Basic Calculus - 1

Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras

Lecture 17 - Part 1

Maxima and Minima - Part 1

This is lecture 17 of Basic Calculus 1. In the last lectures, we had discussed about differentiation and how to differentiate functions involving power functions and trigonometric functions. Today we will be discussing a nice application of differentiation, which is about finding maxima and minima of functions.

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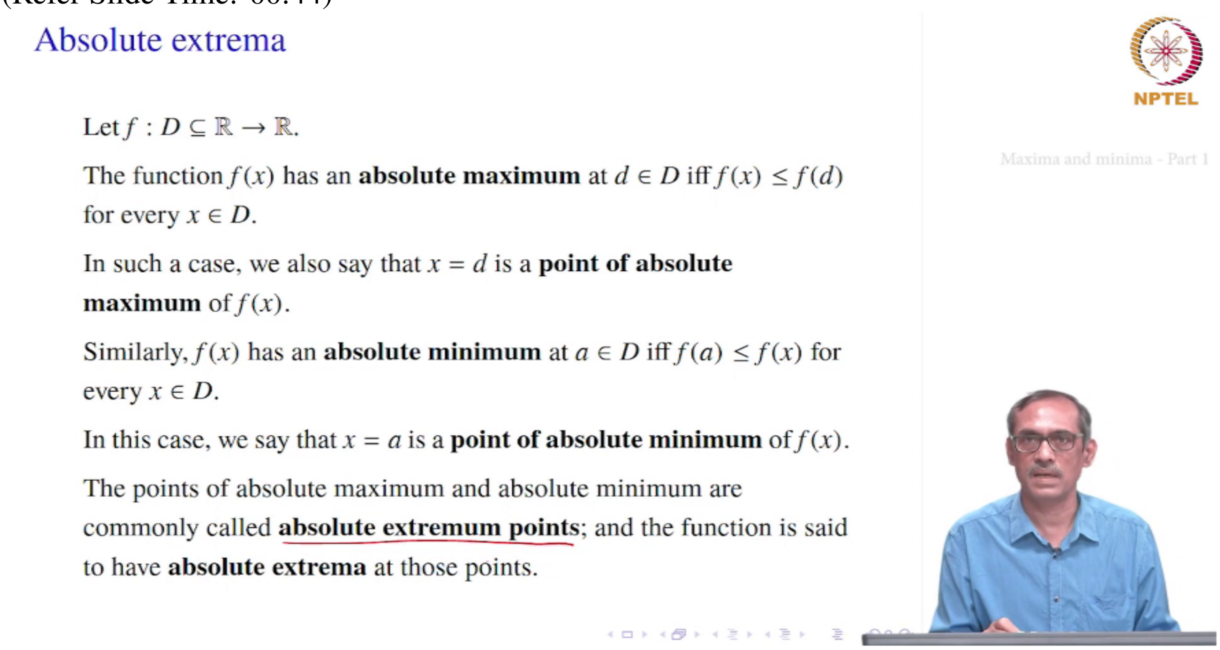

Let us give first the idea of what is this maxima and minima. As usual, it is the maximum of the functional value, which is called the maximum of the function $f(x)$. But always maximum may not exist. So, we have to be really careful while formulating it. We start with a function $f : D \to \mathbb{R}$, where D is a subset of $\mathbb{R}$. In fact, we will not be considering any abstract subset, but union of intervals. So, you may think of this D as an interval.

Now, f is a function from D to $\mathbb{R}$. We will say that the function $f(x)$ has an absolute maximum at $d \in D$ if and only if $f(x) \leq f(d)$ for every $x \in D$. That is easy, because what we want is $f(d)$ is the maximum value of all the functional values. So, you take any $f(x)$; that should be less than or equal to $f(d)$. Instead of just maximum, we will say absolute maximum because something else is coming.

This $f(d)$ will be called the absolute maximum of the function and it is achieved at $d \in D$. Then the point $x = d$ is called a point of absolute maximum of $f(x)$. Absolute maximum is the maximum value and where it is attained that point is called a point of absolute maximum. Similarly, you can define absolute minimum. Suppose a is a point where you think there is absolute minimum.

That means $f(a)$ should be less than or equal to every other functional value. So, $f(a) \leq f(x)$ for every $x \in D$. Again, the point a is called the point of absolute minimum of the function $f(x)$.

These are fairly straightforward ideas, coming from the English language. If a point is either a point of maximum or a point of minimum, then it is called an extreme point or an absolute extreme point. That is, both a and d will be called absolute extreme points here. And the function is said to have absolute extrema at these points. Similarly, it has absolute extrema or an extremum at these points. This is just a terminology.

