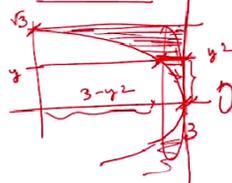


Basic Calculus - 1
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Lecture 34 - Part 2
Volumes by Cylindrical Shells - Part 2

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Exercise 1

Using the cylindrical shells method find the volume of the solid generated by revolving the region enclosed by the curve $x = 3 - y^2$, the line $x = 3$ and the line $y = \sqrt{3}$ about the x -axis.



Volumes by cylindrical shells - Part 1



Let us take a problem. It is specifically mentioned to use the cylindrical shells method for finding the volume of the solid which is generated by revolving the region enclosed by the curve $x = 3 - y^2$, the line $x = 3$, and the line $y = \sqrt{3}$ about the x -axis.

First, what is the region? We have $x = 3 - y^2$; that looks something like this, a parabola. Where does it cross the y -axis? Where does it cross the x -axis? When $x = 0$, $y = \sqrt{3}$ on the top and $y = -\sqrt{3}$ on the bottom. But let us take one side only; we can take any one of them since its revolution would generate the solid. So, let us take $y = \sqrt{3}$, which is in the first quadrant.

We want to see the region enclosed by the curve $x = 3 - y^2$, the line $x = 3$ and the line $y = \sqrt{3}$. The region bounded by them is this shaded one; it is not inside that parabola, it is outside. This is the region. And, this region is revolved about the x -axis. So, there is no need to take below this, only one side is enough to generate the solid. The solid would look something like this.

Then, what is the shell radius? What is the shell height? It is revolving about the x -axis. So, we should start with taking a point y . Let us go back to the picture. You take any y here. That gives this distance as your shell height. And what is the shell radius? It is measured from the axis. So, that will be the shell radius. The shell radius is y . The shell height is the point on the curve which gives you x corresponding to that y ; it is $3 - y^2$; this distance is $3 - y^2$. Since the total distance is 3, this particular distance, which is the shell height should be y^2 . So, the shell radius is y and the shell height is y^2 .

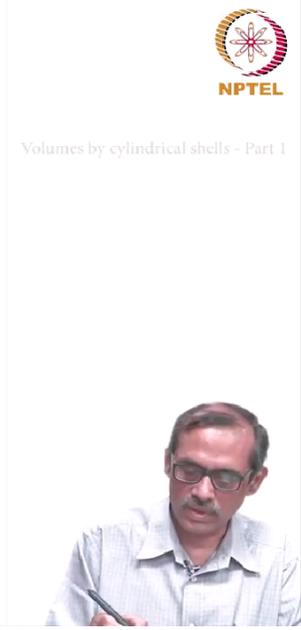
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Exercise 1

Using the cylindrical shells method find the volume of the solid generated by revolving the region enclosed by the curve $x = 3 - y^2$, the line $x = 3$ and the line $y = \sqrt{3}$ about the x -axis.

Ans:

$$\begin{aligned} V &= \int_0^{\sqrt{3}} (2\pi)(y)(3 - (3 - y^2)) dy \\ &= \int_0^{\sqrt{3}} 2\pi y^3 dy \\ &= 2\pi \frac{y^4}{4} \Big|_0^{\sqrt{3}} \\ &= \frac{9\pi}{2} \end{aligned}$$



So, the volume is the integral where y varies from 0 to $\sqrt{3}$ and the integrand is 2π times the shell radius times the shell height. This gives the volume as $\int_0^{\sqrt{3}} 2\pi y^3 dy$. That is easy to integrate. The answer is $y^4/4$ evaluated at 3 and 0, then subtracted. That simplifies to $2\pi(\sqrt{3})^4/4 = 9\pi/2$.

That is how we solve it by the cylindrical shell method. You can also do it by the disk method and verify whether the answer is same or not.

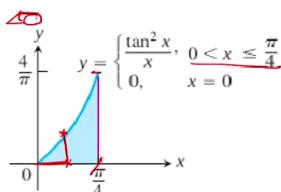
Let us go to next problem. We have the function given as $g(x) = [\tan^2 x]/x$ for $x \neq 0$ and $g(0) = 0$. Since $[\tan^2 x]/x$ is not defined at $x = 0$, g at 0 is redefined with the value as 0. This is the function. We have the region bounded by $y = g(x)$, the lines $x = \pi/4$ and $y = 0$. This region is revolved about the y -axis to generate a solid. Then we want to find the volume of the solid.

