Basic Calculus - 1

Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras

Lecture 25 - Part 2

Definite integral - Part 2

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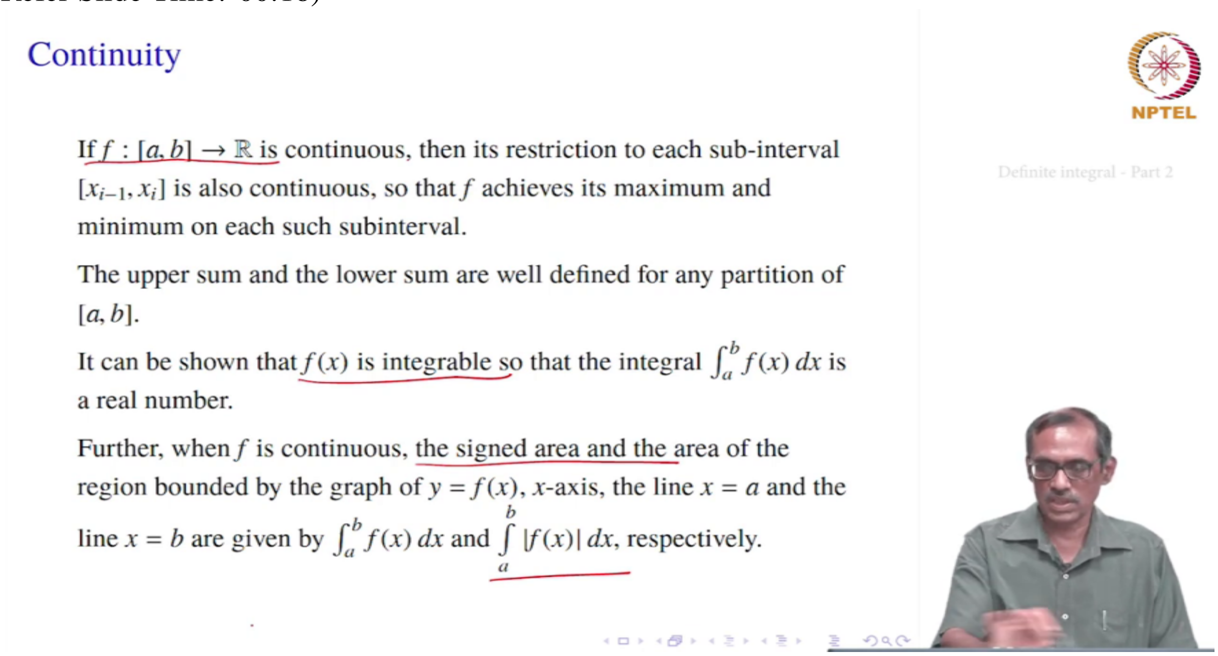

And exactly using the same technique an important theorem is proved. We will not prove the theorem here, but give its background. It goes as follows. Suppose f is a continuous function from the closed interval $[a, b]$ to $\mathbb{R}$. Automatically f is bounded; we know that. Its restriction to any sub-interval is also continuous. So, f achieves its maximum and minimum inside every sub-interval. Then the upper sum and the lower sum are well defined because maximum and minimum exist in any sub-interval of $[a, b]$.

The theorem states that if $f : [a, b] \to \mathbb{R}$ is continuous, then it is integrable, that is, the integral is a real number. The main result uses our notions of upper and lower sums. It can be shown that when f is continuous, the limit of $\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}$ is equal to 0 as the norm of the partition goes to 0. This can be proved by using continuity of the function f on the closed interval $[a, b]$. It then follows that when $f(x)$ is continuous on $[a, b]$, it is integrable. We accept this result.

Further, we are really getting the signed area in that case. And what is that signed area? Area of which region? It is the region bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. The integral is the signed area of this region, When you compute the proper area, the geometric area, that will be $\int_a^b |f(x)|\,dx$. That also should exist, because you can consider the sign of the function in parts of the region and then after taking the absolute values of the function, add the respective areas. This is the basic idea of the area and integrability.

