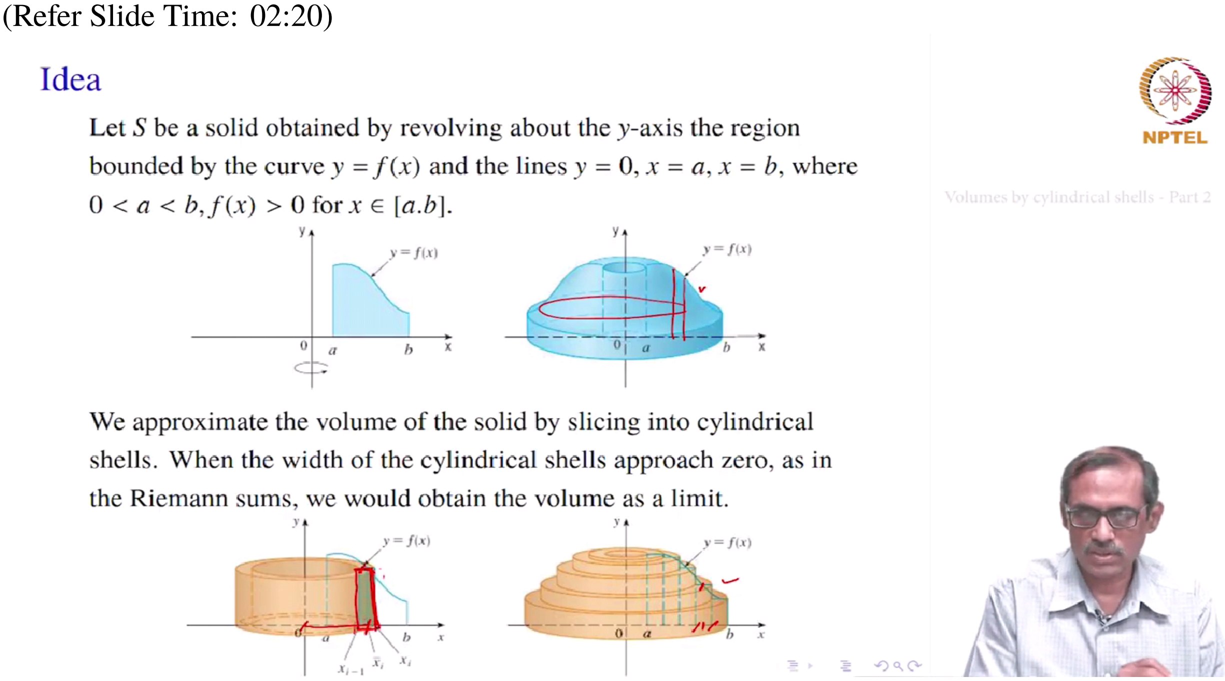
Basic Calculus - 1
Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras
Lecture 34 - Part 1
Volumes by Cylindrical Shells - Part 1

Well, this is lecture 34 of Basic Calculus - 1. We were discussing how to compute the volumes of solids of revolution. The last thing we discussed was the washer method.

Today we will have an alternative method to compute the volume of solids of revolution. This is called the method of cylindrical shells. In fact, we will be going through the Riemann sums again, but in a different way. We will see how to address this issue.

Let us consider this. We have a curve $y = f(x)$, which is continuous, and it remains positive inside the interval $[a, b]$. We have the area of the region bounded by $x = a$ on the left, $x = b$ on the right, the x-axis below, and the curve $y = f(x)$ on the top. Now, this region is revolved around y-axis, which is the line $x = 0$. We obtain the solid which is on the second picture, and it is having a hole. The hole is cylindrical; it is generated from $-a$ to a. This is how it looks. For this solid, of course, we can use the washer method as earlier. It will be the sum of all these things. But instead of that, we want to apply another idea.

