Basic Calculus - 1
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Lecture 19 - Part 2
Using Rolle's Theorem and Mean Value Theorem - Part 2

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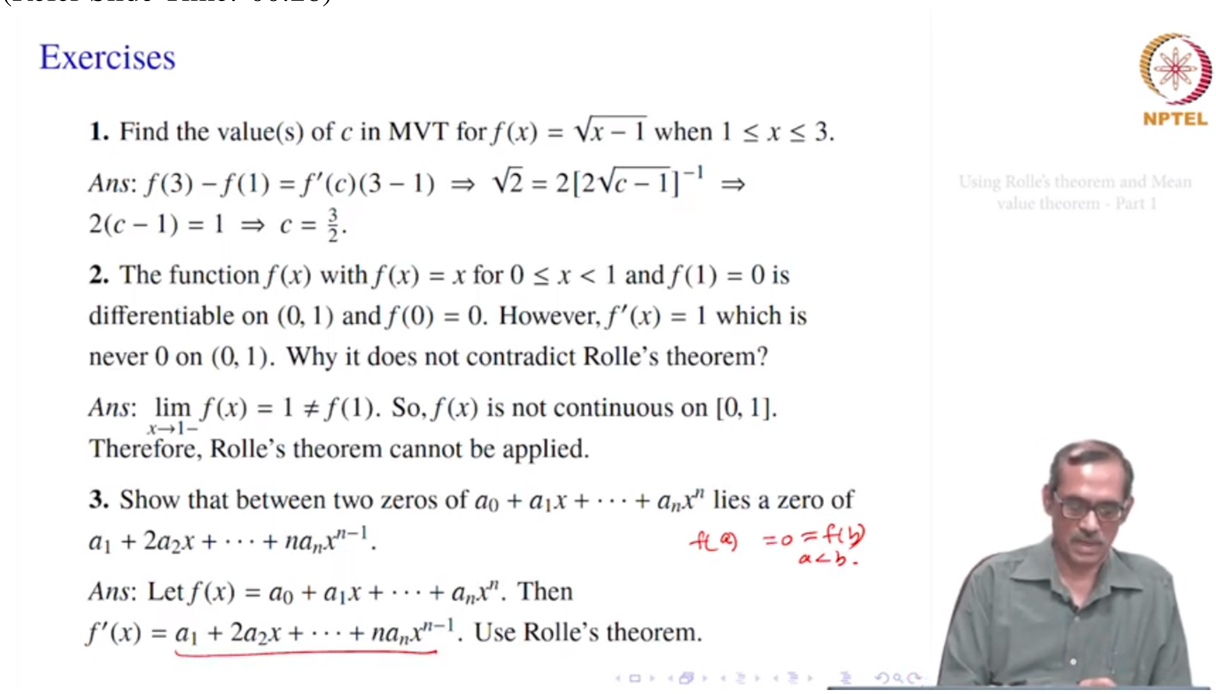

Let us solve some more problems. First problem: find the value of c or values of c, we do not know how many, in the Mean Value Theorem for this function $f(x) = \sqrt{x-1}$ when $1 \le x \le 3$ What does it mean? Here the function $f(x)$ is $\sqrt{x-1}$ on this closed interval. It is continuous on $[1,3]$ and on the open interval $(1,3)$ it is differentiable. So, you can use the Mean Value Theorem on this interval. That is, $[f(3) - f(1)]/(3-1) = f'(c)$ for some c. If possible, we want to find that c or many such cs. That is what we want.

Now, $f(3) - f(1) = f'(c)(3-1)$. That is what the conclusion of the Mean Value Theorem says. We find that $f(3) = \sqrt{3-1} = \sqrt{2}$ and $f(1) = \sqrt{1-1} = 0$. So, left side is $\sqrt{2}$, we have to find out f' for the right side. Well, $f'(x) = 1/[2\sqrt{x-1}]$. Evaluated at c gives $f'(c) = [2(c-1)]^{-1}$. And that is equal to $3 - 1 = 2$. We get this equation, which we solve to find our c. That gives $c = 3/2$.

So, there is a unique value for this function. For different functions, there can be other cs in the interval where the values of the function, its derivative, are same.