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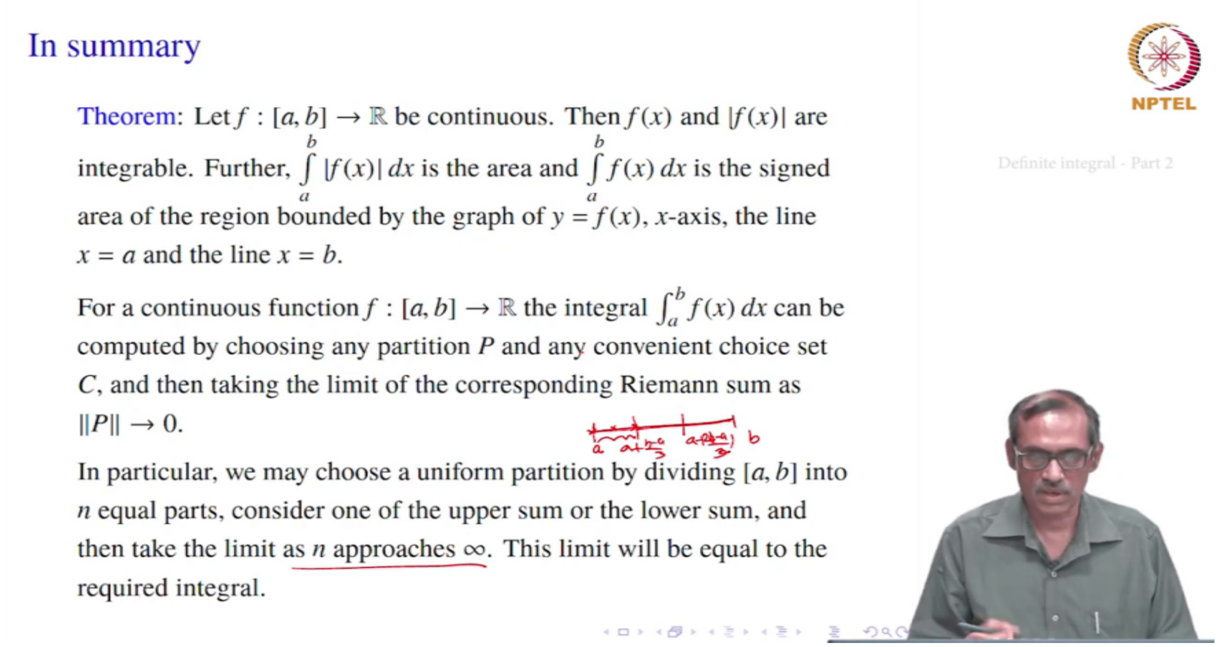

We will see what does it mean. We can now state our result. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, both $f(x)$ and $|f(x)|$ are integrable. The integral $\int_a^b |f(x)|\, dx$ is the area and the integral $\int_a^b f(x)\, dx$ is the signed area of the region bounded by the graph $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. That is what we have discussed.

We will be looking at some problems basing on this. But wait a bit. how are we going to compute this area? You can always compute this area by choosing a suitable partition P. First we choose any partition P and then once you know that f is continuous, it is integrable; so, you choose any choice point c_i in between that x_{i-1} and x_i. The theorem says that in the limit you get the same value for all possible choices of c_i. That means, we can take any partition, we choose any convenient point c_i inside each sub-interval, take the Riemann sum, and then take the limit as norm of P goes to 0, that is, as each $(x_i - x_{i-1}$ goes to 0. This is what we do for computing the integral.

You may also choose as you like. As a particular case, you can choose your partition to be a uniform partition; that is, you can divide $[a, b]$ into n equal parts; so, each part is having the length as $(b-1)/n$. Suppose you want to divide $[a, b]$ into 3 parts. Then, you set the length as $(b-1)/3$ The partition points x_i are a, $a + (b-1)/3$, $a + 2(b-a)/3$ and b.

Next, you can choose any set of choice points. May be the left endpoint x_{i-1} of each sub-interval can be a choice point c_i; or, may be the right endpoints, that is, $c_i = x_i$. We have to make that choice. We can choose anything that is convenient. We can do that provided the function is integrable. In that case, the limit of the Riemann sum will be same for all such choices.

If you divide $[a, b]$ into n parts, then the norm of the partition goes to 0 will means that $(b-1)/n$ must go to 0. That is, n should go to infinity. That means if you take this Riemann sum after this choice and you take n approach ∞, then the limit will be equal to the required integral. This is a particular case of that. Sometimes we choose like this if it is easier, or sometimes we choose other c_i which may be convenient for us for computing the integral.