(Refer Slide Time: 02:20)

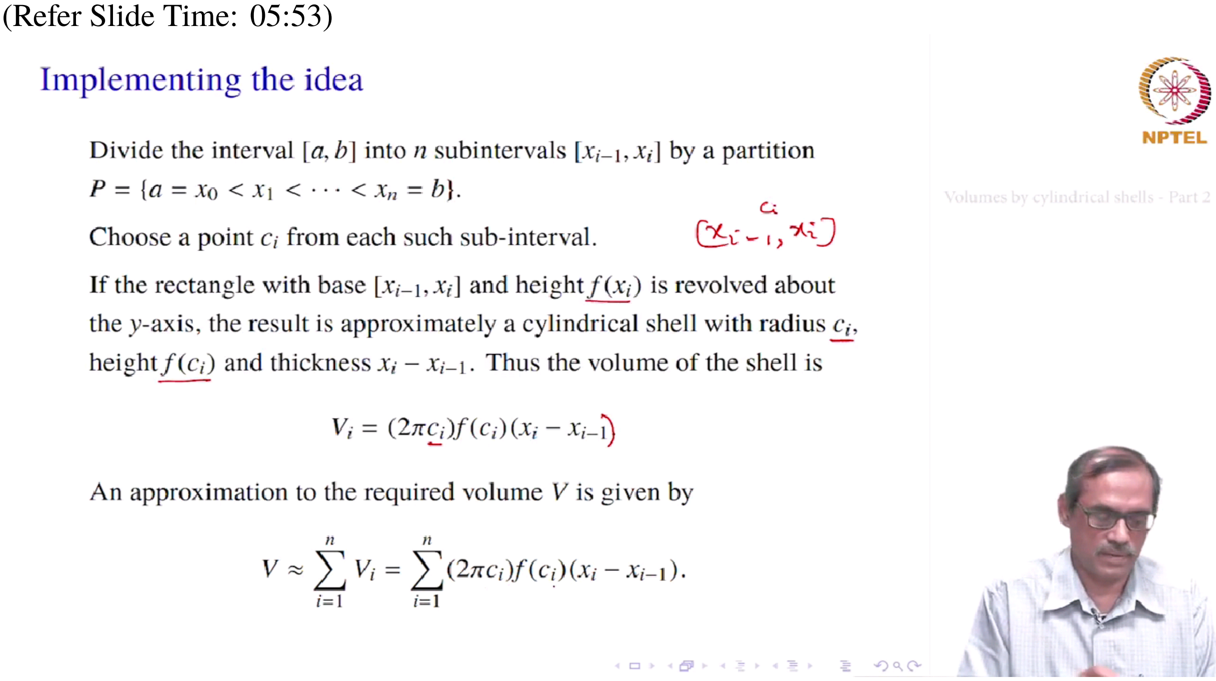


We think of approximating the volume of the solid in this fashion; look at the fourth picture down. We divide the interval $[a, b]$ into many smaller parts. From each smaller part, the sub-interval $[x_{i-1}, x_i]$, we choose a point c_i. Then, we approximate the volume below that curve and bounded by this by taking in step-wise fashion the smaller cylinders. Look at the third picture. You will find one typical cylindrical shell there. That shell really approximates the volume, or rather, the slice of the volume, which is generated over this. Only that slice is approximated, because we have taken this rectangle instead of the whole region. This rectangle has width c_i and height as $f(c_i)$.

Now, we get this cylindrical shell, whose height is $f(c_i)$ and width is $x_i - x_{i-1}$. We see that this volume which we want to compute is approximated by the sum of these cylindrical shells; that is the idea. We should remember that when the length of the sub-interval $x_i - x_{i-1}$ goes to 0, and since c_i is in between them, the sum of these shells should be giving us the volume. That is the idea of the procedure.

We should be able to compute the volume of each cylindrical shell. Look at the arrangement in step-wise fashion. Each shell is raised on a sub-interval of the type $[x_{i-1}, x_i]$. How do we implement this idea? How do we get this approximation? Of course, the volume of the cylindrical shell on $[x_{i-1}, x_i]$ has height equal to $f(c_i)$, which is this height, and it has the width as the length $x_i - x_{i-1}$. And then, how to get the volume of the cylindrical shell? It is really the circumference times this area. To get this circumference, you have to get this radius, which is computed from the line of the axis of revolution, which is the y-axis here. We compute this length from the y-axis. Then, that is the radius of the shell, the height is $f(c_i)$ and the width is $x_i - x_{i-1}$. That should give us the volume of the cylindrical shell.