Let us go to the second problem. Here, we have a function $f(x)$, which is defined as $f(x) = x$ for x between 0 and 1, including 0; and $f(1) = 0$. That means f is defined at least on the closed interval $[0,1]$. So, at 1 it is 0, at everywhere else, it is like the identity function $f(x) = x$.

Now this function is differentiable in the open interval $(0, 1)$. That is easy to see. Because, on the open interval it is $f(x) = x$. This function also has the property that $f(0) = 0 = f(1)$. That is correct because $f(x) = x$ for $0 \leq x < 1$. However, $f'(x) = 1$ in the open interval $(0, 1)$. So, it is never 0 on the open interval $(0, 1)$. So, like in the Rolle's Theorem, you have $f(1) = f(0)$, which of course happens to be 0 here, f is continuous and differentiable on $(0, 1)$. So, why does it not contradict Rolle's Theorem?

As you see, Rolle's Theorem requires that f should be continuous on the closed interval $[0, 1]$, but here it is not. So, that is the problem; it is not continuous. Because when you take the limit of $f(x)$ as $x \to 1-$, from the left hand side, you get it to be 1, but $f(1)$ is given to be 0. There is a jump there; and one of the conditions in Rolle's theorem is not satisfied, namely, "f should be continuous on the closed interval" is not true here.

Let us go to the third problem. It asks us to show that between any two zeros of this polynomial, there lies another zero. Suppose this polynomial vanishes at two points, say, at real numbers a and b. It says that the second polynomial vanishes at a point between a and b. This is very easy to see if you recognize the second polynomial as the derivative of the first polynomial, and then apply Rolle's Theorem. That is what we do.

Let $f(x)$ be the first polynomial. Then $f'(x)$ is equal to this second polynomial. So, Rolle's Theorem says that if a is a root, that is, if $f(a) = 0 = f(b)$, for $a < b$, then between a and b there is a point c where its derivative should be 0. Therefore, there is a zero of the second polynomial between any two zeros of the first polynomial. Fine.

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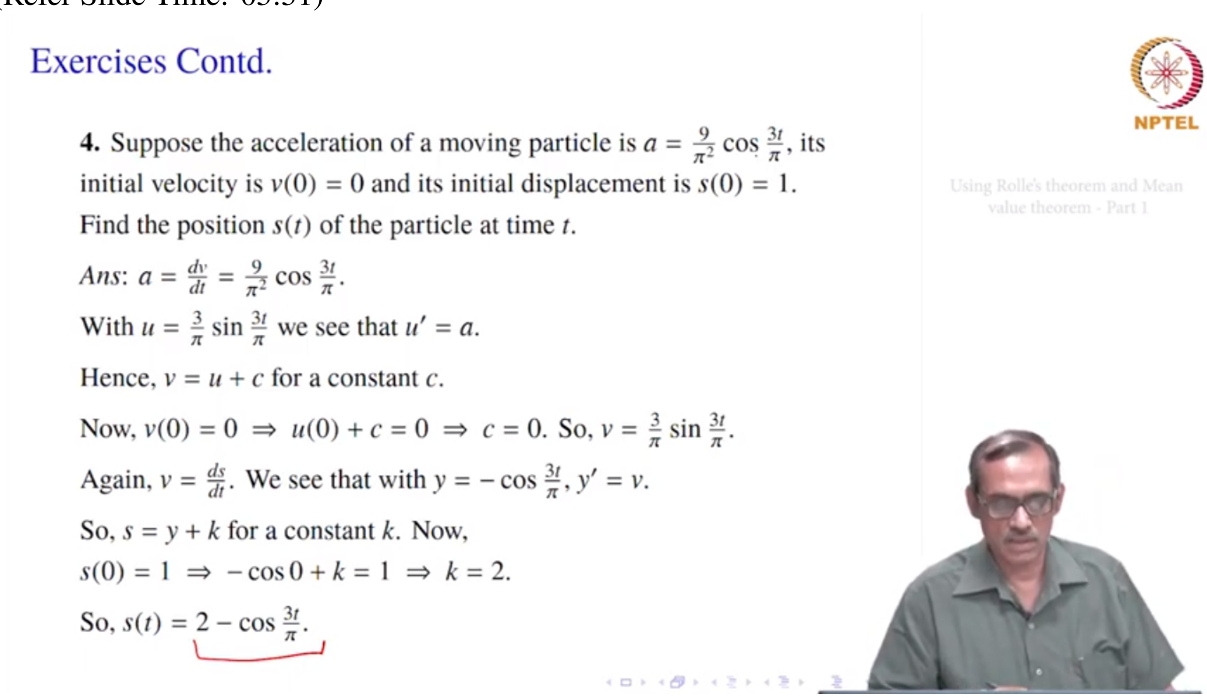