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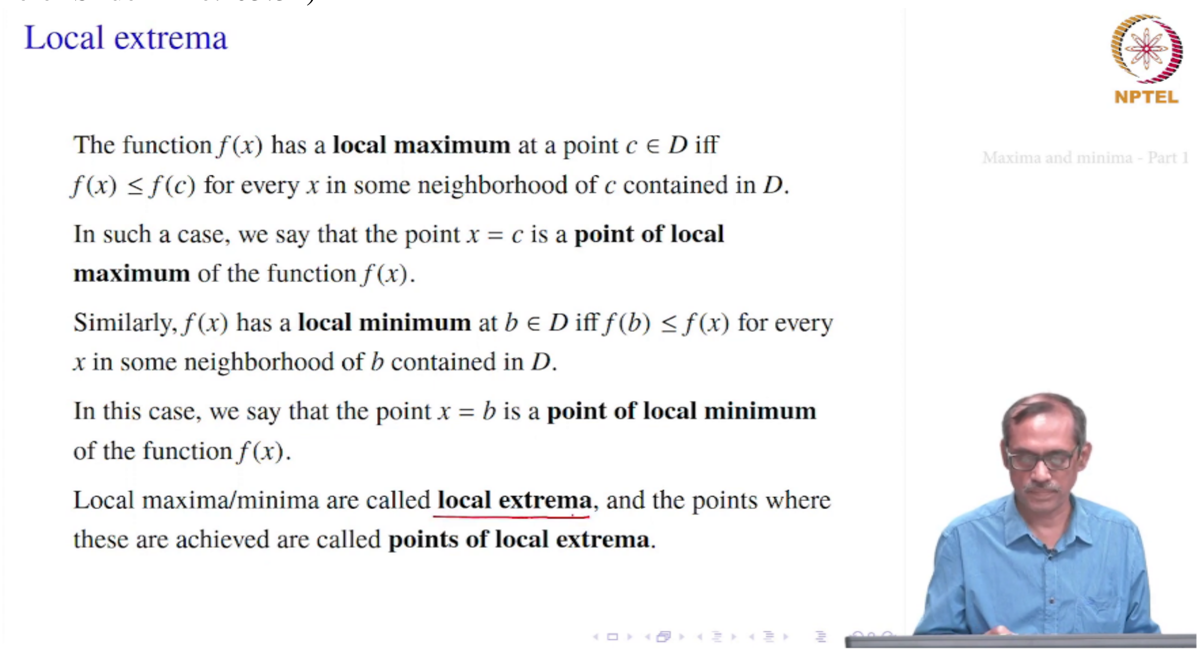

As we have told, these notions go with the word 'absolute'. There is something which is called a local maximum. A local maximum is slightly different from it, but that maximum thing should hold in some sense. Sometimes absolute maximum is also called a global maximum; absolute minimum is called a global minimum; absolute extrema is called global extrema. Correspondingly, we have notions called local maximum, local minimum and local extrema. So, what happens here?

Let c be a point in D. We will say that the function $f(x)$ has a local maximum at this point c when every value $f(x)$ is less than or equal to $f(c)$, but now, x does not vary over the full domain D, it varies only in some neighborhood of that point c. Once for every point in some neighborhood of c, this inequality $f(x) \leq f(c)$ holds, we will say that the function has a local maximum at the point c. In such a case, the value $f(c)$ is called a local maximum value and the point $x = c$ c is called a point of local maximum of the function $f(x)$.

Similarly, local minimum can be defined. Suppose b is a point inside D. We will say that the function $f(x)$ has a local minimum at b, if $f(b) \leq f(x)$ where x varies over some neighborhood of b. That neighborhood of course should be contained in D in both the cases. For local maximum and local minimum both the neighborhoods should be contained inside D. Otherwise, we cannot talk of $f(x)$. This is a notion of local maximum and local minimum.

Again, we say that $x = b$ is a point of local minimum of the function $f(x)$. Again, both local maxima and local minima are called local extrema. So, local extrema means it is the values of the function which are local maximum or local minimum. Similarly, those point c and b are called points of local extrema.

