Basic Calculus - 1

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Lecture 28 - Part 2

Applications of Fundamental Theorem of Calculus - Part 2

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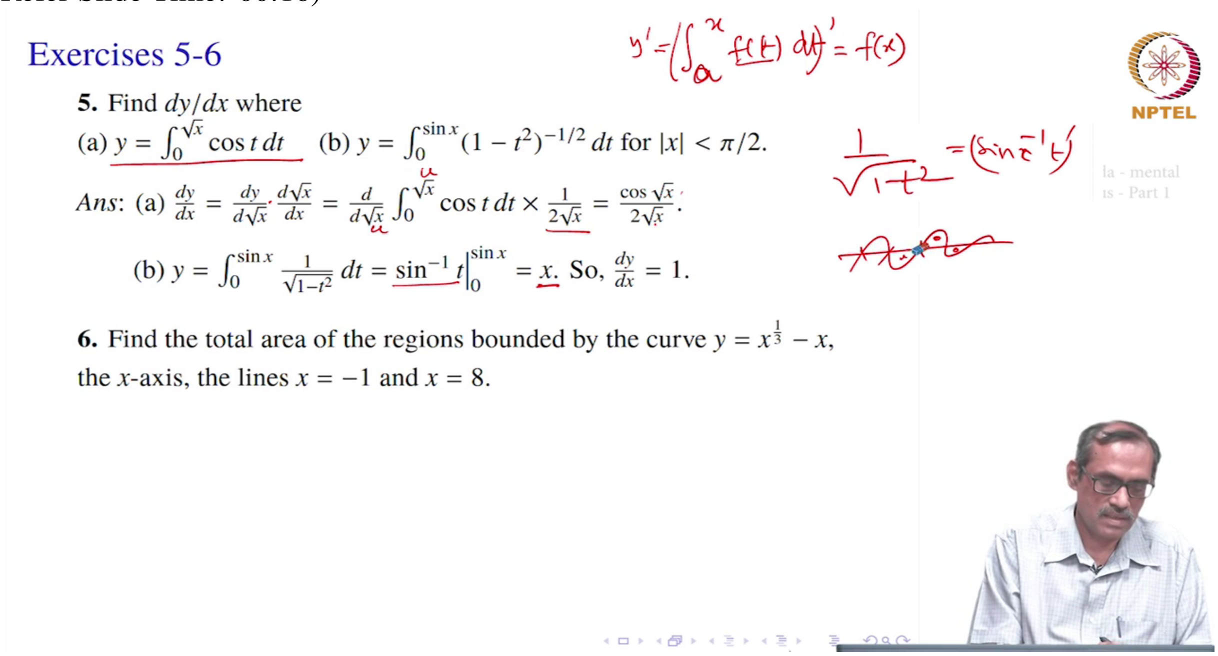

Let us take another problem. Find dy/dx where y is given as the integral $\int_0^{\sqrt{x}} \cos t\, dt$. There are two problems. The second one is the integral from 0 to $\sin x$ of a similar function. We will com back to this after the first one.

So, y is given as $\int_0^{\sqrt{x}} \cos t\, dt$. Our fundamental theorem says that if y is an integral from a to or x of $f(t)$, $y' = dy/dx$ will be equal to its $f(x)$, provided this function is continuous, and so on.

Now, what do we do? When the integral is from a to x, we cantake its derivative; it is straight forward. But here, the upper limit is not x; it is $\sqrt{x}$. The lower limit a can of course be any constant; here, we can take it to be 0. If you take $dy/d\sqrt{x}$, then the result will be $\cos\sqrt{x}$. So, we use the chain rule.

Now, dy/dx is $dy/d\sqrt{x}$ times $d\sqrt{x}/dx$, which is equal to $d/d\sqrt{x}$ times $(1/2)x^{-1/2} = 1/(2\sqrt{x})$. Taking $u = \sqrt{x}$, we would get $dy/dx = dy/du$ times du/dx, where $dy/du = \cos u$ as $y = \int_0^u \cos t\, dt$. Therefore, the answer is $\cos(\sqrt{x})/(2\sqrt{x})$.

