

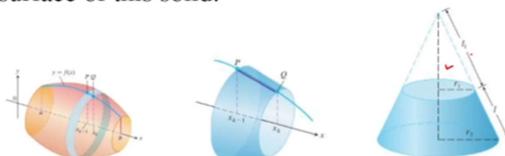
Basic Calculus - 1
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Lecture 36 - Part 1
Areas of Surface of Revolution - Part 1

Well, this is lecture 36 of Basic Calculus - 1. In the last lecture we were discussing how to compute the volume of a solid of revolution. Today we will be talking about the surface area of such a solid. Here, the assumption is that our solid is generated by revolving some region in the plane, around the x or y -axis. Since we are interested only in the area of the surface of revolution; we do not really need the whole solid; we need only its surface. You may think of this scenario: there is a curve given in the plane and that is being revolved about x -axis (or y -axis). And we want to find the area of that surface.

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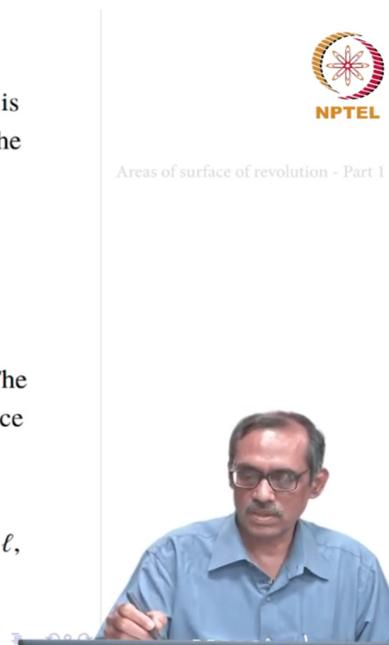
Frustum

Suppose an arc of a smooth curve $y = f(x), f(x) \geq 0$ for $a \leq x \leq b$, is revolved about the x -axis to generate a solid. We wish to compute the area of the surface of this solid.



Partition $[a, b]$ into n subintervals $[x_{i-1}, x_i]$. Write $\Delta x_i = x_i - x_{i-1}$. The norm of the partition is $\max_i \Delta x_i$. When each Δx_i is small, the surface area corresponding to this subinterval is approximately same as the area on the frustum of a right circular cone.

Recall that a right circular cone with base radius R and slant height ℓ , has surface area $\pi R\ell$.



Suppose we have a smooth curve which is given by a function $y = f(x)$; where $f(x) \geq 0$; that is, the curve is lying on the top half plane; and where x varies from a to b . That is how the curve is defined. This curve is revolved about the x -axis; let us take the case of x -axis first; it is revolved about the x -axis. So, you get the solid; well, it is not really a full solid; it is a hollow one. If you take this region and revolve about the x -axis; you would get the solid. But, we are interested only in the surface; so we just take that curve to revolve about the x -axis.

Now, this hollow solid is a surface. We want to compute the area of this surface. As earlier we will be dividing this interval $[a, b]$ into smaller portions. In each of these smaller portions, we will try to approximate the surface area; which is painted in light blue here. We want to approximate that, and then take the sum of all such small bands, to get the area of the surface. That means we take a partition of this interval $[a, b]$ by dividing it into, say, n sub-intervals. The sub-intervals

need not be of uniform length. This is done by choosing some points $x_0 < x_1 < \dots < x_n$ such that $x_0 = a, x_n = b$ as usual for a partition.

In each of these sub-intervals a band is generated. Let us write the length of this of sub-interval as Δx_i , which is equal to $x_i - x_{i-1}$. As earlier, we will say the norm of the partition is the maximum of this Δx_i . We want to take this sum and later take the limit, when this maximum of Δx_i goes to 0. That would give us the area of the surface. During this process, we have to see what exactly is the area of this band. The band looks something like a frustum. Look at the second picture. You see that it looks like a frustum.

Frustum is a part of the cone; it is a cone from where portions from the top as well as from the bottom are cut in a way that the new surfaces are parallel to the base. How do we get the area of this surface which is the frustum here? We do not include the areas of the top and bottom disks. You imagine that once extended it becomes a cone. Look at the third picture. We want to approximate the surface area which is painted blue. How do we approximate that? It is the frustum of a right circular cone; that is what we think. As you know the surface area of the cone is $\pi R\ell$, R is the radius of the base and ℓ is the slant height. Using that, we will compute this area of the frustum.