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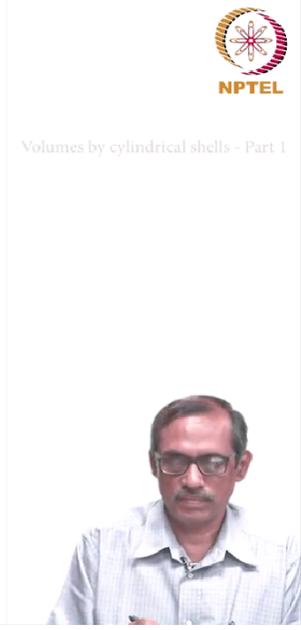
Exercise 2

Let $g(x) = (\tan^2 x)/x$ for $x \neq 0$ and $g(0) = 0$. Using cylindrical shell method find the volume of the solid generated by revolving the region bounded by $y = g(x)$, lines $x = \pi/4$ and $y = 0$ about the y -axis.

Ans:



$$\begin{aligned} xg(x) &= \tan^2 x \text{ for } 0 \leq x \leq \pi/4. \text{ The required volume is } \int \sec^2 x dx = \tan x \\ V &= \int_0^{\pi/4} (2\pi)(x)(g(x)) dx = 2\pi \int_0^{\pi/4} \tan^2 x dx \quad (\sec^2 x - 1) \\ &= 2\pi [\tan x - x]_0^{\pi/4} = 2\pi(1 - \frac{\pi}{4}). \end{aligned}$$



Let us look at the region. The function is $y = [\tan^2 x]/x$ including the origin, and we have the lines $x = \pi/4$ and the x -axis. We limit all our focus to $0 \leq x \leq \pi/4$ and also to one of the quadrants. Notice that $g(x)$ remains positive for x positive. The line $x = \pi/4$ and the curve cross at the point $(\pi/4, 4/\pi)$ since $x = \pi/4$ implies $y = [\tan^2(\pi/4)]/(\pi/4) = 4/\pi$. So, this is how the region looks like. Now, this region is revolved about the y -axis. That means, you can use the disk method to get the volume of the solid so generated.

But let us use the cylindrical shell method. If you take the radius of revolution at any point, say x , then it is this one, just x . The shell height will be this, which is up to $[\tan^2 x]/x$. So, the volume is the integral with the integrand as $g(x) \times x = \tan^2 x$. Notice that at $x = 0$, $g(x) \times x$ is 0, which is also $\tan^2 0$. Then, the volume is $\int_0^{\pi/4} 2\pi \tan^2 x \, dx$.

How do we compute the integral of $\tan^2 x$? We know how to integrate $\sec^2 x$; it is $\int \sec^2 x \, dx = \tan x$. So, we express this $\tan^2 x$ in terms of $\sec^2 x$; that is, $\tan^2(x) = \sec^2 x - 1$. Then, the integral of $\sec^2 x$ gives you $\tan x$ and the integral of -1 gives you $-x$. So, it is $2\pi(\tan x - x)$ evaluated at 0 to $\pi/4$. That simplifies to $2\pi(1 - \pi/4)$. As you see the cylindrical shell method is not very difficult to apply.

Let us solve another problem. Find the volume of the solid generated by revolving the region enclosed by the curves $2x = y^2$ and $4x^2 = y^4 - 2y^2$ in the upper half plane about the line $y = 5$. In the upper plane we have the region enclosed by $2x = y^2$ and $4x = y^4 - 2y^2$. We are revolving this region about the line $y = 5$. We want to find the solid of revolution.

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Exercise 3

Find the volume of the solid generated by revolving the region enclosed by the curves $2x = y^2$ and $4x = y^4 - 2y^2$ in the upper half plane about the line $y = 5$.

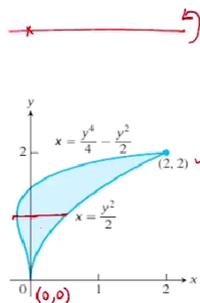
Ans:

The curves intersect at $4x = 2y^2 = y^4 - 2y^2$

$\Rightarrow y^2(4 - y^2) = 0 \Rightarrow y = 0$ or $y = 2$, since $y \geq 0$. Then $x = 0$ or 2 .