(Refer Slide Time: 05:53)

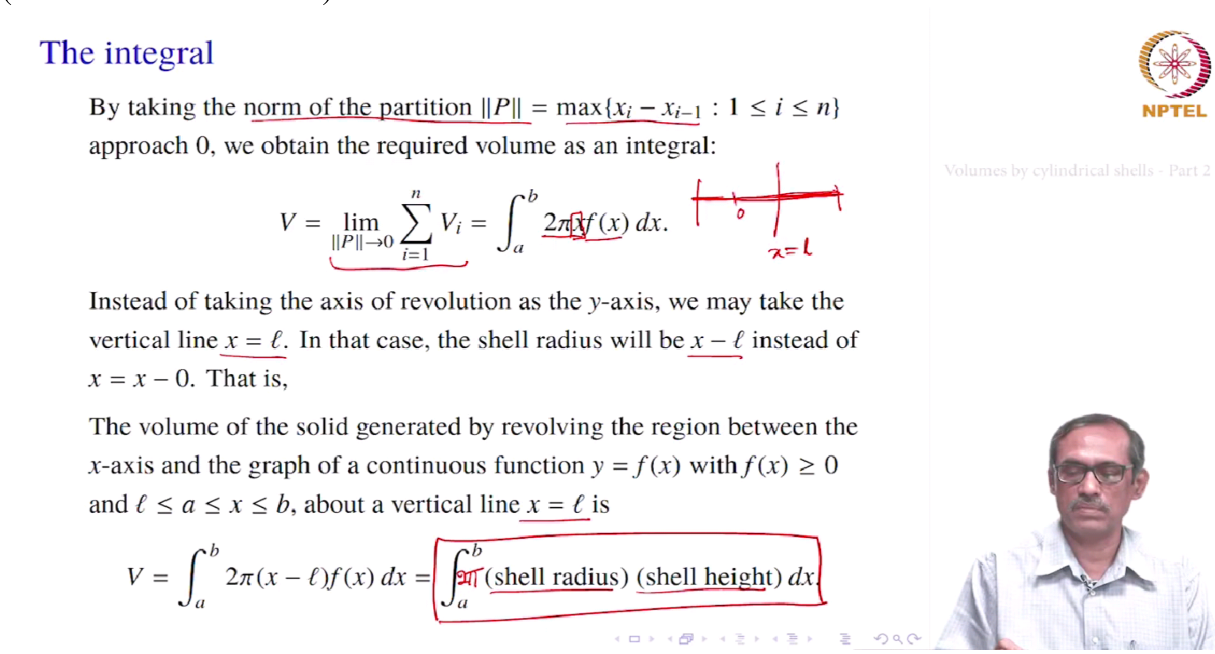


Let us go to the beginning. We divide the interval $[a, b]$ into n sub-intervals. That means we choose a partition with points as $a = x_0 < x_1 < x_2 < \cdots < x_n = b$. From each sub-interval $x_{i-1}, x_i]$ we choose a point c_i, just as in your definition of the definite integral. These points c_i make up the choice set C.

Then we see that the cylindrical shell has the height as $f(c_i)$, thickness $x_i - x_{i-1}$, and the radius of that cylindrical shell is c_i. Therefore, the circumference corresponding to that point c_i of the base of the cylinder is $2\pi c_i$. Then, the volume of the cylindrical shell will be $2\pi c_i f(c_i)(x_i - x_{i-1})$. What happens, you have taken c_i because the radius is really this one, which is on the x-axis, where the height is $f(c_i)$ and width is $x_i - x_{i-1}$. That gives the volume of the cylindrical shell as $2\pi c_i f(c_i)(x_i - x_{i-1})$.

