(x)$ never vanishes and therefore u is well defined and measurable. Claim, $u \in L^\infty$ and $||u||_\infty \leq ||\phi||$. Let $C > ||\phi||$. Then take $A = \{x \in \Omega \,|\, u(x) > C||\}$. To show, A has measure 0. Then essentially you will have $|u(x)| \leq C$ and therefore $||u||_\infty \leq C\, \forall\, C > ||\phi||$, and therefore $||u||_\infty \leq ||\phi||$, that is the argument. We have to show that. Assume the contrary. $\exists\, B \subset A$with finite positive measure. If A has a finite measure, then we can of course consider subset of A which still has positive measure and which is also finite.

Now consider $f(x) = +1$ if $x \in B,\ u(x) > 0$

and $f(x) = -1$ if $x \in B,\ u(x) < 0$

$f(x) = 0$ if $x \in \Omega\backslash B,\ u(x) > 0$

So, f is a bounded function. See,$|f| = 1$ on a set of positive measure and therefore $f \in L^2(\Omega)$ ($|f| = 1$, $|f| = 0\ on\ \Omega\backslash B\ \&\ B$ has finite measure). So, we can use this f in previous inequality. Here, $v = uw$. So $\int_\Omega u\,w\,f\,dx \leq ||\varphi|| \int_\Omega |\mathrm{wf}|$.

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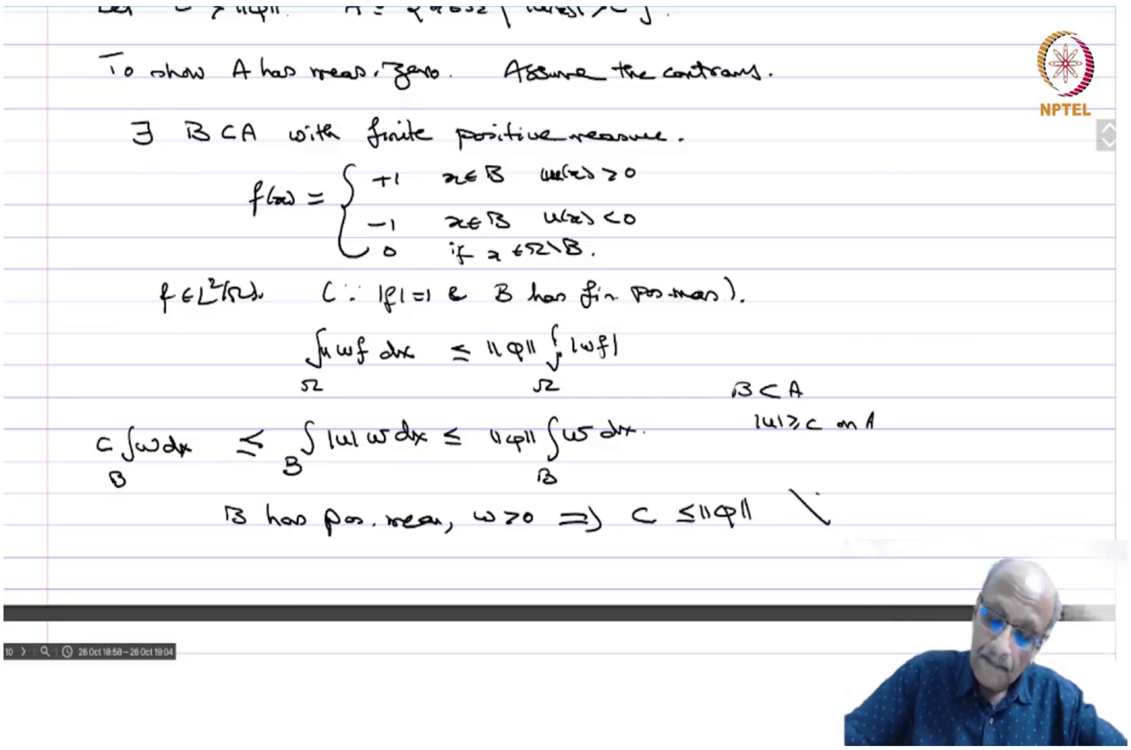