The area of the frustum is the full area of the cone minus that one which is white here. We subtract from this full area, the white portion of the cone. Suppose the small cone (in white) has base radius r_1 and slant height ℓ_1 . Then, its surface area is $\pi r_1 \ell_1$. Suppose the slant height of the frustum is ℓ and the base radius is r_2 . Then, the full cone has the slant height as $\ell_1 + \ell$ so that its surface area is $\pi r_2 (\ell_1 + \ell)$. The surface area of the frustum is $\pi r_2 (\ell_1 + \ell) - \pi r_1 \ell_1$. This is equal to $\pi [(r_2 - r_1)\ell_1 + r_2\ell]$.

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Smaller frustum

The area of the frustum is

$$A = \pi r_2 (\ell_1 + \ell) - \pi r_1 \ell_1 = \pi [(r_2 - r_1)\ell_1 + r_2\ell].$$

Using similarity of triangles we get: $\frac{\ell_1}{r_1} = \frac{\ell_1 + \ell}{r_2}$. This gives $r_2 \ell_1 = r_1 \ell_1 + r_2 \ell \Rightarrow (r_2 - r_1)\ell_1 = r_2 \ell$. So,

$$A = \pi (r_1 \ell + r_2 \ell) = \underline{2\pi r \ell}, \quad \text{where } r = \frac{r_1 + r_2}{2}.$$

We use this formula on the frustum corresponding to the sub-interval $[x_{i-1}, x_i]$. Here, the slant height is approximated by

$$\sqrt{\Delta x_i^2 + \Delta y_i^2}, \quad \text{where } \Delta x_i = x_i - x_{i-1}, \quad \Delta y_i = y_i - y_{i-1} = f(x_i) - f(x_{i-1}).$$

The average radius is $r = (r_1 + r_2)/2 = [f(x_i) + f(x_{i-1})]/2$. Hence the area A_i of the frustum corresponding to $[x_{i-1}, x_i]$ is

$$\underline{A_i} = 2\pi \frac{f(x_{i-1}) + f(x_i)}{2} \sqrt{\Delta x_i^2 + \Delta y_i^2}.$$

We would like to express this formula where ℓ_1 does not appear because that is not in the frustum. For this, we go back to the picture and look at the two triangles there. One is this smaller one on the top small cone, and the other is the bigger one on the full cone. They are similar



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triangles. Therefore, we infer that $\ell_1/r_1 = (\lambda_1 + \ell)/r_2$. We cross multiply and bring it to the form $(r_2 - r_1)\ell_1 = r_1\ell$. Thus, the surface area of the frustum is equal to $\pi[(r_2 - r_1)\ell_1 + r_2\ell] = \pi[r_1\ell + r_2\ell]$. The same can be written as $2\pi r\ell$, where $r = (r_1 + r_2)/2$. That is, the surface area of the frustum (leaving the areas of two bases) is 2π times the average of the top and bottom radii into the slant height.

Now we have to sum up all these areas taking the sub-intervals $[x_{i-1}, x_i]$. For the sub-interval $[x_{i-1}, x_i]$, the corresponding slant height of the frustum is now approximated by $\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2}$. Look at the picture again. Here, we want to approximate this ℓ . We find that it is the hypotenuse in this right angles triangle with sides Δx_i and Δy_i . That is the reason for the slant height to be as mentioned above. Then, we plug in this slant height to the formula of the area of the frustum. There, the radii of the bigger base is $r_2 = f(x_i)$ and the radius of the smaller base is $r_1 = f(x_{i-1})$. Hence, $r = (r_1 + r_2)/2 = [f(x_{i-1}) + f(x_i)]/2$. Therefore, the area of the frustum corresponding to the sub-interval $[x_{i-1}, x_i]$ is $A_i = 2\pi r\ell = 2\pi\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} [f(x_{i-1}) + f(x_i)]/2$. As we have said, our approximation to the whole area will be the sum of these A_i s. But, before that let us have some simplification of this.