Since axis of revolution is $y = 5$, the shell radius is $5 - y$ and the shell height is $y^2/2 - (y^4/4 - y^2/2) = y^2 - y^4/4$. The volume is

$$V = \int_0^2 2\pi(5 - y)(y^2 - y^4/4) \, dy = 2\pi \left[\frac{5y^3}{3} - \frac{5y^5}{20} - \frac{y^4}{4} + \frac{y^6}{24} \right]_0^2 = 8\pi.$$



Volumes by cylindrical shells - Part 1



Let us look at how does it figure out? It is a revolution about $y = 5$. So, you should be able to express x in terms of y to get the integral. The first curve is $2x = y^2$ or $x = y^2/2$. It is this one, the down one which in the upper half plane; it joins the origin to $(2, 2)$, which is a point on this curve. Since the region is enclosed by two curves we should find out what are the points of intersection in the upper half plane. You can easily get that. Let us eliminate x from both the equations. It gives

$4x = 2y^2 = y^4 - 2y^2$, which implies $y^4 - 4y^2 = 0$. That factorizes to $y^2(y^2 - 4)$. So, $y = 0$ or $y = 2$. Since we are interested in the upper half plane, $y \geq 0$ so that we do not consider the solution $y = -2$. Then the values of x corresponding to $y = 0, 2$ are $x = 0$ and $x = 2$, respectively. So, we have two intersection points; one is the origin and other is $(2, 2)$. Now, y lies between 0 and 2, x lies between 0 and 2 of course. We need the values of y .

We go to the solid generated by the revolution of this region about the line $y = 5$. The line $y = 5$ will be somewhere here. The region is revolved around this line. We need the shell radius. If you take any y , then the difference between that $y = 5$ and y , which is $5 - y$ is the shell radius. And what is the shell height? Now, you have two ways to proceed from this point. One, you can find the volumes of two solids here, and add them later. Two, you may take both into consideration simultaneously. Let us compute the second way.

The shell radius is $5 - y$ as we have seen. For the shell height, look at the region from the y -axis. The upper curve is $x = y^2/2$ and the other one is the lower curve. So, the shell height is their difference, which is $y^2/2 - (y^4/4 - y^2/2)$. This is the shell height, which simplifies to $y^2 - y^4/4$.

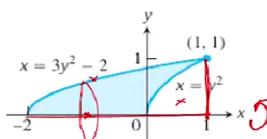
Then our volume is the integral where y varies from 0 to 2 of 2π into the shell radius, which is $5 - y$, into the shell height, which is $y^2 - y^4/4$. That is, $\int_0^2 2\pi(5 - y)(y^2 - y^4/4) dy$. This 2π goes out and we multiply the two factors to get $5y^2 - y^3 - (5/4)y^4 + y^5/4$. Integrating we get $2\pi[(5/3)y^3 - (5/20)y^5 - (1/4)y^4 + (1/24)y^6]$. This is to be evaluated at 0 and 2, and then subtracted out. That simplifies to 8π .

We go for the next problem. Here, we use all the three methods such as the Disk method, the Washer method, and the Cylindrical Shell method to find the volume of the solid generated by revolving the region about the x -axis. What is the region? It is bounded by the curves $x = y^2$ and $x = 3y^2 - 2$, and the line segment joining $(-2, 0)$ to the origin.

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Exercise 4

By using all the three methods such as disk, washer, and cylindrical shell, find the volume of the solid generated by revolving the region bounded by the curves $x = y^2$, $x = 3y^2 - 2$ and the line segment joining $(-2, 0)$ to the origin, about the x -axis.



$$y^2 = 3y^2 - 2$$

$$2y^2 - 2 = 0$$

$$y = \pm 1$$

Ans: (a) Disk method: Since the axis of revolution is the x -axis, we need to compute two volumes and subtract one from the other. It is

$$V = \int_{-2}^1 \pi \frac{x+2}{3} dx - \int_0^1 \pi x dx = \pi \left[\frac{x^2}{6} + \frac{2x}{3} \right]_{-2}^1 - \pi \left[\frac{x^2}{2} \right]_0^1 = \pi$$