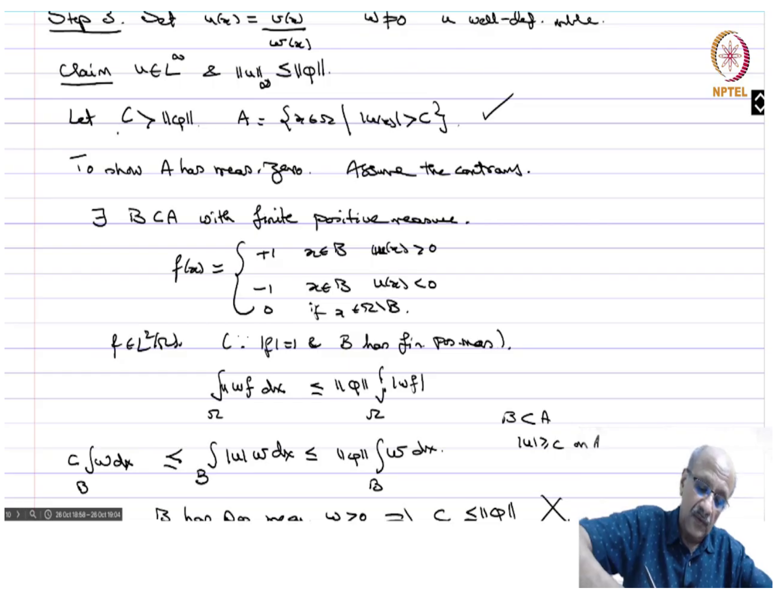

Now, $f = 0$ outside B, so, all the integrals come only to B which has got positive measure and $uf = |u|$. So, $\int_B |u|\, w\, dx \le ||\phi|| \int_B w\, dx$ (as $|f| \le 1$ & w is non negative)

What do you know about u on B? $B \subset A$ and $|u| \ge C$ on A. Therefore, you can write,

$C\int_B w\,dx \leq \int_B |u|\,w\,dx \leq ||\phi||\int_B w\,dx$, B has positive measure and w is strictly positive and therefore $\int w$ can be cancelled so this implies $C \leq ||\phi||$ which is a contradiction because we have taken $C > ||\phi||$, and therefore, $C \leq ||\phi||$ becomes a contradiction.

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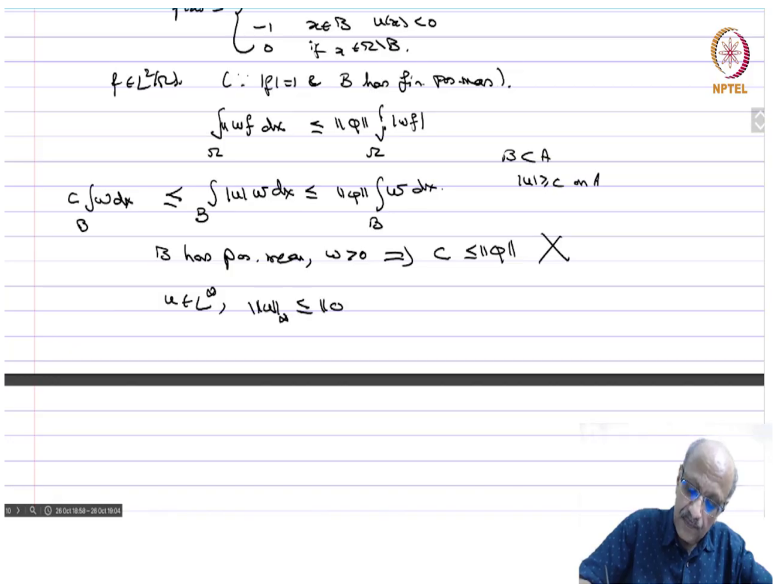

Therefore, you have that $u \in L^{\infty}$ and $||u||_{\infty} \leq ||\phi||$.