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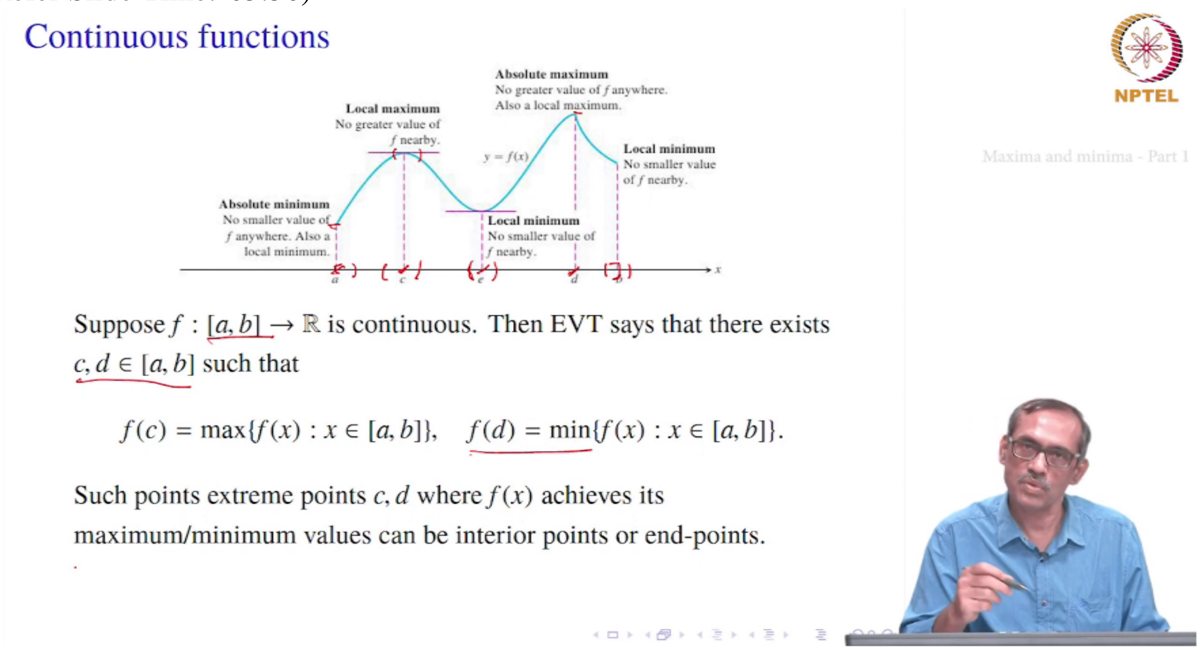

If you draw a graph, then it will be easier to see how these notions are translated to in terms of the graph of the function. Suppose we have a function f which is defined on the closed interval $[a, b]$ to $\mathbb{R}$. Assume that it is continuous. So, we have its graph like this. Then what do we see?

Since $[a, b]$ is a closed interval and f is continuous, our Extreme Value Theorem says that it has a maximum, it has a minimum and these values are achieved at some points in $[a, b]$. So, there exist points $c, d \in [a, b]$ such that $f(c)$ is equal to the maximum value of $f(x)$ where x varies over the whole of $[a, b]$. Similarly, $f(d)$ is the minimum of $f(x)$, where x varies over the closed interval $[a, b]$. Again, such points c and d are the extreme points and the maximum and minimum values are attained at these points.

Look at the graph. We have a here and we have c and d here. Let us say that it is maximum here; the absolute maximum occurs at d. That is what we see from the biggest height in the graph of the function. At d, we have the absolute maximum and at a, we have the absolute minimum. That is the smallest value, the smallest vertical line or values of f.

And you see that at c it is not an absolute maximum, it is not also absolute minimum, but it is some sort of maximum. That is what we say: it is local maximum. If you take some small neighborhood (you can find always a neighborhood here of c), then in that neighborhood $f(c)$ is greater than or equal to all values of $f(x)$ where x varies over that neighborhood $(c - \delta, c + \delta)$.

Similarly, if you consider this point e, we have a local minimum here. That is, we can find a neighborhood of that e, say, $(e - \delta, e + \delta)$, where $f(e) \leq f(x)$ for each x inside that neighborhood. So, this is how local maximum, local minimum, absolute maximum, absolute minimum look like.

An absolute maximum can also become a local maximum. But converse is not always true;

some of the local extrema are not absolute extrema, as you see here at c and e. If you look at b, then you cannot take a neighborhood, only the left neighborhood is relevant here. The left neighborhood of b is contained inside d. In that neighborhood consider all points; they are limited to this only, from $b - \delta$ to $b + \delta$. If you take any x in this neighborhood, then $f(x) \geq f(b)$. So, you may think of b as a point of local minimum. Notice that b is not an interior point of the interval. Similarly, on the other side, the point a is a point of absolute minimum. You can see also that it is a point of local minimum. Here, the relevant neighborhood will be a right neighborhood. You can find a right neighborhood where $f(x) \geq f(a)$.