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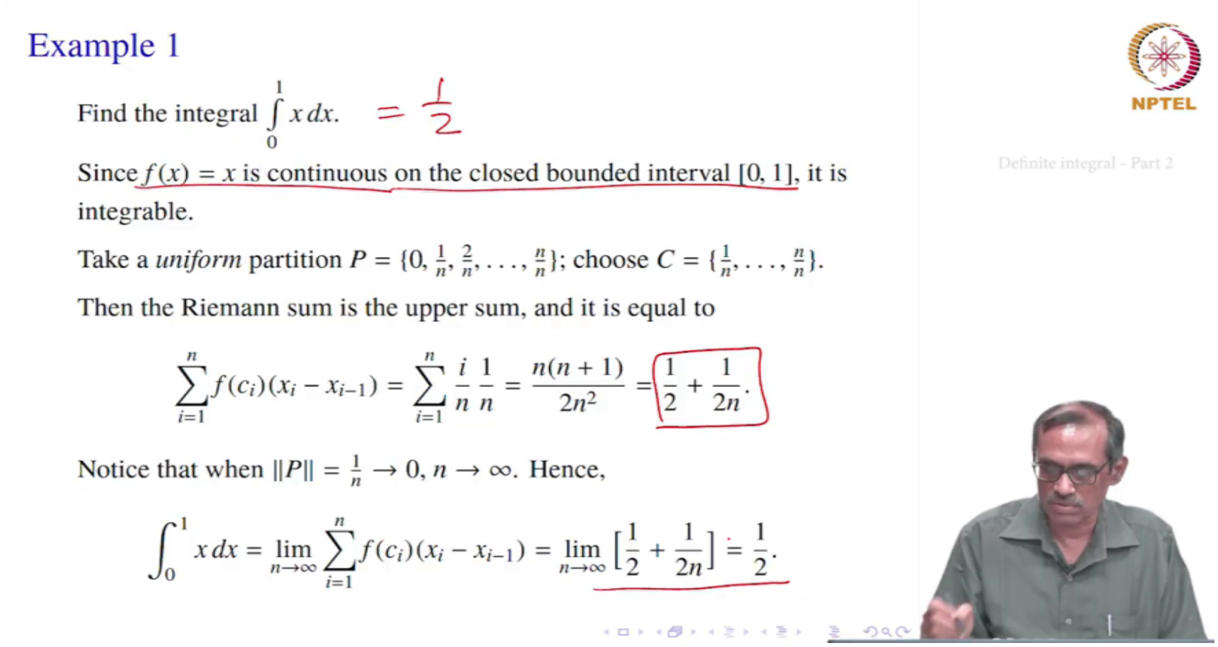

Let us see how to compute the integral $\int_0^1 f(x)\,dx$. How do we proceed? First of all we know that integration is possible; the integral exists. Reason: $f(x) = x$ is is continuous on the closed bounded interval $[0, a]$. We apply our theorem to conclude that it is integrable. Then, we can choose any partition. Let us choose a uniform partition $P = \{0,\ 1/n,\ 2/n,\ \ldots, (n-1)/n,\ n/n\}$. There are $n + 1$ points since you have n number of sub-intervals. Let us chose c_i to be the right end point. Of course, you could have chosen the left endpoint; it does not matter.

Suppose I choose the right endpoints. Then, the Riemann sum is exactly the upper sum because $f(x) = x$ is an increasing function; so the right endpoint will give the maximum value of the function in the sub-interval x_{i-1} to x_i. That is really the upper sum. It is the sum of $f(c_i)(x_i - x_{i-1})$. Now, what is $f(c_i)$? c_i is i/n; so, $f(x) = x$ gives $f(c_i) = c_i = i/n$. And, the length of the sub-interval is $1/n$. Now you see, the Riemann sum is $\sum_{i=1}^{n}(i/n^2)$. It is $(1/n^2)$ into $\sum_{i=1}^{n} i$. This sum is $n(n+1)/2$. Hence the Riemann sum is $n(n+1)/(2n^2) = (n+1)/(2n) = 1/2+1/(2n)$. When n approaches ∞, that should give us the integral; because in that case our norm of the partition which is $1/n$ goes to 0. So, we take the limit of this expression $(1/2) + 1/(2n)$ as $n \to \infty$. That means this integral is equal to $1/2$. That is what we have computed by using the definition and choosing one uniform partition.

But most important is this sentence: "$f(x)$ is continuous on the closed bounded interval $[0, 1]$; therefore, it is integrable". So, we can choose any particular type of partition, like a uniform partition or not uniform, we can choose any particular choice set C, and then take the limit to arrive at the answer. Of course, if you choose different c_i's it should give the same answer.

Let us take another example. Suppose we have the constant function $f(x) = k$ where k is some real number. Then we want to see that this integral will be equal to $k(b - a)$. Intuitively it is very easy to see. But we want to verify whether our intuition is really going with our formal definition or not. So, the function f defined on $[a, b]$ is given by $f(x) = k$. It looks something like this; it is

k throughout. When you take the area, it should be this rectangle, whose area is $k(b-a)$. That is why we show that this is indeed so by using our formal definition.