Similarly we will try Part (b). Here, instead of $\sqrt{x}$ we have $\sin x$. Of course, we can do it an easier; because we find that $1/\sqrt{1-t^2}$ is the derivative of $\sin^{-1} t$. So, we keep integral 0 to *sinx* as earlier instead of applying straight forward the earlier technique as in Part (a). Now, we can integrate it. The term $1/\sqrt{1-t^2}$ gives $\sin^{-1} t$. So, $\sin^{-1} t$ is evaluated at $\sin x$ and at 0; and then subtracted to give $\sin^{-1}(\sin x) - \sin^{-1}(0) = x$.

Remember our inverse functions? This notation gives only the principal value. If you do not get the principal value, then you may have to add some multiples of π. Here, it is $\sin^{-1}(\sin x)$, where x is the principal value. So, we get x back, and at 0 it is $\sin^{-1} 0$, which is 0. Now, you see if you have you taken some other value of $\sin^{-1}$ instead of the principal part, there can be a difference of multiples of π. Anyway, our notation is only for the principal part. So, $\sin^{-1}$ is that number whose sin is equal to this. That means, whichever really remains inside the interval $(-\pi/2, \pi/2]$; that was our definition.

Hence, we get $y = x$. Therefore, $dy/dx = 1$. You can do it the other way also. But that will be slightly complicated here, because you have to write $d(\sin x)/dx$ and then this will be 1 divided by $\sqrt{1 - \sin^2 x}$. That gives you $1/\cos x$; the $\cos x$ cancels, and the answer is 1.

You may wish to see it another way. Write it as $d \sin x/dx$ times the derivative of this with respect to $\sin x$. That will give you $f(\sin x)$; it is $1/\sqrt{1 - \sin^2 x}$. The first one gives $\cos x$ and this gives $1/\cos x$, and you get 1, using the technique used in Part (a).

Let us go to the sixth problem. We want to find the total area of the regions bounded by the curve $y = z^{1/3} - x$, the x-axis, the lines $x = -1$ and $x = 8$.

It is written 'regions'; there might be multiple regions of which total area we want to find. However, there can be multiple regions or maybe only one region, we do not know. Then it will be something this way. We are not plotting it exactly but only conceptually. There can be many points where it will cross the x-axis. Then you have to find out these areas and add them up; that is being asked. So, let us find first where does it cross the x-axis.

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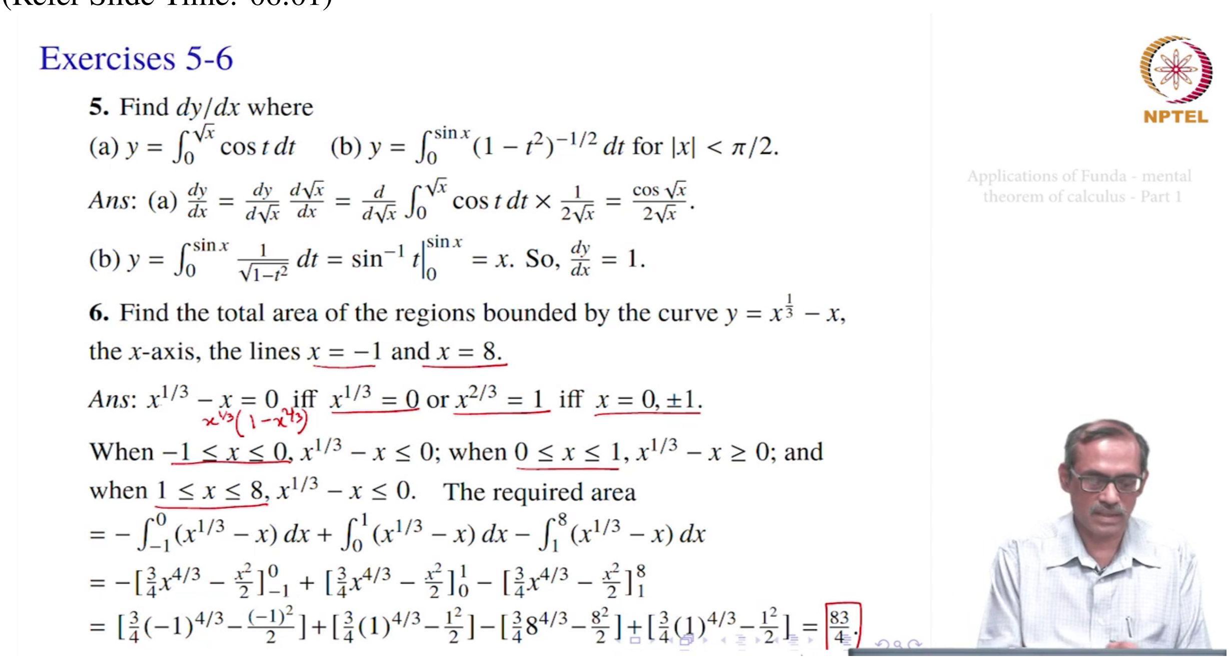