Volumes by cylindrical shells - Part 1



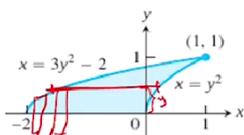
The line segment joining the point $(-2, 0)$ to the origin is this; it lies on the x -axis. We have

the curve $x = y^2$ here, and the curve $x = 3y^2 - 2$ is this one. First of all we should find the points of intersection of these two curves. Let us compute the points of intersection. Eliminating x from $x = y^2$ and $x = 3y^2 - 2$, we have $y^2 = 3y^2 - 2$, or $2y^2 - 2 = 0$. It gives $y = \pm 1$. So, the points of intersection are $(1, -1)$ and $(1, 1)$.

The region is revolved about the x -axis. We take only one of these points to get the region. If you take both the points, then you will get twice the volume required; they will be adding up. So, let us consider the point $(1, 1)$. We have the point $(-2, 0)$ where the curve $x = 3y^2 - 2$ crosses the x -axis, and we have the origin as the third point where $x = y^2$ crosses the x -axis. These three points are the three corners of the region. And the suitable curves joining these points form the border of the region. The points $(0, 0)$ and $(1, 1)$ are joined by the curve $x = y^2$. The points $(1, 1)$ and $(-2, 0)$ are joined by the curve $x = 3y^2 - 2$, and the points $(-2, 0)$ and $(0, 0)$ are joined by the x -axis. We thus obtain this region, which is painted blue. Now this region is revolved about the x -axis to generate the solid.

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Exercise 4 Contd.



(b) Washer method:

This gives subtraction of exactly the same two integrals as in (a).

(c) Cylindrical shell method:

Here, shell radius is y and shell height is $y^2 - (3y^2 - 2) = 2 - 2y^2$.

And y varies from 0 to 1. So,

$$V = \int_0^1 2\pi y(2 - 2y^2) dy = 2\pi \left[y^2 - \frac{2}{4}y^4 \right]_0^1 = \pi.$$



Volumes by cylindrical shells - Part 1



You can use the disk method because the axis of revolution, which is the x -axis, is bordering the region. It will generate disks of this type. So, let us use the Disk method first. In the disk method, what is the radius? At a point y , the curve is $x = 3y^2 - 2$. That gives you the disk area as $\pi y^2 = \pi(x + 2)/3$. This is revolving about the x -axis. Thus, the integral $\int_{-2}^1 \pi(x + 2)/3 dx$ will give the volume of the solid including the volume of the solid generated by this white region, the region bounded by the curve $x = y^2$, the line $x = 1$ and the x -axis. To get the required volume we must subtract this extra volume. This extra volume is the integral $\int_0^1 \pi y^2 dx = \int_0^1 \pi x dx$.

Thus, the required volume is $\int_{-2}^1 \pi(x + 2)/3 dx - \int_0^1 \pi x dx$. Notice that the revolution is about the x -axis; so, we are taking the integral with respect to x . Integrating we have $\pi(x^2/6 + 2x/3)$ to be evaluated at 1 and -2 , and then subtracted minus $\pi(x^2/2)$ to be evaluated at 1 and 0, and subtracted. This gives the answer as π .

Now, if you use the Washer method, there is nothing new here. Because there is no hole. It is the same as the disk method; you will not get anything new. This gives the subtraction of exactly the same two integrals. So, you get the same answer.

Let us go to the cylindrical shell method. In the cylindrical shell method, this is really approximated by the sum of cylinders of this type. Since the revolution is about the x -axis, you have to express everything in terms of y . So, take any y . Corresponding to that y , what is the shell radius? The shell radius is this one, which is y really. And, what is the shell height? The shell height will be the distance between these two points. That means, we have to compute the distance from $3y^2 - 2$ to y^2 . It is $y^2 - (3y^2 - 2) = 2 - 2y^2$. That is the shell height and the shell radius is y .

Therefore, the volume is the integral from 0 to 1 of $2\pi y(2 - 2y^2)$. That is, $\int_0^1 2\pi y(2 - 2y^2) dy$. Integrating we get $2\pi(y^2 - (2/4)y^4)$. And this is to be evaluated at 1 and 0, and then subtracted. When simplified, that again gives the same result π .

Of course, for every problem here you could have used the washer method. You must try and verify the answers. That is an exercise for you. Let us stop here.