We come to the fourth problem. Here is a physical application. Suppose the acceleration of a moving particle is a, which is given as $(9/\pi^2)\cos(3t/\pi)$ at any point t, and its initial velocity is $v(0) = 0$. Though its acceleration and velocity are given, we are not really concerned about them as vector quantities. We are taking their scalar parts, which are really numbers, or rather as functions. It is given that the initial velocity is $v(0) = 0$, and its initial displacement is $s(0) = 1$.

Recall that $v = \dot{s}$ and $a = \dot{v}$, which is $a = \ddot{s}$, where the dot is the derivative with respect to t. We are asked to find the position $s(t)$ of the particular time t. That means we find the function $s(t)$ if these things are given.

Now, $a = dv/dt$, which is given to be $(9/\pi^2)\cos(3t/\pi)$. If you take $u = (3/\pi)\sin(3t/\pi)$, then you can see that the derivative of u with respect to t gives you that. Does it give? It is $(3/\pi)$ times $\cos(3t/pi)$ into the derivative of $3t/\pi$ with respect to t, which gives $3/\pi$. And that makes up $(9/\pi^2)$. So, it is the derivative of the function u. Our corollary to the Mean Value Theorem says that u can differ from v by a constant. So, we can write $v = u + c$ for some constant c.

Now we will use the condition that $v(0) = 0$. That gives $u(0) + c = 0$. As $u(0) = 0$, that gives $0 + c = 0$. Therefore c must be 0. So, we have got $v = (3/\pi)\sin(3t/\pi)$.

Once more, we apply the same method. This v is again ds/dt. We see that if I take a new function y, which is given by $y(t) = -\cos(3t/\pi)$, then its derivative is equal to $(3/\pi)\sin(3t/\pi)$. That is clear; if you differentiate $-3\cos(3t/\pi)$, with respect to t, it gives $\sin(3t/\pi)$ times the derivative of $3t/\pi$ with respect to t, which is $3/\pi$, and we get $(3/\pi)\sin(3t/\pi)$. That means $y' = v$, where $y = -3\cos(3t/\pi)$. Therefore, y can differ from s by a constant.

That is, $s = y + k$ for some constant k. Now we apply the condition $s(0) = 1$. That gives y, which is $-3\cos(3t/\pi)$ evaluated at 0; which is $-\cos 0 = -1$. This is $-1 + k$, and it should be equal to $s(0) = 1$. That gives $k = 2$. And, we get our s as $2 - \cos(3t/\pi)$.

So, we had to apply the same method twice, starting from acceleration to the displacement.