At those points, $x^{1/3} - x$ must be equal to 0. When you take this curve it crosses the x-axis means that the curve y equal to this function must be equal to 0. Now, $x^{1/3} - x = 0$ implies $x^{1/3}(1 - x^{2/3} = 0$. So, either $x^{1/3} = 0$ or $x^{2/3} = 1$. This gives you three points such as -1, 0, and 1. Of course, if you take $x = -1$ here, that gives you $(-1)^{1/3} = -1$ so that $(-1)^{2/3} = 1$. So, $x = -1$ is also possible. Hence, there can be three such points.

That means we have to break the integral into two integrals, one from -1 to 0 and another from 0 to 1. Let us consider those intervals $[-1, 0]$ and $[0, 1]$. In $[-1, 0]$, the function $x^{1/3} - x \leq 0$, since in $(-1, 0)$, $x^{1/3} < x$. In $[0, 1]$, $x^{1/3} - x \geq 0$. We have the lines $x = -1$ and $x = 8$. So, we have another interval $[1, 8]$. In $[1, 8]$ the function $x^{1/3} - x \leq 0$.

You want to find the total area. Since nothing about sign is mentioned, we take the actual area, and not the signed area. The actual area is the sum of all those actual areas. Since $x^{1/3} - x \leq 0$ in $[-1, 0]$, $|x^{1/3} - x| = -(x^{1/3} - x)$. That means, we should put a minus sign before the integral to get the actual area. In $[0, 1]$, $x^{1/3} - x \geq 0$, so its mod is itself. In $[1.8]$, again $x^{1/3} - x \leq 0$; so its mod is $-(x^{1/3} - x)$. So, we put a minus sign before this integral on $[1, 8]$. So, the required area is

$$-\int_{-1}^{0} (x^{1/3} - x)\, dx + \int_{0}^{1} (x^{1/3} - x)\, dx - \int_{1}^{8} (x^{1/3} - x)\, dx.$$

Now, if you differentiate $(3/4)x^{4/3}$, you get $x^{1/3}$ and if you differentiate $x^2/2$, you get x. Hence the indefinite integral of $x^{1/3} - x$ is $(3/4)x^{4/3} - x^2/2$. Then, we evaluate this at -1, 0, 1 and 8, and then add or subtract as necessary. We substitute back all these expressions and simplify. That would give us $83/4$. You can verify it later how this simplifies to $83/4$.

So, we may have to really consider breaking into sub-intervals. We have not plotted it because, in general, you will not be able to plot the function easily, but still your analysis should be able to point out all the details.