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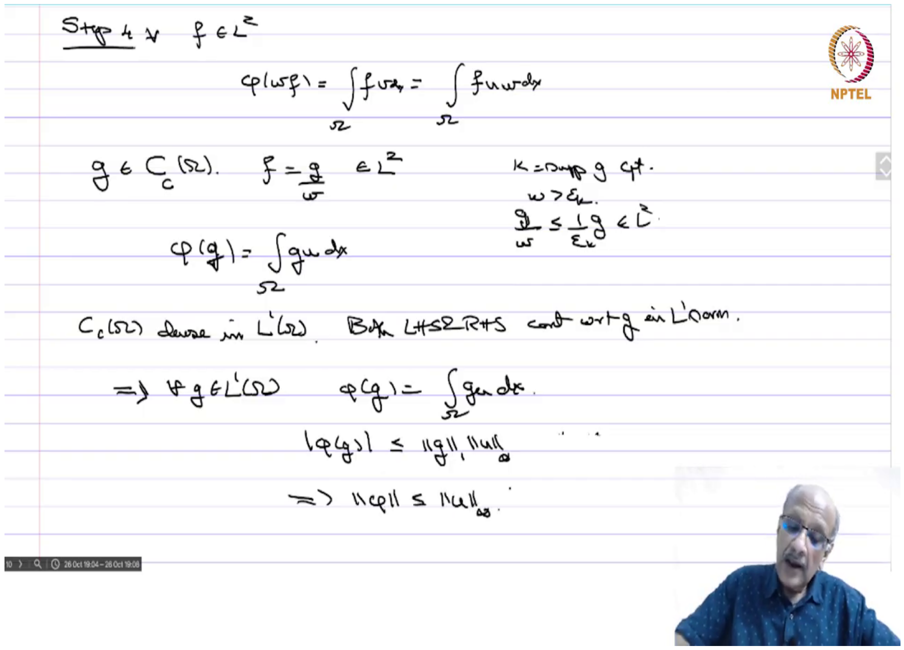

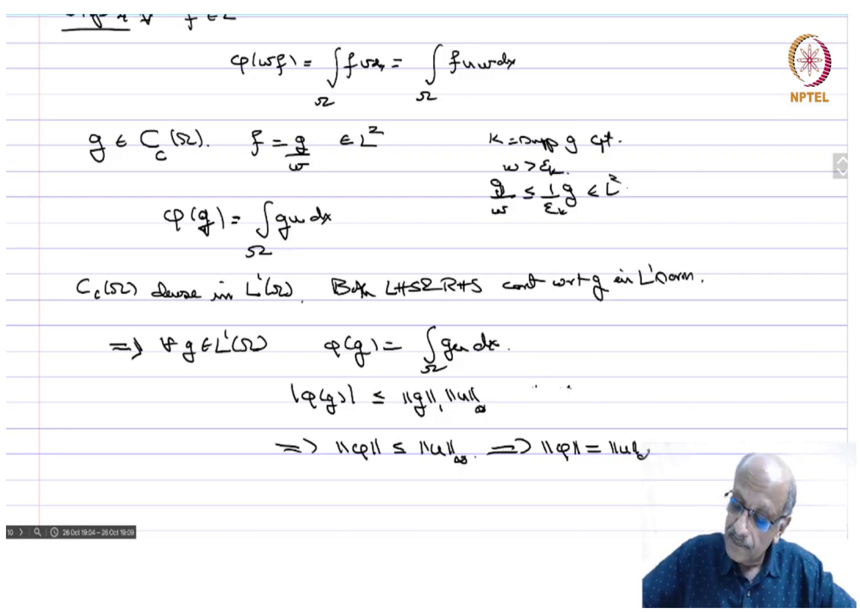

<u>Step 4:</u> $\forall f \in L^2$, we have $\phi(w\, f) = \int_\Omega v\, f dx$. But what is v? v$v = \int f\, u\, w\, dx$ because u is defined as v/w. so v is nothing but uw.

So now, take $g \in C_c(\Omega)$. Now, then if you look at $f = g/w$, now, g is nonzero only on a compact set $\subset \Omega$ and on any compact set $w > 0$. So, $\frac{1}{w}$ is bounded above. $k = supp\, g$ compact so $w > \epsilon_k$, so $\frac{g}{w} \leq \frac{1}{\epsilon_k} g \in L^2$ and therefore $\frac{g}{w} \in L^2$. So, we can substitute this in our equation and therefore $\phi(g) = \int g\, u\, dx$ for every $g \in C_c(\Omega)$. We have that $C_c(\Omega)$ is dense in $L^1(\Omega)$and both L.H.S. and R.H.S. continuous with respect to g in L^1 norm. This implies, $\forall g \in L^1(\Omega)$, $\phi(g) = \int_\Omega g\, u\, dx$. $|\phi(g)| \leq ||g||_1 ||u||_\infty$ (by the Holder inequality) $\Rightarrow ||\phi|| \leq ||u||_\infty$. We already saw in the previous step, that $||u||_\infty \leq ||\phi||$ and therefore $||u||_\infty = ||\phi||$.