Now, to implement the idea, we take the sum of all these volumes of cylindrical shells to get an approximation for the volume of the solid of revolution. So, the approximation is $\sum_{i=1}^{n} 2\pi c_i f(c_i)(x_i - x_{i-1})$.

(Refer Slide Time: 08:17)



Now, we use the limiting process to get the correct volume. As earlier, we take the norm of the partition as the maximum of $x_i - x_{i-1}$ and we take the limit of this sum of all these cylindrical shell volumes, where norm P goes to 0. You recognize that it is really a Riemann sum. The limit turns out to be the definite integral where the function is $2\pi x f(x)$. From this you get $2\pi c_i f(c_i)(x_i - x_{i-1})$, which is the corresponding Riemann sum. If the limit exists, then that limit must be equal to the integral $\int_a^b 2\pi x f(x)\, dx$. What are these quantities here? This $2\pi x$ is really the circumference, which gives $2\pi c_i$ at a choice point $x = c_i$. That means, this x is the radius of the shell, $f(x)$ is the height of the shell. So, it is the integral from a to b of the function 2π times the radius of the shell into the height of the shell.

This abstraction will really help, because sometimes we will not take the y-axis as the axis of revolution, but some other line. For instance, we may take the axis of revolution as the line $x = 1$, which is parallel to y-axis. In that case, the radius will be $x - 1$. How? This is x and the line is $x = 1$ here; then the radius will be $x - 1$ instead of x. If this line is on the other side, say, $x = -1$, then the radius should be $x - (-1) = x + 1$, which is the distance there; that should be the radius.

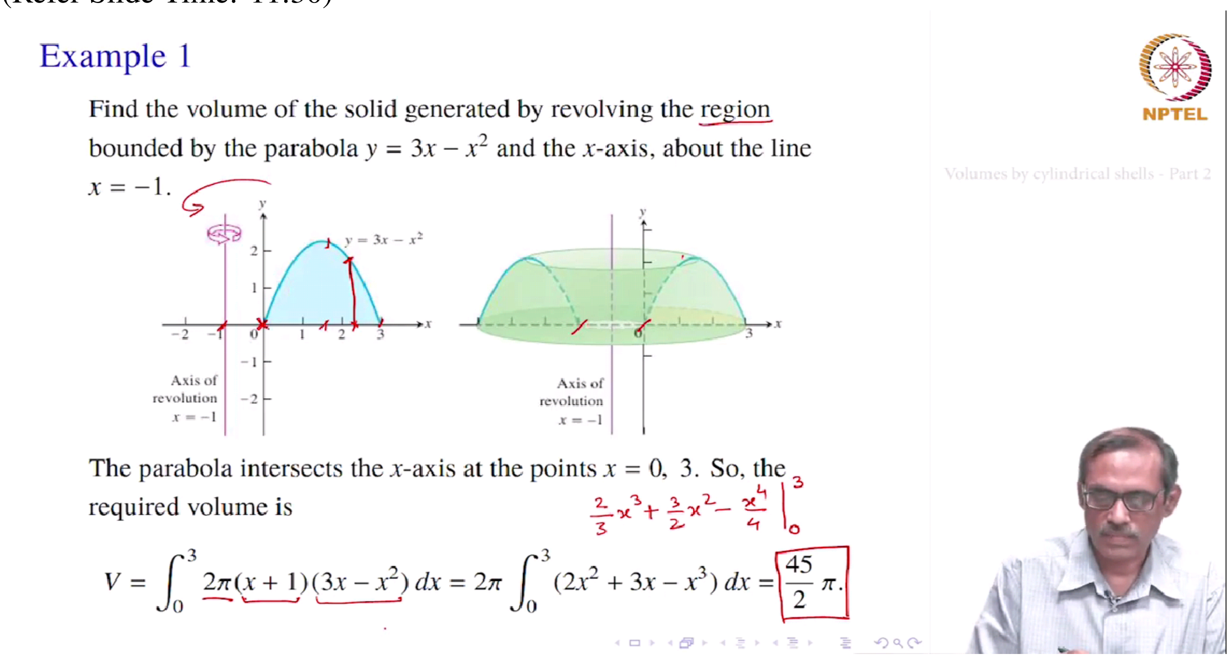
So, this heuristic will help us. Instead of taking the axis of revolution as the y-axis, if you take the vertical line $x = 1$ as the axis of revolution, then the radius of the shell will be $x - 1$. Then how do we compute the volume of the solid so generated?