That is how you see absolute maximum. But we do not know till now how to find one really. We have just defined the notions of absolute maximum, absolute minimum, local maximum, local minimum; and these are all called extrema. We will be interested in finding the points, where these are achieved, and also, what are the corresponding values; what are the maximum; what are the minimum, and so on.

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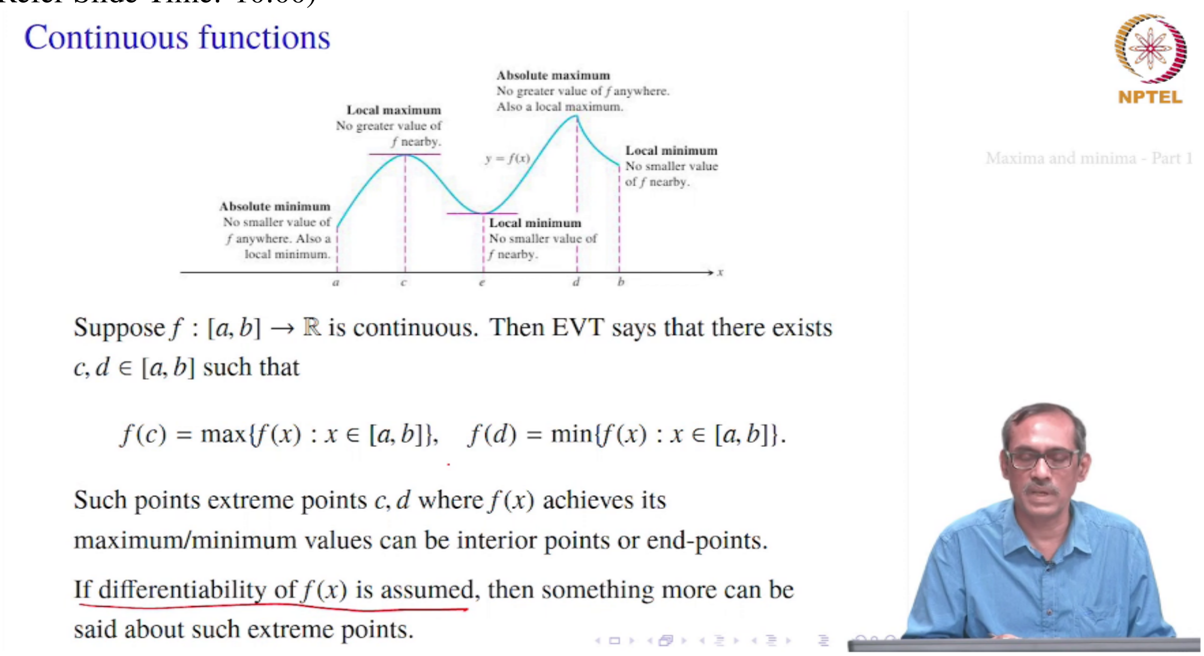

If $f(x)$ is differentiable, then something more can be told about these extreme points. That is how we will apply the notion of differentiation to computing this absolute extrema, or local extrema. First, we will consider absolute extrema. We will come back to local extrema in a later lecture.

In finding out this absolute extrema, we have this particular statement, which becomes very helpful. It is a theorem. What does it say? Suppose $f(x)$ has a local maximum and also has a local minimum. You can take one of them. Say, it has a local maximum or it has a local minimum, that is, $f(x)$ has a local extremum at an interior part c of the domain of f. Suppose $f(x)$ is defined over D, and $D = [a, b]$. Then, any point in the open interval (a, b) is an interior point. This theorem talks about the interior points only. If c is an interior point, then when can you say that that c is an extreme point, or a point of local extremum? The statement says that if $f(x)$ is differentiable at

$x = c$, which is a local extremum point, then $f'(c)$ must be equal to 0.