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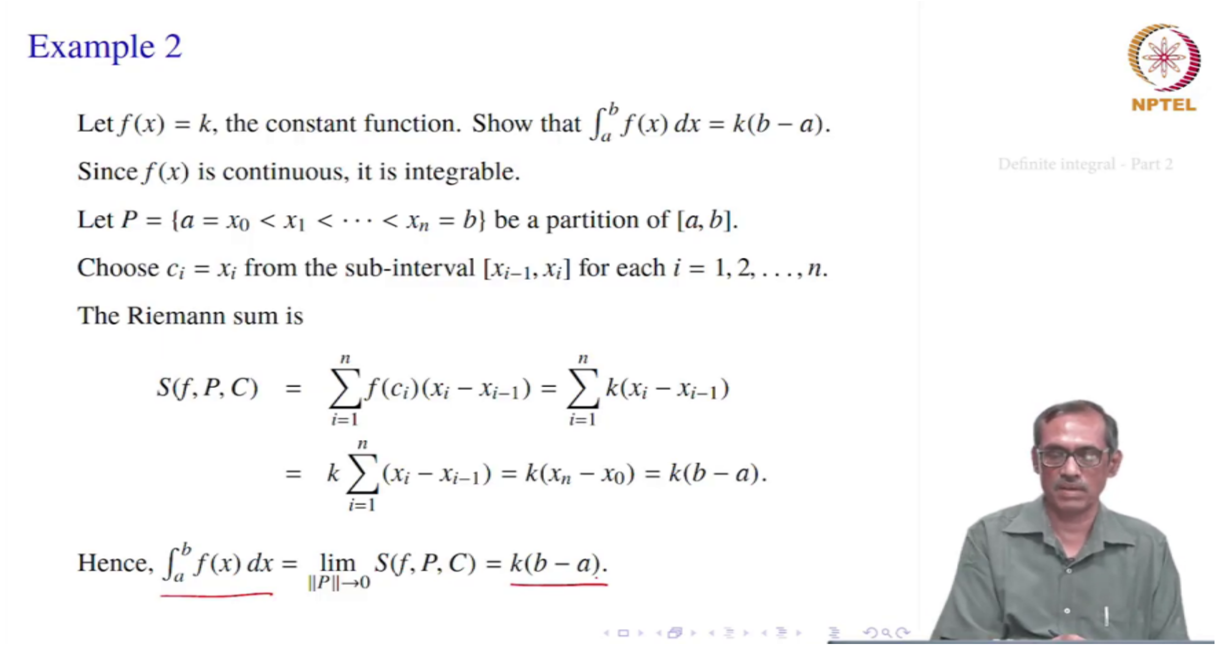

So, what do we do? First we observe that it is continuous. This function is continuous on the closed interval $[a, b]$; so it is integrable. Therefore we can choose any partition. Let us take the partition $P = \{x_0, x_1, \ldots, x_n\}$ in the abstract. We are not giving exactly what are the values of these x_i. You will see that it will not matter. Then we choose $c_i = x_i$, the right endpoint. You can choose left endpoint or even any point between x_{i-1} and x_i, because the function is constant anyway. In each sub interval $[x_{i-1}, x_i]$, we take $c_i = x_i$.

Now, the Riemann sum with respect to this is $\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1} = \sum_{i=1}^{n} k(x_i - x_{i-1})$. But this is easily summable. It is something like if you expand the summation, when $i = 1$, it looks like

$$k(x_1 - x_0) + k(x_2 - x_1) + \cdots + k(x_n - x_{n-1}).$$

You see that this there is x_1 and also $-x_1$ and so on. These terms get canceled and what remains is $kx_n - kx_0$; everything else cancels away. And, that is equal to $k(x_n - x_0) = k(b-a)$.

Therefore, when you take the limit of this as the norm of P goes to 0, you will be get the same thing $k(b-a)$ because it is a constant. That is simple. It is a verification of our definition for this easier case.

So, let us take one which is not very easy. Evaluate the integral $\int_a^b \cos x\,dx$. The first thing is, this integral exists because $\cos x$ is continuous on the closed interval $[a, b]$. It is integrable according to our theorem. Now we choose any partition P, which is : $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. Let that be the partition. We have to choose some c_i so that we can compute the limit. Here, we will choose our c_i in a very specific way. The answer should be same because the function is integrable. We will choose in a particular way and see later why we have chosen it this way.