This volume is obtained by revolving the region between the x-axis, the graph of the continuous function $y = f(x)$, where $f(x) \ge 0$ for all $x \in [a, b]$. As the axis of revolution is $x = 1$, we get the volume as $\int_a^b 2\pi(\text{shell radius}) \times (\text{shell height})\, dx$.

This should be the correct formula to memorize. Of course, understanding comes from the sum of the cylindrical shells, which gives rise to the integral as the result.

Let us take an example to see how our idea is executed. Here, we have a region; the region is bounded by the parabola $y = 3x - x^2$ and the x-axis. This is the parabola $y = 3x - x^2$ in blue. The parabola crosses the x-axis at 2 points $x = 0$ and $x = 3$. the points are, respectively, $(0, 0)$ ad $(3, 0)$. The region is bounded by the parabola and the x-axis. This is the region which is painted blue. This region is revolved about the line $x = -1$ to generate a solid. Th line $x = -1$ is colored pink. That is how you get the hole corresponding to this interval $[0, 1]$ in the solid.

(Refer Slide Time: 11:30)

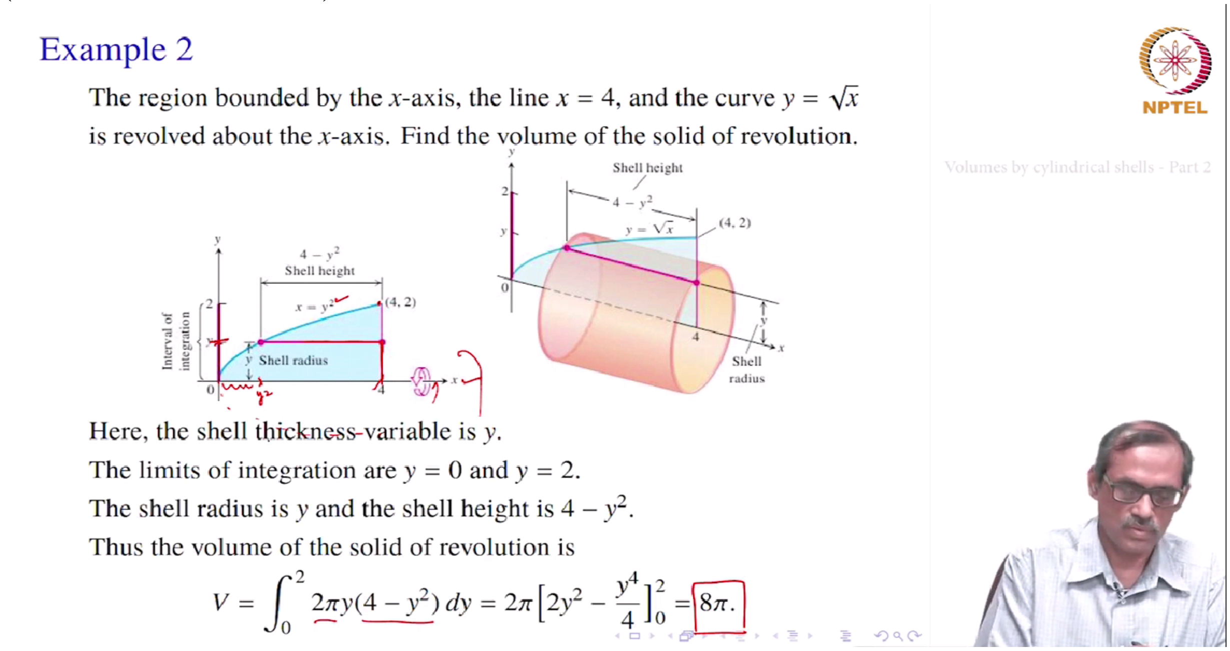


We could have used the washer method for computing this volume of the solid of revolution. But we want to see how it will be approximated or it will be computed by taking the cylindrical shell method.