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Surface area

Due to MVT, there exists $c_i \in [x_{i-1}, x_i]$ such that

$$\Delta y_i = f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}) = f'(c_i)\Delta x_i.$$

Then, $\sqrt{\Delta x_i^2 + \Delta y_i^2} = \sqrt{1 + [f'(c_i)]^2} \Delta x_i$. Also, the surface area of revolution is approximated by $\sum_i A_i$. It is then given by

$$\sum_{i=1}^n A_i = \sum_{i=1}^n 2\pi \frac{f(x_{i-1}) + f(x_i)}{2} \sqrt{1 + [f'(c_i)]^2} \Delta x_i.$$

Its limit when the norm of the partition $\max_i \Delta x_i$ approaches 0, is the area of the surface of revolution.



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We use the mean value theorem here. Since we have assumed that $f(x)$ is smooth, we can use the mean value theorem to conclude that $f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1})$, where c_i is some point between x_{i-1} and x_i . Notice that the left side is Δy_i and $x_i - x_{i-1} = \Delta x_i$. Hence, $\Delta y_i = f'(c_i)\Delta x_i$. Then, $(\Delta x_i)^2 + (\Delta y_i)^2 = (\Delta x_i)^2(1 + [f'(c_i)]^2)$. Once this is obtained, we get

$$A_i = 2\pi\sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} [f(x_{i-1}) + f(x_i)]/2 = 2\pi\sqrt{1 + [f'(c_i)]^2} \Delta x_i [f(x_{i-1}) + f(x_i)]/2.$$

Then we take the sum of all these A_i s by varying $i = 1, 2, \dots, n$. When the limit of this sum as the norm of the partition, $\max_i \Delta x_i$, approaches 0, we obtain the area of the surface of revolution.

Now, what will happen when we take the limit? Once $\max_i \Delta x_i$ goes to 0, you see that c_i goes

to x_i . So, what do we get in the limit? It will be the integral of $2\pi y\sqrt{1 + [f'(x)]^2}$.
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Revolution about y-axis

The surface area of revolution is given by

$$A = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2} dx = \int_a^b 2\pi y\sqrt{1 + [y']^2} dx.$$

Write $ds = \sqrt{1 + [f'(x)]^2} dx$. Then $A = \int_a^b 2\pi y ds$, where the limits of integration are for $x = a$ and $x = b$.

When the curve is given as $x = g(y)$ and its arc corresponding to $c \leq y \leq d$ is revolved about y-axis, the surface area is given by

$$A = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2} dy = \int_c^d 2\pi x\sqrt{1 + [x']^2} dy.$$

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So, we take $2\pi \int_a^b f(x)\sqrt{1 + [f'(x)]^2} dx$ as the surface area of the solid of generated by revolving the curve $y = f(x)$, $a \leq x \leq b$ about the x -axis. For this to happen, we have assumed that $f(x)$ is greater than equal to 0, and the curve does not intersect itself. This latter assumption is used when we approximated the surface area as the sum of surface areas of frustms. If the curve $y = f(x)$ intersects itself, then this will not be the surface area. This formula holds only when the curve is non-intersecting and thus it is traversed only once.

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Caution

In the first case, when $y = f(x)$ is revolved about the x -axis, we assumed that $f(x) \geq 0$ for the arc of the curve.

If $f(x) \leq 0$ then we should consider $y = -f(x)$ to obtain the area of the surface of revolution.

Geometrically, they are same, but algebraically, they are different; one is the negative of the other.

Since area is non-negative, we consider $y = -f(x)$ instead of $y = f(x)$ for computation purposes.

Similar comments hold true for the other cases as well.

Alternatively, we may consider $|f(x)|$ uniformly.

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While x varies from a to b , the curve revolves about the x -axis. In that case, the integrand has $2\pi y$ times the square root of $1 + [f'(x)]^2$. If the curve revolves about the y -axis, then the integrand

will involve x instead of y . Because this factor y in the above formula corresponds to the radius r in the frustums. There is a common mistake one may commit. Why do we think of $\sqrt{1 + [y']^2}$ in the integrand instead of just y ? The reason is, the length of the curve is revolving, so, this factor times dx is simply ds . Similarly, when the curve revolves about the y -axis, we may take $\sqrt{1 + [x']^2} dy$ since that is also equal to ds .