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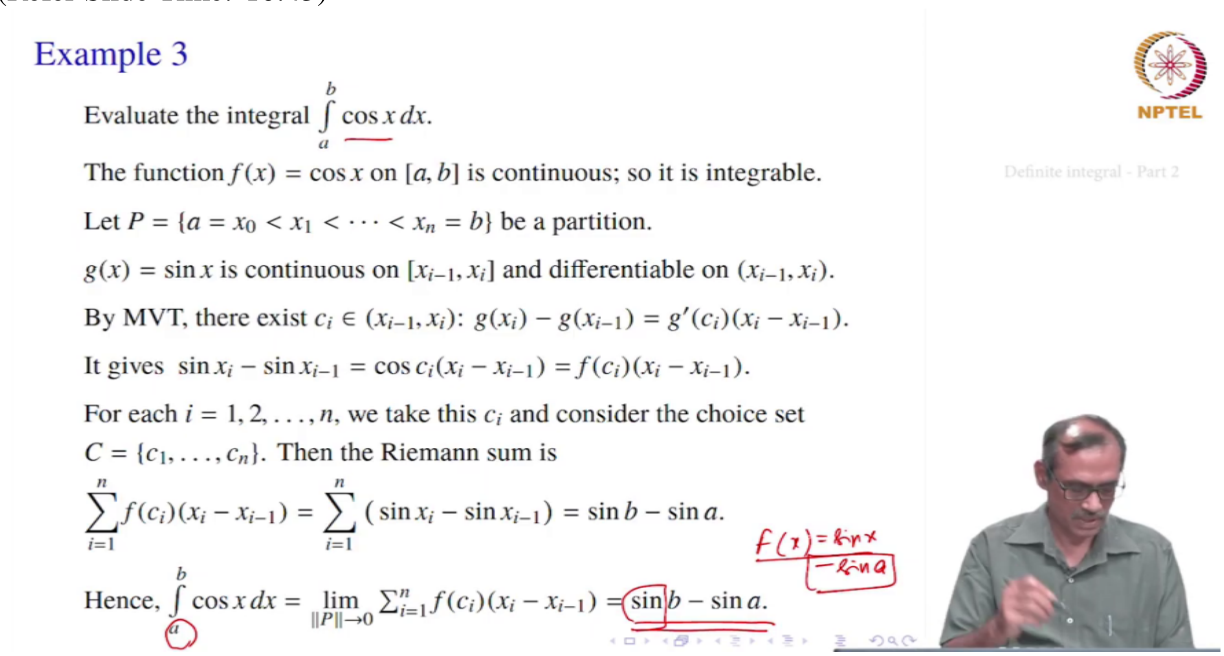

Look at the function $g(x) = \sin x$. This is just a trick to get that Riemann sum evaluated. Otherwise, you may have to refer to difficult limits. Now, look at the function $g(x) = \sin x$. This is continuous on each of the sub-intervals $[x_{i-1}, x_i]$; and also differentiable on the open interval (x_{i-1}, x_i). Therefore, we can use the Mean Value Theorem. By the Mean Value Theorem, there exists a point c_i in between x_{i-1} and x_i such that $g(x_i) - g(x_{i-1}) = g'(c_i)$. Is that clear? That is the Mean Value Theorem. Since $g(x) = \sin x$, $g'(x) = \cos x$. So, we have $\sin(x_i) - \sin(x_{i-1}) = \cos c_i$. Now what is $\cos c_i$? Our $f(x) = \cos x$. So, $\cos c_i = f(c_i)$.

This is the advantage in taking this c_i. We need to take the Riemann sum. So, we choose our point c_i as these c_is. Fine? With this choice set $C = \{c_1, c_2, \ldots, c_n\}$, the Riemann sum, which is $\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$ is equal to $\sum_{i=1}^{n} \cos(c_i)\,(x_i - x_{i-1})$ and that is equal to $\sum_{i=1}^{n} [\sin(x_i) - \sin(x_{i-1})]$. This sum is same as $\sin(x_1) - \sin(x_0) + \sin(x_2) - \sin(x_1) + \cdots + \sin(x_n) - \sin(x_{n-1})$.

Many things cancel out; only the first and the last remain so you get $\sin(x_n) - \sin(x_0)$ which is equal to $\sin b - \sin a$. Once you are able to evaluate the sum, you get $\sin b - \sin a$, which is a number. When you take the limit of the Riemann su,m that will be equal to this number itself. So, $\int_a^b \cos x\, dx = \sin b - \sin a$.

You see, our question was which function $F(x)$ it is whose derivative will be equal to this $f(x)$? Now, you see that if $f(x) = \cos x$, then the function $F(x)$ should be $\sin x$; that is what it says. Specifically, in this integral if b is a variable, a variable point, then the result will be $\sin x - \sin a$. That is our $F(x)$, which we can take to be $\sin x - \sin a$. It will really solve our purpose, but up to a constant; here this constant $-\sin a$ comes from the lower limit a.

It is going nicely because our aim was to find such a function $F(x)$ whose derivative is this. And by defining the area we are reaching this integral, and that integral may solve this problem. But we have to prove it separately that the integral really serves the purpose, that it is a reverse process of differentiation.