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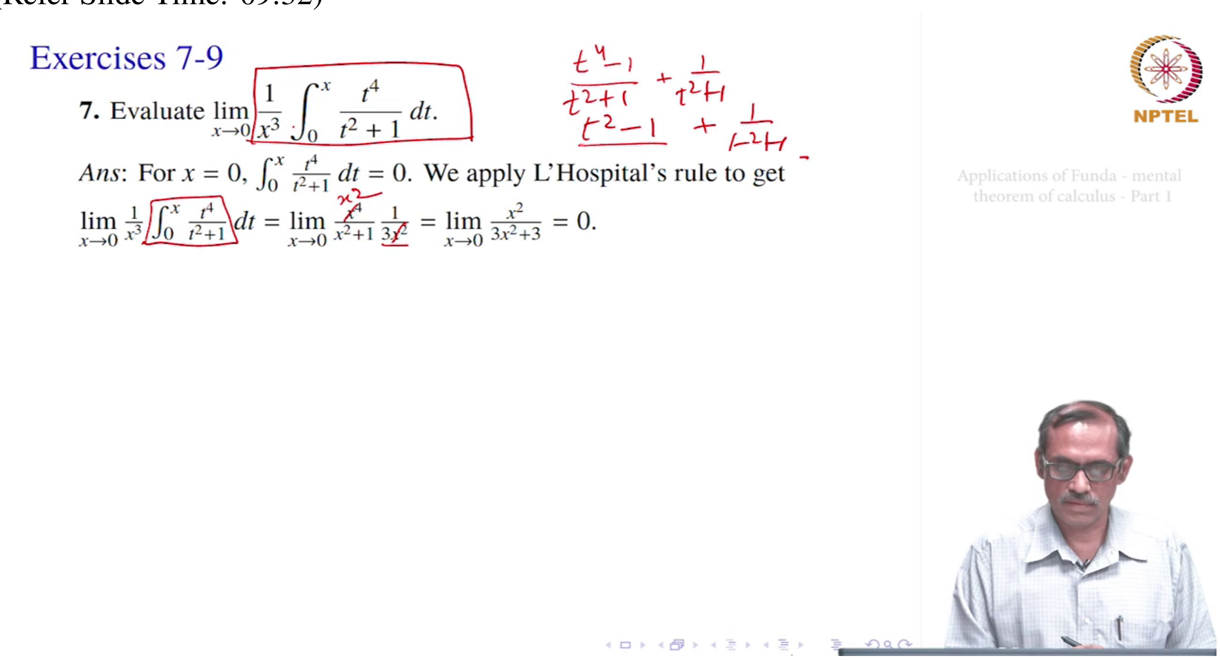

Let us find out what happens in this problem. Here, we want to evaluate a limit as x goes to 0 of the function $(1/x^3)$ times an integral. Of course you can integrate this, its integrand is $t^4/(t^2 + 1)$. you can take write it as $(t^4 - 1)/(t^2 + 1)$ plus $1/(t^2 + 1)$. The first is $t^2 - 1$ and the the second one is kept as it is. The integral of the first one gives $t^3/3 - t$ and of the second one is $\tan^{-1} t$. So, you can really integrate, substitute x and then go for the limits. But that will be a bit lengthy.

We rather look at this expression. As x goes to 0, its denominator x^3 goes to 0. Now, the limit does not exist if the numerator does not go to 0 when x goes to 0. But what is the numerator? It is $f(x) = \int_0^x t^4/(t^2+1)\,dt$. Since the integral $f(x)$ is a continuous function, as x goes to 0, the limit of this will be $f(0)$. But $f(0)$ is equal to $\int_0^0 t^4/(t^2+1)\,dt$. Hence, it is 0. As the integral becomes 0 when $x = 0$, that is limit of the numerator, which is 0.

We know that the limit of the denominator is also 0; so it is $0/0$ form. So, you can use L'Hospital's rule. Provided this right side limit exists, this is equal to the limit of the derivative of this function divided by the derivative of x^3. The derivative of x^3 is $3x^2$ and the derivative of the integral, by the fundamental theorem, is $x^4/(x^2+1)$. It is the limit of $x^4/3(x^2+1)x^2$ as $x \to 0$. Now, x^2 cancels, and it is $x^2/3(x^2+1)$. As $x \to 0$, its limit is 0.

So, there can be many ways; we have to really find out which one is simpler. Sometimes the other method can be simpler, where we evaluate the integral and then do it. Let us see the next problem.