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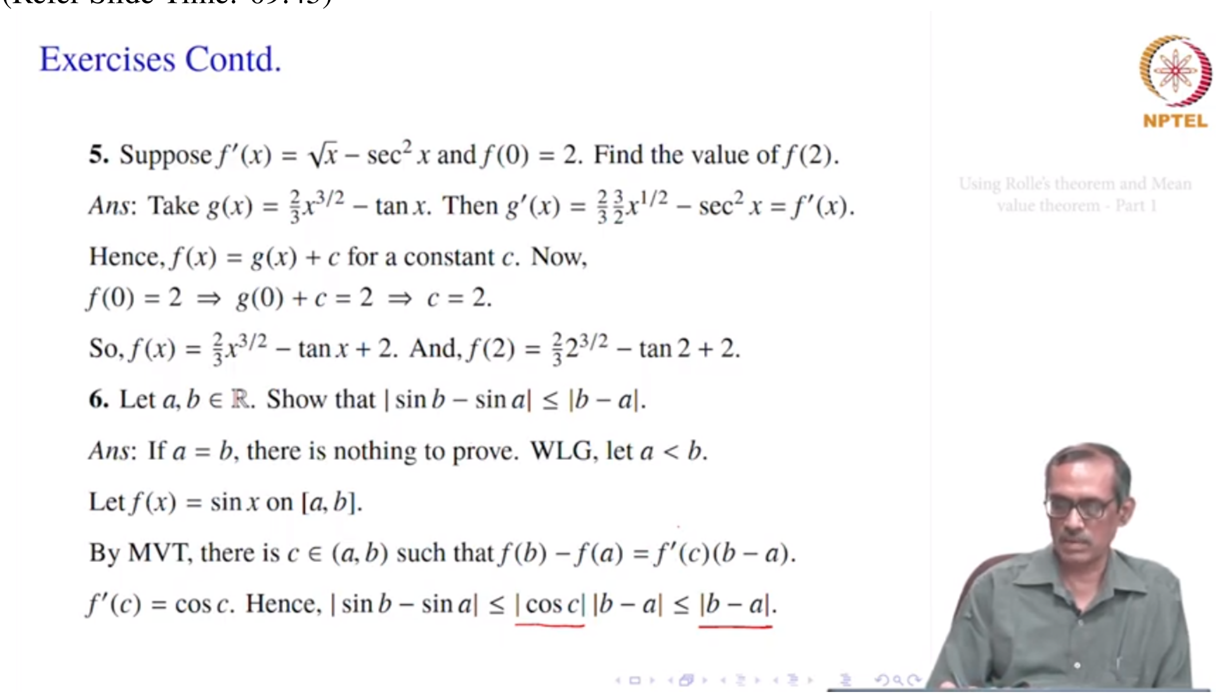

Let us take the next problem. Here, there is some function. Assumed that there is a function $f(x)$, which is defined on its domain to real numbers such that $f'(x) = \sqrt{x} - \sec^2 x$. We are given the information on f'; and also we are given that $f(0) = 2$. We need to find the value of $f(2)$.

It is similar to the earlier problem. Here the derivative is given. That means you have to guess some function whose derivative will be $\sqrt{x} - \sec^2 x$. That should be our starting point. So, we start with this.

We know that $\sqrt{x}$ will come from $x^{3/2}$ by differentiation. If I take $x^{3/2}$ and differentiate, I would get $(3/2)x^{1/2}$; that is the derivative will be $(3/2)x^{3/2-1}$ and that is $(3/2)x^{1/2}$. Since I have only $x^{1/2}$, and not this constant, I multiply this constant here. That is, I start with $(2/3)x^{3/2}$, and I know that $\sec^2 x$ comes from $\tan x$ when you differentiate. So, we start with $g(x) = (2/3)x^{3/2} - \tan x$. To find out such a g is a guesswork; we should s find g whose derivative should be equal to that given one.

Now you find that $g'(x) = (2/3)(3/2)x^{1/2} - \sec^2 x$, which is $f'(x)$. That is, $g'(x)$ matches with $f'(x)$. Therefore we conclude that $g(x)$ and $f(x)$ will differ by a constant. We say that $f(x) = g(x) + c$ for a constant c. Then you use the condition $f(2)$. No, we have to find $f(2)$. We will use the condition $f(0) = 2$. When you substitute $x = 0$, you get $f(0) = 2$, and then $g(0) + c = 2$. But $g(0) = -\tan 0 = 0$. The left side gives c, the right side gives 2; so $c = 2$. Therefore, $f(x) = g(x) + 2$, which is this.

So, once $f(x)$ is obtained, we evaluate it at $x = 2$ to get our answer, because we are interested in finding the value of $f(2)$. When you substitute $x = 2$, it is $(2/3)2^{3/2} - \tan 2 + 2$, and that is the answer.

Let us look at the next problem. Suppose a and b are two real numbers. We want to show that the absolute value of $\sin b - \sin a$ is less than or equal to the absolute value of $b - a$. It is not just $\sin x \le x$ or $|\sin x| \le |x|$, This is what we want to show. First, we notice that a can be equal to b so that $\sin b = \sin a$. In that case, we would get $0 \le 0$; and there is nothing to prove. So, assume that $a < b$.