This is the region which has revolved about $x = -1$. Since this is the way it has revolved, if you take any point c inside this interval, the radius of revolution will be $c + 1$. That is, the radius of revolution at any point x is $x + 1$. What is the shell height? It is obtained from the solid, or rather, from the curve; that is, $f(x)$ at any point x. Here, it is $3x - x^2$. This is the shell height. Then, what are the limit for x? Here, x varies from 0 to 3. So, we have the volume as the integral $\int_0^3 2\pi(x + 1)(3x - x^2)\, dx$. That is how we are looking at it. The integrand expands to $2\pi(2x^2 + 3x - x^3)$ When you integrate, you get $2\pi(2x^3/3 + 3x^2/2 - x^4/4)$. This expression is to be evaluated at 0 and 3, and then subtracted. At 0 it is 0, and at 3, it turns out to be $45\pi/2$. This is the answer.

At this point, you should take time and compute the same volume by washer method. See that by the washer method you also get the same answer.

(Refer Slide Time: 15:12)



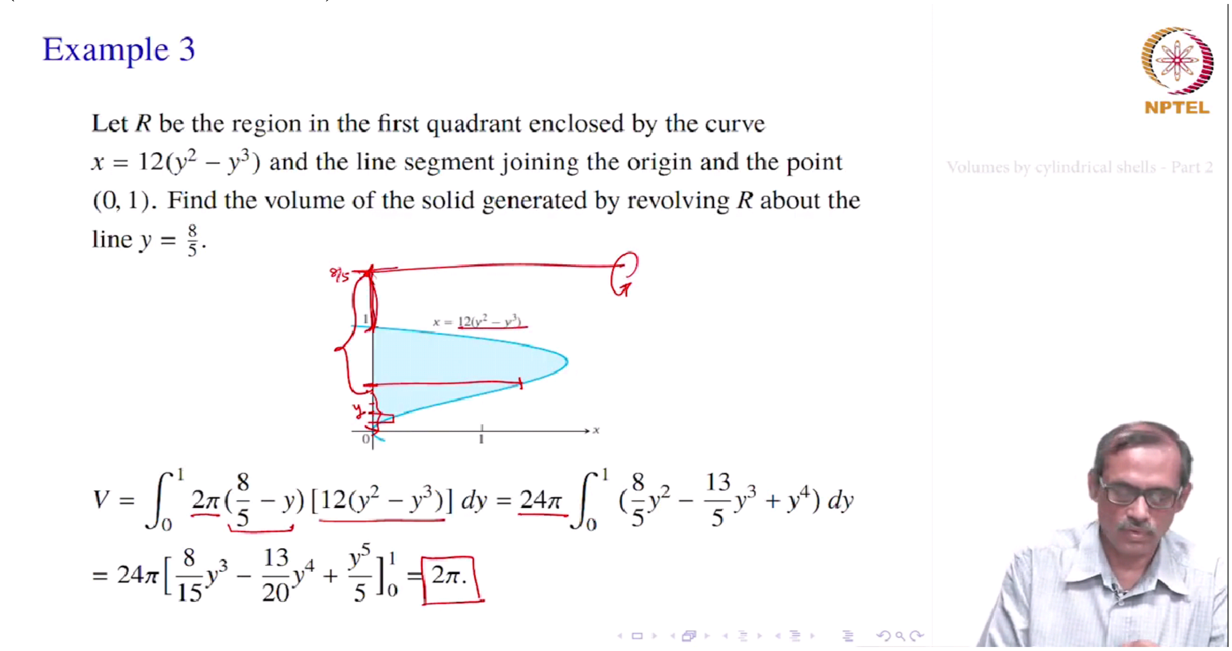
Let us take another example. In this example, we have the region bounded by the x-axis, the line $x = 4$, this is the line, and the curve $y = \sqrt{x}$. This is the curve $y = \sqrt{x}$; it is the parabola $x = y^2$ in the first quadrant, because in $y = \sqrt{x}$, y should be positive (non-negative) and there is nothing to the left. Also, we are considering the interval $[0, 4]$ for x; so it is automatically in the first quadrant. This region which is painted blue here is revolved about the x-axis. It is not about the y-axis; it is revolved about the x-axis now. This is the direction of revolution. We get the solid on the right side. We want to find the volume of this solid of revolution. How do we get this volume?