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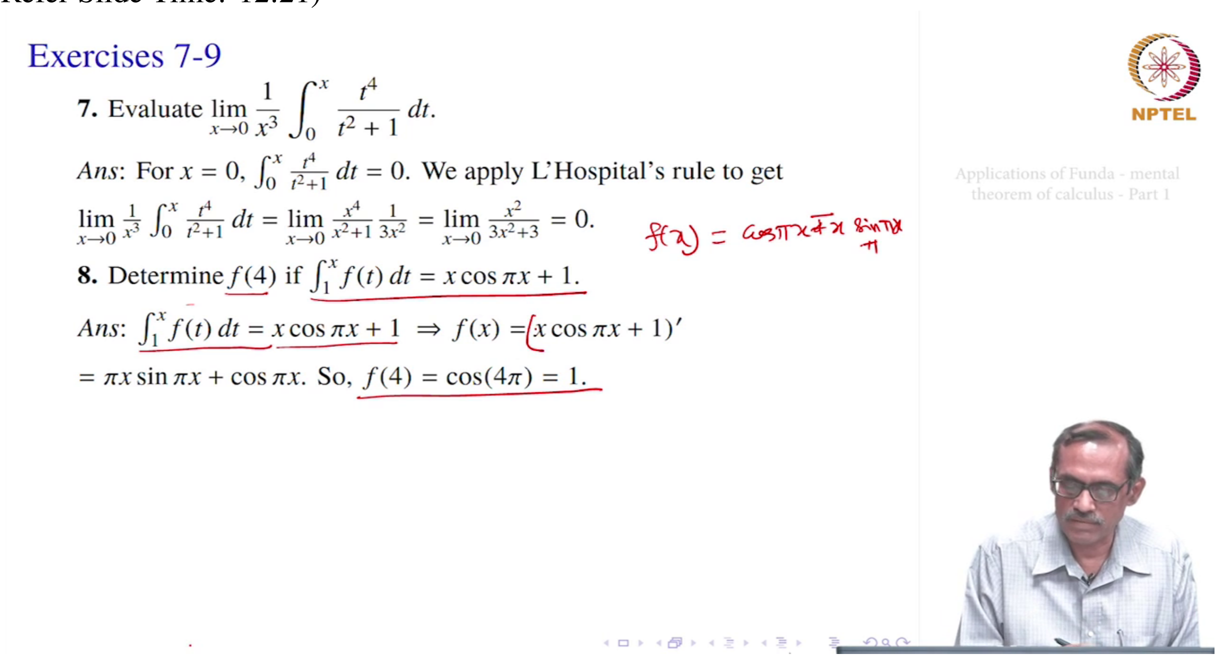

Here, we want to determine $f(4)$ given that $\int_1^x f(t)\,dt = x\cos(\pi x) + 1$. We have no idea on f, but we are given that integral of f is something. So what do we do?

We can find out $f(x)$ here directly by differentiating this. If you differentiate the left side, you would get $f(x)$ by the fundamental theorem. If you differentiate the right side, you get the derivative of 1 as 0, the derivative of $x\cos(\pi x)$ as $1 \times \cos(\pi x) + x \times (-\sin(\pi x) \times \pi$. So, that will be $f(x)$. Then you can substitute $x = 4$ to get $f(4)$. That is what we do to find that $f(4) = 1$.

The first technique of direct integration will fail here since we do not know what $f(t)$ is. So, we use differentiation of the integral. This way, we see that the derivative of the integral, which is $f(x)$ must be equal to the derivative of $x\cos(\pi x) + 1$.