One of them is less than the other. So, without loos of generality, assume that a, b. Consider the function $f(x) = \sin x$ on this closed interval $[a, b]$. We take this closed interval because this one is $f(b) - f(a)$ and the other side is $b - a$. We think we can apply the Mean Value Theorem.

Well, consider $f(x) = \sin x$ on the closed interval $[a, b]$. Then, it is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). By the Mean Value Theorem there exists one c inside the open interval (a, b), in between a and b, such that $f(b) - f(a) = f'(c)(b - a)$. But what is $f'(c)$? As $f(x) = sinx$, $f'(x) = \cos x$. So, this is $\cos c$; its absolute value is bounded by 1. Therefore, $|\sin b - \sin a| \le |\cos c| \times |b - a|$ As $|\cos c| \le 1$ for any c, this is less than or equal to $|b - a|$.

Notice that the same thing also holds for cosine. You can continue the same as for the sine, because the derivative of $\cos x$ will be $-\sin x$, which afgain will be bounded by 1.

Let us consider the next problem. We are given with a quadratic. It is a polynomial of degree 2: $px^2 + qx + r$. It is defined on everywhere; so we are considering this as a function with domain $[a, b]$. Of course we assume that $a < b$. It is given that this coefficient of x^2 is not 0; that is $p \ne 0$. We need to show that the c in the Mean Value Theorem is unique.

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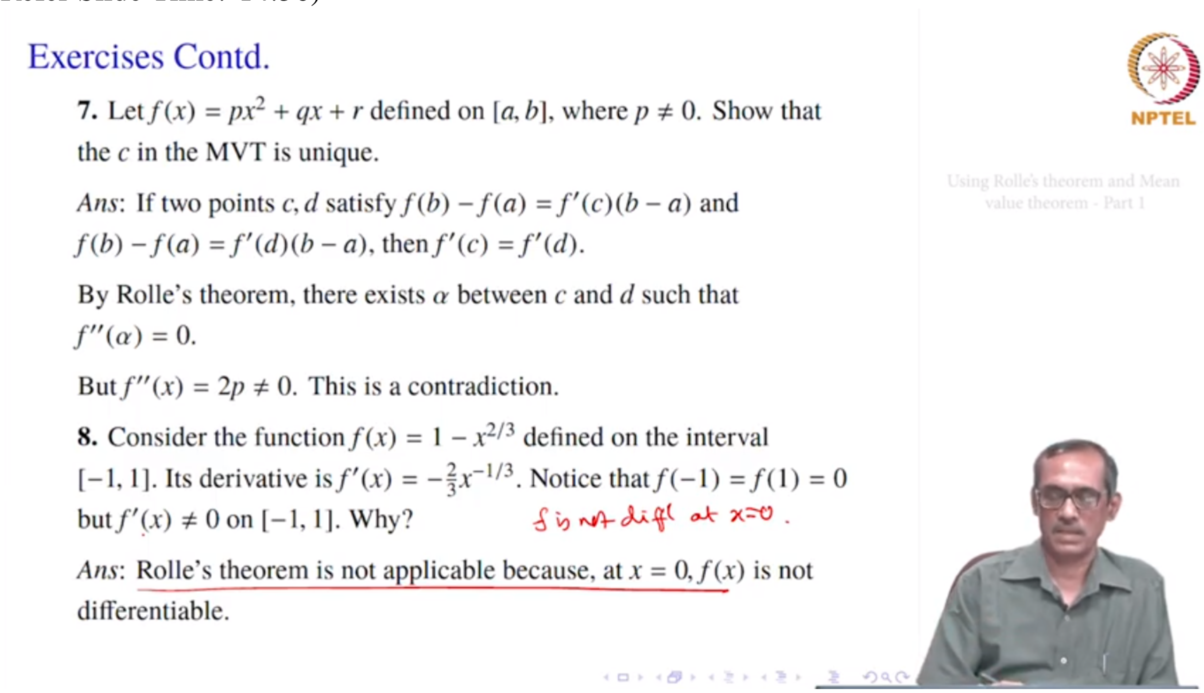

So, what is the meaning of "c in the Mean Value Theorem"? We know that if you apply the Mean Value Theorem (if it is at all applicable), then its conclusion will be $f(b) - f(a) = f'(c)(b - a)$ for some c between a and b. So, that is the c. We are asked to show that this c, which is in between a and b, is unique. There cannot be two different cs satisfying the same condition, if it is a quadratic with $p \neq 0$.