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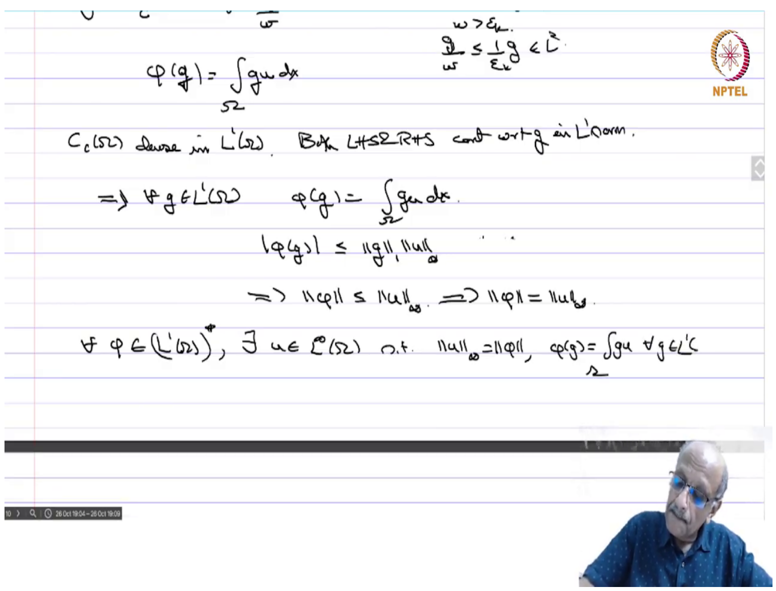

So, $\forall\, \varphi \in (L^1(\Omega))^*$, $\exists\, u \in L^\infty(\Omega)$ such that $||u||_\infty = ||\varphi||$ and

$\varphi(g) = \int_\Omega g\, u \,\forall\, g \in L^1(\Omega)$.

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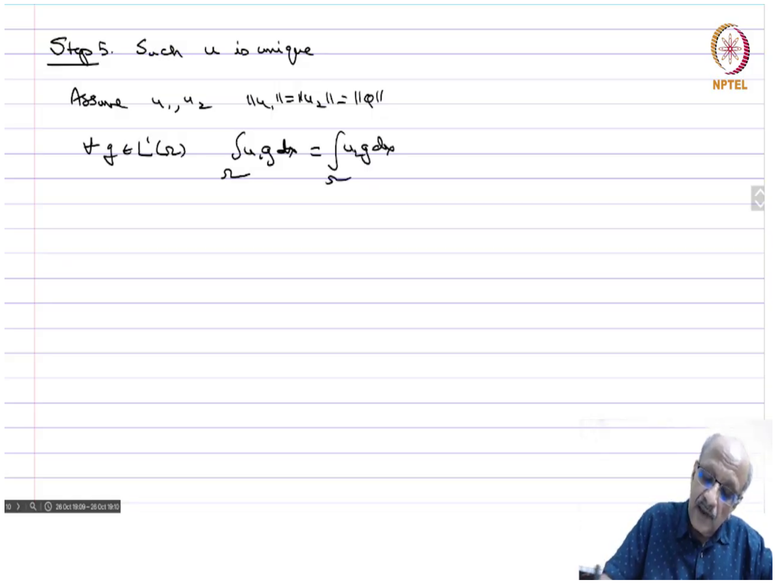

Now, this is a well defined thing.

<u>Step 5,</u> such u is unique. Assume we have two, assume u_1, u_2 such that $||u_1|| = ||u_2|| = ||\varphi||$ $\forall g \in L^1(\Omega), \int_\Omega u_1 g\, dx = \int_\Omega u_2 g\, dx = \varphi(g)$.