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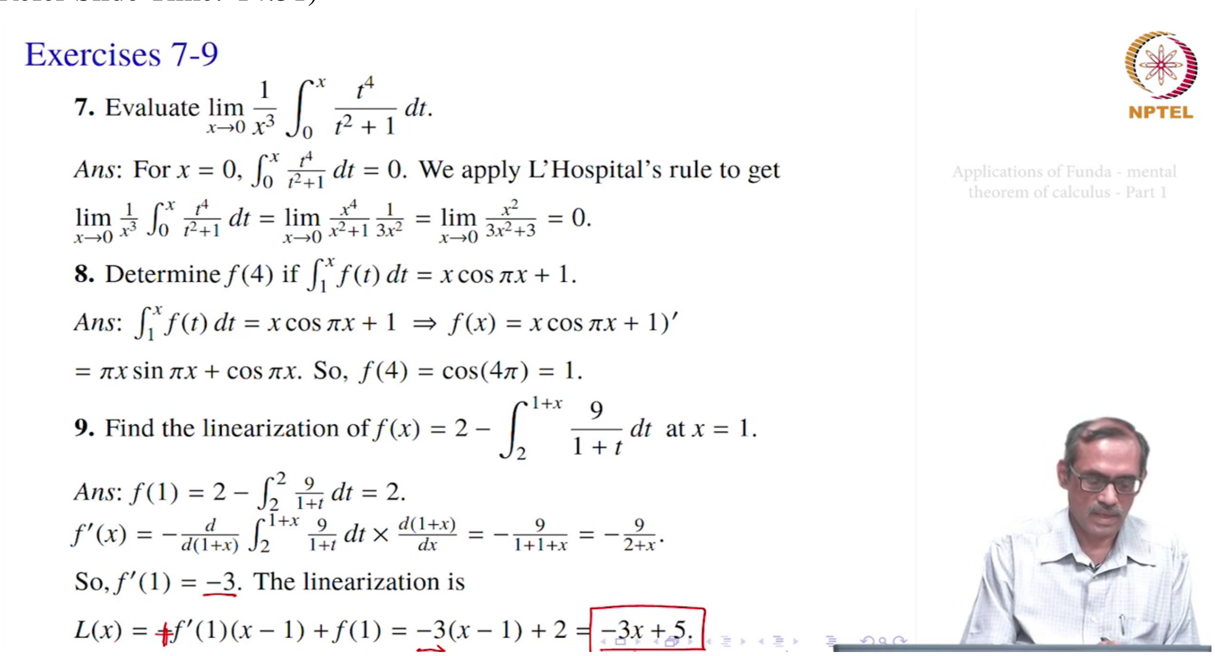

Here in the next problem, we want to find the linearization of a function. Recall that the linearization of a function $f(x)$ at $x = a$ is written as $L(x)$, where $L(x) = f(a) + f'(a)(x - a)$. That was our linearization. In this case, $f(x)$ is given in terms of an integral. It is $f(x) = 2 - \int_2^{1+x} 9/(1+t)\, dt$. And we want to find the linearization of this at $x = 1$.

That means we need $f(a)$, $f'(a)$ where $a = 1$. It may be easy to get $f(1)$ by substitution, and $f'(1)$ by differentiating this integral with the help of the fundamental theorem of calculus. It is pretty straightforward. First thing is, $f(1) = 2 - \int_2^{1+1} 9/(1+t)\, dt = 2$ since the integral is from 2 to 2 and gives the value 0. Then we will take $f'(x)$. Here the variable is $1 + x$ as the top limit in the integral. All that we know is if the integral is from a to x, then its derivative with respect to x is equal to $f(x)$. Here, it is not x but $1 + x$. So, we should find the derivative of the integral with respect to $1 + x$ first. That is not a big thing; it will be same because derivative of $1 + x$ with respect to x is 1, and you use the chain rule. Now, the derivative of 2 is 0, with respect to x and the derivative of this integral is the derivative with respect to $1 + x$ times the derivative of $1 + x$ with respect to x. Since the top limit is $1 + x$, the derivative of the integral with respect to $1 + x$ is $f(1 + x)$, which is that is $9/(1 + 1 + x)$. So, that is what the answer is: $f'(x) = -9/(2 + x)$. Hence, $f'(1) = -9/3 = -3$.

Then, the lineariztion of $f(x)$ at $x = 1$ is $L(x) = f(1) + f'(1)(x - 1) =$. As $f(1) = 2$ and $f'(1) = -3$, we get $L(x) = 2 - 3(x - 1)$. This simplifies to $2 - 3x + 3$ or $5 - 3x$. That is the linearization of the given function at $x = 1$. Let us stop here.