So, suppose on the contrary that there are two such points. Notice that such a point exists is guaranteed by the Mean Value Theorem. Because, this is a continuous function on the closed interval $a, b]$, differentiable on the open interval (a, b), the Mean Value Theorem gives at least one such c. So, suppose there are two points c and d such that $f(b) - f(a) = f'(c)(b - a)$ and $f(b) - f(a) = f'(d)(b - a)$. Since we are assuming that $b > a$ and $f(b) - f(a)$ is same on both the sides, by dividing $b - a$ we can say that $f'(c) = f'(d)$. Both the things are same. But what is f'? Of course, you can find out from there. It is $f'(x) = 2px + q$. This is our f'. This evaluated at c is equal to this evaluated at d, that is given.

Now you consider the function f' itself. This satisfies $f'(c) = f'(d)$ and it is also differentiable. has the condition now f prime c equal to f prime d, and it is again differentiable. As you can see it is a linear function, or rather linear polynomial: $2px + q$. Its derivative exists inside the open interval (a, b). You need of course this to happen inside the open interval (c, d) only. Since c, d lie in between a and b, (not necessarily the whole of (a, b)), $f'(x)$ is differentiable in (c, d). Then you can apply Rolle's Theorem to conclude that $f''(\alpha) = 0$ for some α between c and d. See, if α is in between c and d, then it is also between a and b. So, there is one α between a and b such that $f''(\alpha) = 0$.

But $f''(x) = 2p$ as $f'(x) = 2px + q$. When you differentiate it, it is a constant. So, that goes away, and this x also goes, you get $2p$; so, $f''(x) = 2p$. Since $p \neq 0$, $2p \neq 0$. So, $f''(x)$ is never equal to 0 inside the interval (a, b). It contradicts our conclusion that $f''(\alpha) = 0$ which we obtained by Rolle's Theorem. That means there is something wrong. What is wrong? Our extra assumption that between a and b there are two points, at least two points c and d which work as the c in the Mean Value Theorem. That is wrong. Therefore, c in the Mean Value Theorem is unique.

We consider another problem. We have a function $f(x)$ given as $1 - x^{2/3}$. You see whether it is defined on the interval $[-1, 1]$. Yes it is. So, consider this function f which is defined from the closed interval $[-1, 1]$ to $\mathbb{R}$ and is given by $f(x) = 1 - x^{2/3}$. If you differentiate it, you would get $0 - (2/3)x^{-1/3}$ So, this is correct: $f'(x) = -(2/3)x^{-1/3}$. That is its derivative. If you substitute $x = -1$. you see that $f(-1) = 1-(-1)^{2/3} = 1-(-1)^2 = 1-1 = 0$. Also, $f(1) = 1-1^{2/3} = 1-1 = 0$. If you think of Rolle's Theorem or even the Mean Value Theorem, it should say that $f'(x) = 0$ for some x in between -1 to 1. However, if you take $f'(x)$, it is $(-2/3)x^{-1/3}$; it is not equal to 0 for any $x \in (-1, 1)$. Why do you say it is not equal to 0? Well, it is not defined at $x = 0$. So, you cannot say that $f'(x)$ is equal to 0 on $(-1, 1)$. Why?

The reason is simple. We say that Rolle's Theorem is not applicable because $f(x)$ is not differentiable in the open interval $(-1, 1)$ since at $x = 0$, $f(x)$ is not differentiable. (It is easy to see that $f(x)$ is not differentiable at $x = 0$.) Therefore, $f(x)$ is not differentiable on the open interval $(-1, 1)$. So, you cannot apply Rolle's Theorem. We stop here.