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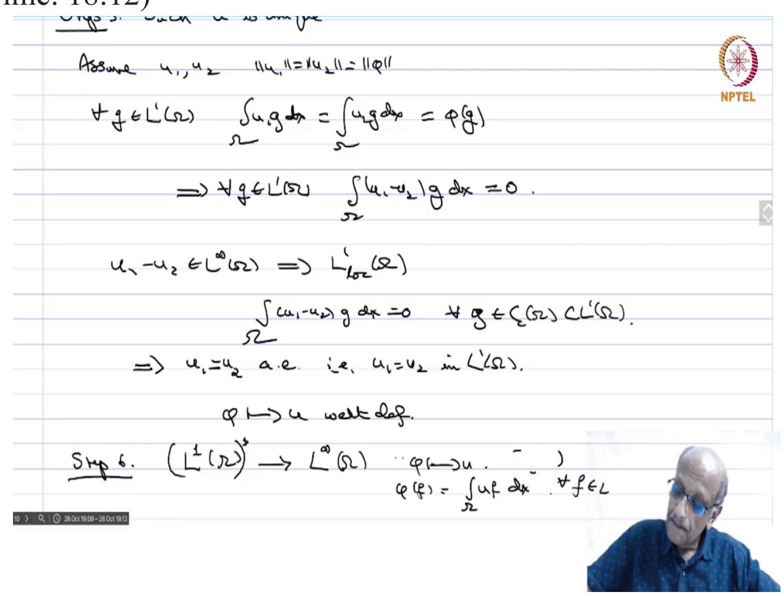

So, this implies $\forall\, g \in L^1(\Omega)$, you have $\int_\Omega (u_1 - u_2) g\, dx = 0$. So, and so this means what? Now, $(u_1 - u_2)$ is, so $(u_1 - u_2) \in L^\infty(\Omega)$, so they are bounded so on any compact set, they are integrable, therefore they belong to $L^1_{loc}(\Omega)$. Because they are bounded functions on any compact set of finite measure, and therefore, it is integrable on, so this is in L^1_{loc} and $\int_{\Omega}(u_1 - u_2)g\,dx = 0\ \forall\, g \in L^1$, therefore in particular, $\forall\, g \in C_c(\Omega)$, and therefore by our previous proposition, this will imply that $u_1 = u_2$ almost everywhere. That is, $u_1 = u_2$ in $L^1(\Omega)$. So, the mapping $\phi \mapsto u$ is well defined.

<u>Step 6</u>. So, you take, $(L^1(\Omega))^* \to L^{\infty}(\Omega)$, so, $\phi \mapsto u$, where

$$\phi(f) = \int_{\Omega} uf\,dx, \forall f \in L^1(\Omega).$$

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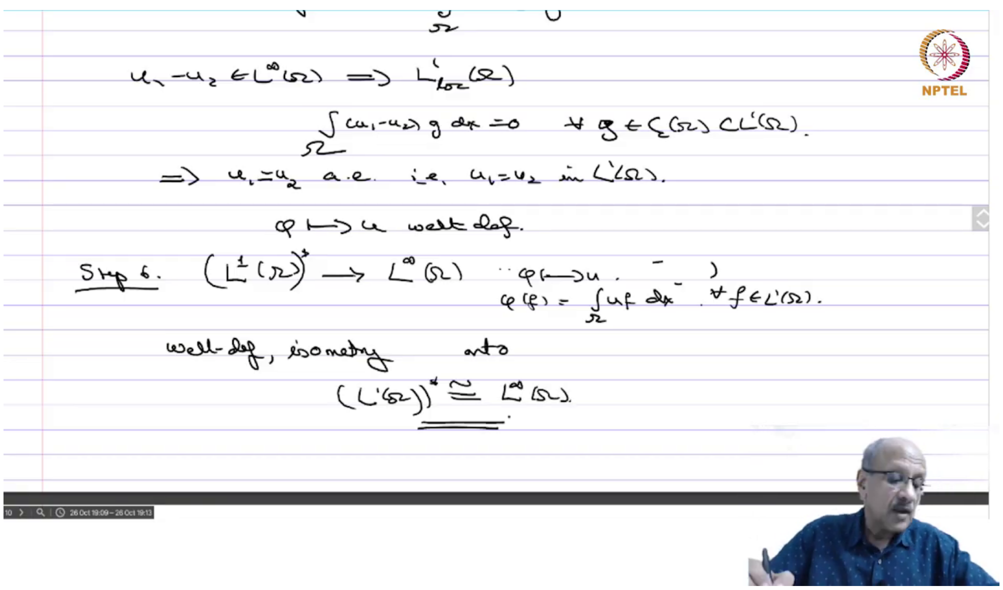

This map is well defined isometry and one-one and onto. So, it is one-one because it is an isometry and it is also onto. And therefore, you have that $(L^1(\Omega))^*$ is isometrically isomorphic to $L^{\infty}(\Omega)$. So, this proves the Riesz Representation Theorem.