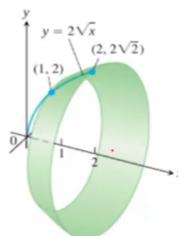
Using this notation, the surface area of revolution about the x -axis can be expressed as $\int_a^b 2\pi y ds$, where the curve is given by $y = y(x)$, $a \leq x \leq b$; the limits of integration are written here for $a \leq x \leq b$. Here, s is thought of as a function of x . When the revolution is about the y -axis, and the curve is given by $x = x(y)$, $c \leq y \leq d$, then the same may be expressed as $\int_c^d 2\pi x ds$, where the limits of integration are written for $c \leq y \leq d$. Here, s is thought of as a function of y .

Look at our assumptions. They are: $y = f(x)$ is revolved about the x -axis, with $f(x) \geq 0$ for $a \leq x \leq b$. Then, you get the arc of the curve. Similarly, if you take $f(x) \leq 0$, then $-f(x)$ satisfies the assumptions. In this case, we work with the curve $y = -f(x)$. That would give rise to the area of the surface of revolution correctly. What do we do if both the things happened? That is, suppose in some sub-interval $[a, c]$, $f(x) \leq 0$ and in the other sub-interval $[c, b]$, $f(x) \geq 0$. Then you may have to compute both of them separately, and then add them up to get the total area of surface of revolution.

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Example 1

Find the surface area of the solid obtained by revolving about x -axis, the arc of the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$.



$$y = 2\sqrt{x}, y' = 1/\sqrt{x}, \sqrt{1 + [y']^2} = \sqrt{1 + 1/x}.$$

The surface area of revolution is equal to

$$\begin{aligned} \int_1^2 2\pi y \sqrt{1 + [y']^2} dx &= \int_1^2 2\pi 2\sqrt{x} \sqrt{1 + 1/x} dx \\ &= \int_1^2 4\pi \sqrt{x+1} dx = 4\pi \frac{2}{3} (x+1)^{3/2} \Big|_1^2 = \frac{8\pi}{3} (3\sqrt{3} - 2\sqrt{2}). \end{aligned}$$



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Let us see an example. We have the curve $y = 2\sqrt{x}$. And, we are taking an arc of this curve, which is defined by varying x between 1 and 2. So, it is this portion when $1 \leq x \leq 2$. The curve $y = 2\sqrt{x}$ is the blue one as you see in the picture, where we are interested only in this arc corresponding to $1 \leq x \leq 2$. This arc is revolved around the x -axis to obtain this surface, which is painted green here. We want to find the surface area. Of course, we will be using the formula, but first let us see what the function is. It is $y = 2\sqrt{x}$. We need its derivative. The derivative is $1/\sqrt{x}$. It is $2x^{1/2}$ when differentiated gives $2(1/2)x^{1/2-1}$ and that simplifies to $1/\sqrt{x}$. Okey. Then,

$1 + [y']^2 = 1 + 1/x$. Then, the surface area will be $2\pi \int_1^2 y\sqrt{1 + [y']^2} dx$.

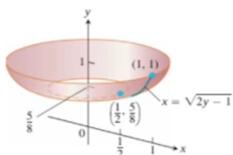
Why do we take this y ? It corresponds to the radius r of the frustum. In fact, you get twice the average of the radii of the top disk and the bottom disk of the frustum. And that average brings in this y .

So, it is the integral $\int_2^2 \pi y 2\pi y \sqrt{1 + [y']^2} dx$. The limits for x are 1 and 2, since the curve is given only for $1 \leq x \leq 2$. The integral is $\int_1^2 2\pi \times 2\sqrt{x}\sqrt{1 + 1/x} dx$. If you multiply that x taking it inside the square root; then it gives you $\sqrt{x + 1}$ so that the integral is $4\pi \int_1^2 \sqrt{x + 1} dx$. You can integrate it thinking of $x + 1$ as u ; and then $u^{1/2}$. That gives to $(2/3)(x + 1)^{3/2}$ as the integral of $\sqrt{x + 1}$. We have the factor 4π multiplied with this. So, it is $(8\pi/3)(x + 1)^{3/2}$, which is to be evaluated at 1 and 2; and then subtracted out. And if you simplify that gives you $(8\pi/3)(3\sqrt{3} - 2\sqrt{2})$. That is the surface area of revolution.