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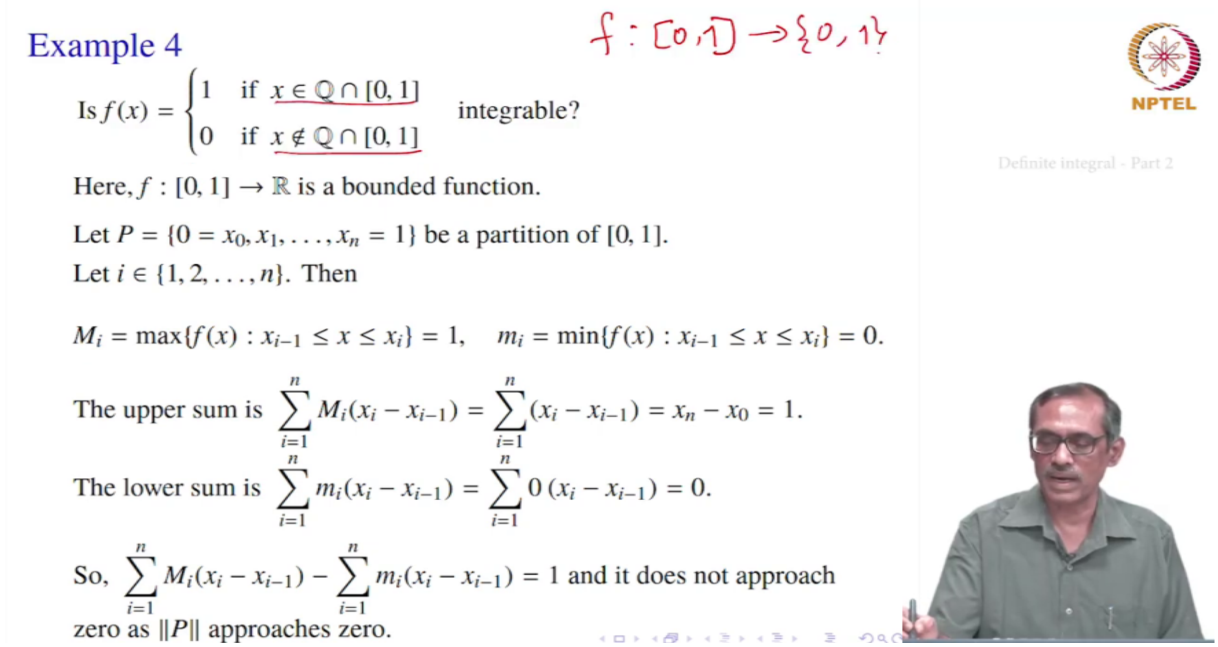

Before proving that, let us solve some more problems and see how our definition goes. The idea is, if f is continuous on the closed bounded interval $[a, b]$, then it is integrable. But there can be a case where f is not integrable. Of course, if f is not continuous, then it does not mean that it will not be integrable. But here is an example where it is not continuous and it is not integrable. Consider the function $f(x) = 1$ if x is a rational number between 0 and 1, and $f(x) = 0$ if x is an irrational number between 0 and 1. So, f is defined from $[0, 1]$ to $\mathbb{R}$. Its range has only 0 and 1, nothing else.

This is a bounded function, because it assumes only two values 0 and 1. Let us take any partition. You can choose any partition; it does not matter. Let $i \in \{1, 2, \ldots, n\}$. You can see that the maximum of $f(x)$ is equal to 1 in every sub-interval because in every sub-interval there are rational numbers, and its minimum is equal to 0 with a similar reason that in every sub-interval $[x_{i-1}, x_i]$, there is an irrational number. So for f, the minimum becomes 0 and the maximum becomes 1. Then the upper sum is 1 and the lower sum is 0, and their difference is 1. Or, directly, you can compute the difference as follows:

$$\sum_{i=1}^{n}(M_i - m_i)(x_i - x_{i-1}) = \sum_{i=1}^{n}(1 - 0)(x_i - x_{i-1}) = \sum_{i=1}^{n}(x_i - x_{i-1}) = x_n - x_0 = 1 - 0 = 1.$$

Then in the limit as norm of P approaches 0, is 1; it will never go to 0. Therefore it is not integrable. It does not matter whatever way you choose your partition P.

So, here is one function which is not integrable; you cannot find $F(x)$ such that $F'(x)$ will be equal to this function. That is the basic idea; we will come back to it soon.

Let us take another problem. Here it is asked to find the lower sum and the upper sum for this function f defined on the closed interval $[0, 1]$. What is the function f? It is $f(x) = x^2$. We want to find the lower sum and the upper sum, by dividing this interval $[0, 1]$ into n equals sub-intervals. This problem is in between our problem of going for the computation of the integral.