As earlier, we have to find the shell and the sum of those shells will give rise to the integral. It is revolving around the x-axis. So, we have to express everything in terms of y. Now, the curve is $x = y^2$. Since it is revolved this way, the solid is on this side. We take any point y. At that point the shell radius is this distance,which is y itself. So, y is the shell radius. What is the shell height? It is computed from this line. It is the length of this line that gives the shell height. We have the curve as $x = y^2$. At this point, this is y^2, and this distance is the total that is 4 minus y^2. So, the shell radius at y is $4 - y^2$. The shell radius is y which varies from 0 to 2.

Therefore, the volume can now be written as an integral directly. The integral is $\int_0^2 2\pi y(4 - y^2)\,dy$; this is the volume. Now, we compute this. The term 2π goes out, $4y$ has the integral as 22^2, $-y^3$ has the integral as $-y^4/4$. This is evaluated at 2 and at 0, and then subtracted essentially gives its value at 2. That turns out to be 8π.

Again, you should verify the result by using the Washer method. Here, it is not really the washer method; it is the general disk method. You may use the disk method because there is no hole in this. The x-axis is bordering the region so that the disk method is applicable. Verify by the disk method that the answer is really 8π.

(Refer Slide Time: 18:42)



Let us take another example. Here R is the region in the first quadrant that is already given. In the first quadrant, the region R is enclosed by the curve $x = 12(y^2 - y^3)$ and the line segment joining the origin and the point $(0, 1)$. The point $(0, 1)$ is on the y-axis, so, the line segment lies on the y-axis. This is the curve $x = 12(y^2 - y^3)$. Thus, this is the region R, which is painted blue here. We want to find the volume of the solid generated by revolving this region R about the line $y = 8/5$. The line $y = 8/5$ stays somewhere here, about which R is revolved. Then, there will be a hole in the solid corresponding to the line segment from $y = 1$ to $y = 8/5$. We can then use the washer method. But we want to use this new method of cylindrical shells.

The cylindrical shells will be described by taking small partitions here, something like this. The shells will be generated like this, and they need to be approximated. We require the shell radius. For that take any point y. We find that this is the center of that shell. Measuring from there, we get the shell radius as this distance. If this is y, then we subtract that from the total of $8/5$ to get the shell radius as $8/5 - y$. What is the shell height? It is that y, the point on the curve, and the corresponding $x = 12(y^2 - y^3)$ is really the shell height. And where does y vary? It varies from 0 to 1.

Therefore, the volume of the solid is equal to the integral $\int_0^1 2\pi(8/5 - y)12(y^2 - y^3)\,dy$. That is the integral we get for the volume of the solid of revolution. Now, it is just a matter of integrating it. Let us integrate; it is 12 and there is a 2; we take 24π outside th integral. We multiply the factors to get $(8/5)y^2 - (13/5)y^3 + y^4$. On integration, y^2 gives $y^3/3$, y^3 gives $y^4/4$ and y^4 gives $y^5/5$. So, we get $24\pi\left[(8/15)y^3 - (13/20)y^4 + y^5/5\right]$. At 0 its value is 0. So, essentially, the answer will be this expression evaluated at 1. And that is $24\pi\left[(8/15) - (13/20) + 1/5\right]$. It turns out to be 2π.

Again, it is an exercise for you to verify that you obtain the same answer when you use the washer method. In the washer method there is a hole in the solid corresponding to $y = 1$ to $y = 8/5$. So, the washers will be generated this way and then summed over to get the volume.