This happens when we are revolving a curve about the x -axis. How to take care when a curve revolves about the y -axis? Let us take the second example. Here we want to find the area of this surface generated by revolving the arc of the curve $y = (1 + x^2)/2$. And what is that arc? The arc is described when you take x lying between half and 1. That corresponds to this blue one here as depicted in the picture. That arc is revolved about the y -axis. Then, you get the picture this way. Notice that we need to express x in terms of y .

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Example 2



Find the area of the surface generated by revolving the arc of the curve $y = (1 + x^2)/2$ for $1/2 \leq x \leq 1$, about the y -axis.

Revolution is about y -axis. We write $x = x(y)$.

$y = (1 + x^2)/2$ and $x > 0$ iff $x = \sqrt{2y - 1}$.

$1/2 \leq x \leq 1$ iff $[1 + (1/2)^2]/2 \leq y \leq [1 + 1^2]/2$ iff $5/8 \leq y \leq 1$.

The curve $x = \sqrt{2y - 1}$, $5/8 \leq y \leq 1$ is revolved about the y -axis.

$x' = dx/dy = 1/\sqrt{2y - 1}$. Then the surface area of revolution is

$$\int_{5/8}^1 2\pi \sqrt{2y - 1} \sqrt{1 + 1/(2y - 1)} dy = \int_{5/8}^1 2\pi \sqrt{2y}^{1/2} dy$$

$$= 2\sqrt{2} \pi \frac{2}{3} y^{3/2} \Big|_{5/8}^1 = \frac{\pi}{12} (16\sqrt{2} - 5\sqrt{5}).$$



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As you see, the curve is lying in the upper half of the plane, that is, $y \geq 0$. When you take x in terms of y ; it will be $x = x(y) = \sqrt{2y - 1}$ for $x \geq 0$. Since x varies from half to 1, $x \geq 0$. That is how we get this curve. Our integral will involve limits for y but we only know that x varies from half to 1. We have to find out limits for y . In $y = (1 + x^2)/2$, we substitute the values of x ; when $x = 1/2$, $y = (1 + 1/4)/2 = 5/8$, and when $x = 1$, $y = (1 + 1)/2 = 1$. So, y varies from $5/8$ to 1, x is a function of y , which we may say $x = g(y) = \sqrt{2y - 1}$ for $5/8 \leq y \leq 1$. We want to find the area of surface of revolution, when the curve is revolved about the y -axis.

The area will be equal to the integral $\int_{5/8}^1 2\pi g(y) \sqrt{1 + [g'(y)]^2} dy$. Now that $g(y) = \sqrt{2y - 1}$, we use the chain rule to differentiate it. That is, when you differentiate with respect to y ; first you differentiate with respect to $2y - 1$ and next $2y - 1$ with respect to y . It gives $x'(y) = g'(y) = (1/2)(2y - 1)^{-1/2} \times 2$, which is equal to $1/\sqrt{2y - 1}$. Then, $\sqrt{1 + [g'(y)]^2} = \sqrt{1 + 1/(2y - 1)} = \sqrt{2y/(2y - 1)}$. Therefore, the surface area will be $\int_{5/8}^1 2\pi \sqrt{2y - 1} \times \sqrt{2y/(2y - 1)} dy$ which is equal to $\int_{5/8}^1 2\sqrt{2} \pi \sqrt{y} dy$. Integration of \sqrt{y} gives $(2/3)y^{3/2}$. Then, the integral is equal to $4\sqrt{2} \pi/3 y^{3/2}$ evaluated at $5/8$ and 1 , and then subtracted. That simplifies to this number; which is $(\pi/12)(16\sqrt{2} - 5\sqrt{5})$.

This is how we proceed when a curve is revolving about the y -axis; and it is given as x is a function of y .