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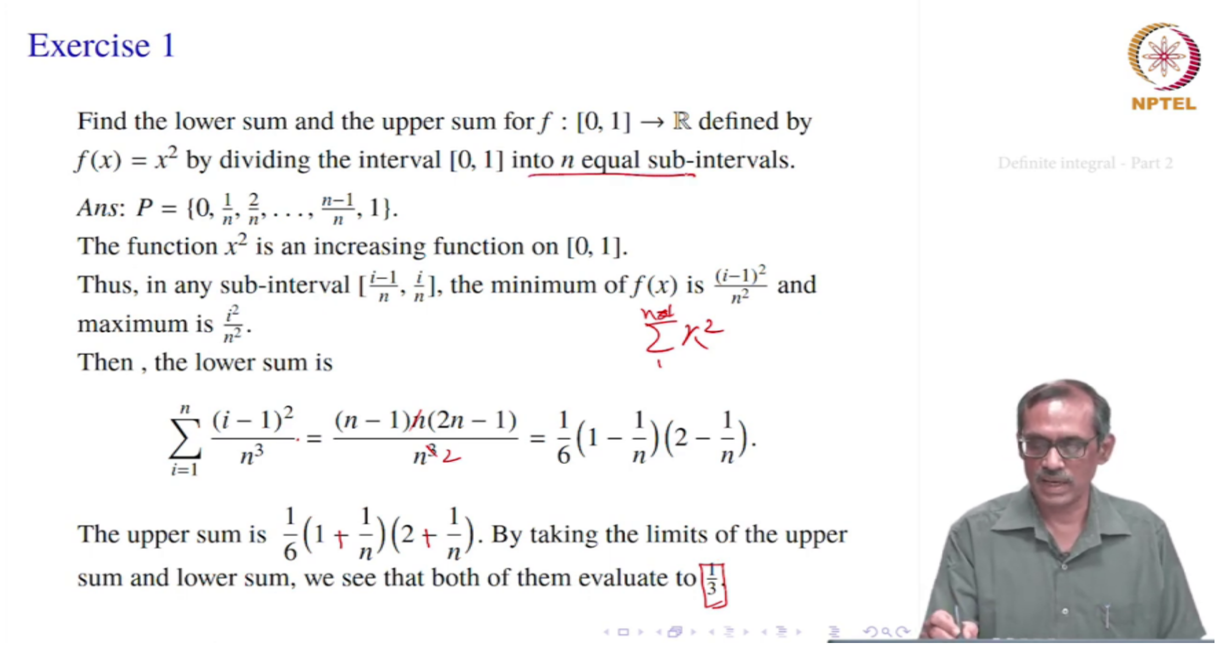

So, let us find the lower and upper sums. We have the partitions of $[0,1]$ as $0,\ 1/n,\ 2/n,\ \ldots,(n-1)/n,\ n/n=1$. since in this uniform partition, each sub-interval has the length $1/n$. Now, x^2 is an increasing function on $[0,1]$. It is increasing on $[0,1]$, that is what we know. Therefore, when you take the lower sum it will have the choice points c_i as the left endpoint x_{i-1}, and when you take the upper sum, the choice point c_i will be the right endpoint x_i. That is, in the sub-interval $[(i-1)/n, i/n]$, the minimum is achieved at $(i-1)/n$, and the maximum is achieved at i/n.

Therefore, the lower sum is summation of f of that lower endpoint where minimum is achieved, that is, $\sum_{i=1}^{n}[(i-1)^2/n^2](1/n)$. Tha ts gives $\sum_{i=1}^{n}(i-1)^2/n^3$. Th term n^3 comes out. And the sum is $1/n^3$ times $\sum_{k=0}^{n-1}k^2$ with $k=i-1$. You apply that formula for summation of k^2; that gives $(n-1)(n-1+1)(2(n-1)+1)/(6n^3)$. It simplifies to $(1/6)(1-1/n)(1-2/n)$.

When you take the upper sum, it will be i^2 here instead of $(i-1)^2$. So, it is $\sum_{i=1}^{n} i^2/n^3$ which is $n(n+1)(2n+1)/(6n^3)$. And it simplifies to $(1/6)(1+1/n)(1+2/n)$.

Now, if you take the limit of the lower sum and the upper sum, of course you get to $1/3$. Both the limits are same. Therefore this function is integrable. But we have got this only for a uniform partition! If you take some other sub-intervals it will also be like that. It will be having similar terms because you can take x_i and x_{i-1}. However, we cannot evaluate the summation since we do not know the values of x_i. Of course, you can show this later that the sum will be $1/3$ even if you take other partitions. But the problem here does not ask that. The problem asks for upper and lower sums with n equal sub-intervals.

Let us go to the second problem. We have a function $f(x)$, which is continuous. We have not given the domain of $f(x)$; it can be any domain. If nothing is given we may think of $f:\mathbb{R}\to\mathbb{R}$ or any convenient interval which will be required. We want to show that $\sum_{i=1}^{n}(1/n)[f(1+i/n-1/(2n))-f(1+i/n)]$ has the limit 0 as $n\to\infty$.

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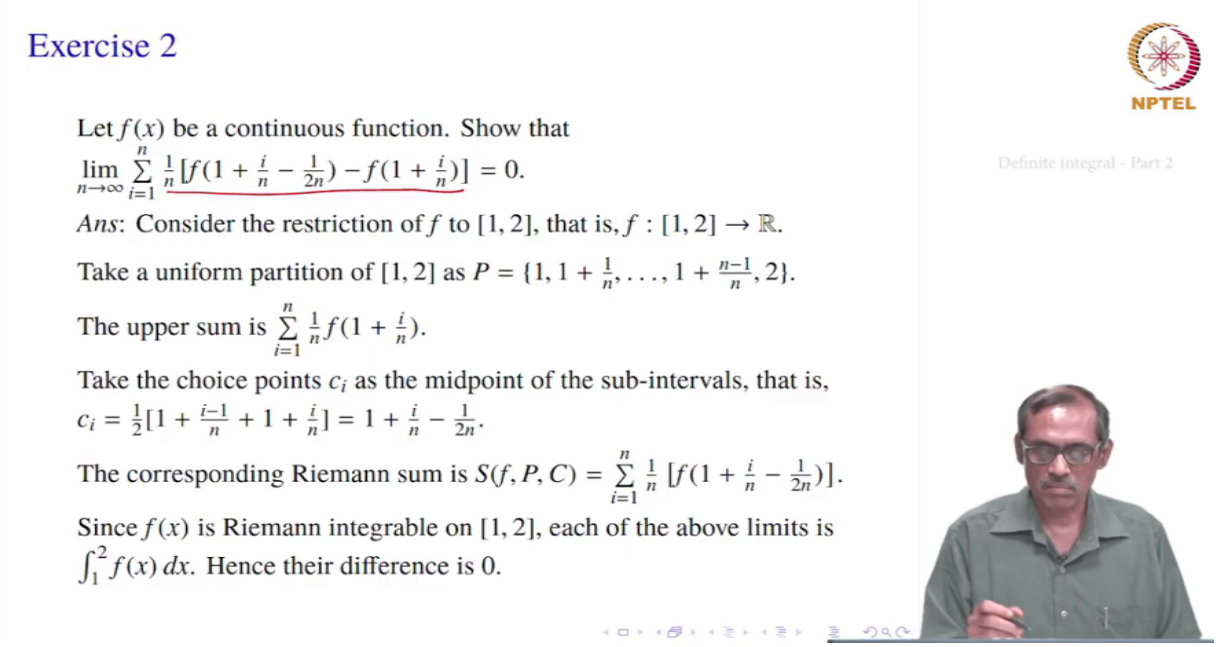

We guess that the given sum is the difference between the upper sum and the lower sum of $f(x)$ with some fixed partition of an interval I, which is the domain of $f(x)$. As i varies between 1 and n, if you take $i=n$, that should correspond to the maximum value of any point in I. That is, the right endpoint of I should be the point $1+i/n-1/(2n)$ and the left endpoint should be $1+i/n$. As $n\to\infty$ $1+i/n-1/(2n)$ goes to 2 and $1+i/n$ goes to 1. So, it looks that we have to define the function from the interval $[1,2]$. Let us try. Let $f:[1,2]\to\mathbb{R}$ be a function. You may alternatively take $f:\mathbb{R}\to\mathbb{R}$ and consider its restriction to $[1,2]$. Now, we take a uniform partition of $[1,2]$, which is $1,\ 1+1/n,\ 1+2/n,\ ,\ldots,1+(n-1)/n,\ 1+n/n=2$ and formulate the upper sum.

The upper sum is $(1/n)\sum_{i=1}^{n} f(1+i/n)$. But this is not really the upper sum; it is the Riemann sum obtained by taking c_i as the right endpoint of the ith sub-interval. Because upper sum means the function should achieve a maximum value at that point. Here, we do not know whether f is increasing or not. So, it is not exactly the upper sum; it is one of the Riemann sums. So, consider the Riemann sum where we choose $c- = x_i$, the right endpoint. The Riemann sum is $(1/n)\sum_{i=1}^{n} f(1+i/n)$.

When you take the choice point c_i as the midpoint of the sub-interval, that is, the midpoint of $[x_{i-1},x_i]$, which is $[(1+(i-1)/n)+(1+i/n)]/2$. And this is equal to $1+i/n-1/(2n)$. So, with this choice for c_i, the Riemann sum turns out to be the first term in the given summation. So, we can see that the expression we have here is the Riemann sum with the mid-points of the sub-intervals. The second summation where we have $f(1+i/n)$ corresponds to the Riemann sum where we choose c_i as the right end-point of the sub-intervals. So, the given summation is the difference between two Riemann sums; one is obtained by taking c_is as the mid-points and the other is obtained by choosing c_is as the right-endpoints.

Now that $f(x)$ is a continuous function on the closed interval $[1,2]$, it is integrable. Once it is integrable, you choose any two Riemann sums for the same function defined over that interval, then their difference goes to 0 as the norm of the partition goes to 0. Since $f(x)$ is Riemann integral on $[1, 2]$, each of the above limits exists and it is equal to the integral $\int_1^2 f(x)\,dx$. Therefore that difference must be equal to 0. That is how we will be solving some problems. Identifying this expression in n with some Riemann sum is really the clever part here. You will get acquainted with that slowly. So, let us stop here.