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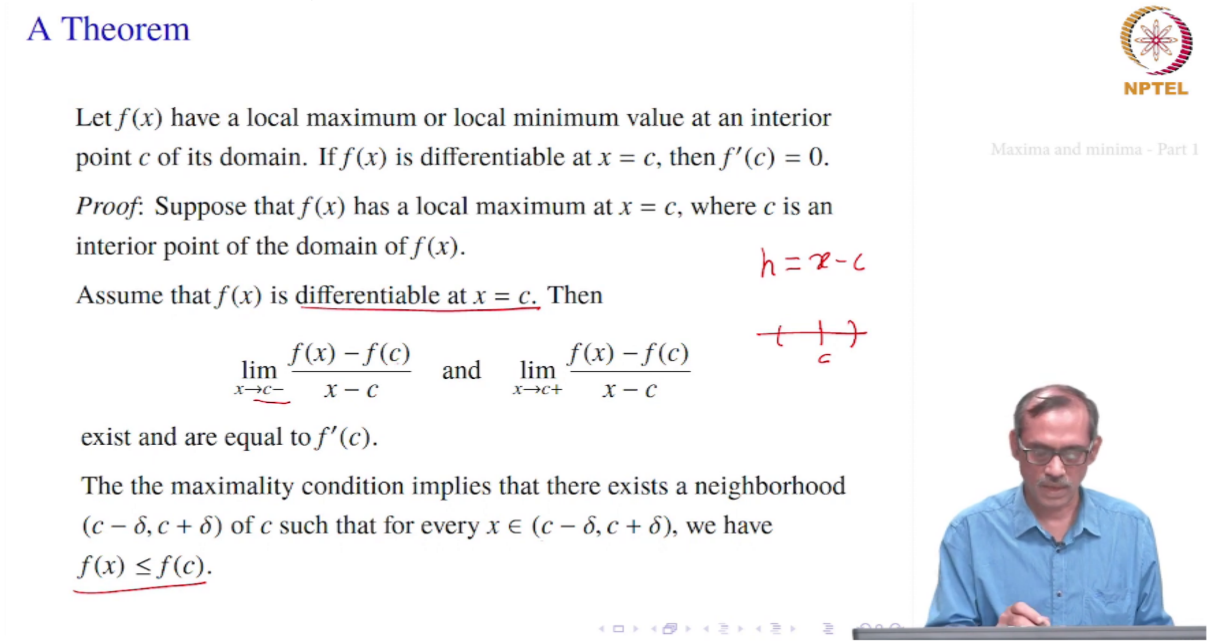

That is what we have seen in the last picture. If we have a local maximum at c, then we see that a tangent to the curve at that point c is horizontal. If we have a local minimum at e, then at e, the tangent is also horizontal. These mean that f' at those points is 0. Notice that we need $f(x)$ to be differentiable, and these points are interior points. It says that if you have a point of local extremum at an interior point c and f is differentiable at that c, then $f'(c) = 0$. We will see how to prove this.

Suppose $f(x)$ has a local maximum at $x = c$. For local minimum, the argument will be similar; only the inequalities will be reversed everywhere. And we assume that c is an interior point of the domain of the function $f(x)$. These are to be assumed. All that we want is to show that $f'(c) = 0$. Again our assumption is $f(x)$ is differentiable at $x = c$. Since it is differentiable at $x = c$, the limit as $x \to c$ of $[f(x) - f(c)]/(x - c)$ exists.

We are writing x as $c + h$ so that in the limit, $h \to 0$. In our earlier definition of the derivative e take h as $x - c$. So, the limit as $x \to c$ of $[f(x) - f(c)]/(x - c)$ exists. That means your left-hand side limit and the right hand side limit must exist and they are equal; and we are writing that common value as $f'(c)$.

The left hand side derivative is obtained when you take $x \to c-$. That means, if c is here, then x will be somewhere here, in some neighborhood $(c - \delta, c)$ of c. When you say $c - \delta$ to c, then at all those points $f(x)$ must be less than or equal to $f(c)$. In fact, since c is an interior point, you can take the whole neighborhood $(c - \delta, c + \delta)$ also, and whenever x varies over that neighborhood, you will have the inequality $f(x) \leq f(c)$. We will take both the cases of the left hand limit and the right-hand limit separately and use this inequality $f(x) \leq f(c)$ whenever it is applicable.

Let us consider the left-hand limit first. Then, $x < c$; it is varying over a left neighborhood of c. Since $f(c)$ is a local maximum, we have $f(x) \leq f(c)$; that is, $f(x) - f(c) \leq 0$. The numerator here is less than or equal to 0. In fact, it cannot be equal; (unless f is a constant function) let us say it is less than 0. And the denominator $x - c$ is also less than 0. So, you get $[f(x) - f(c)]/(x - c) > 0$. Of course, if it is equal, then we have no problem, the same argument goes through. So, for all x with $x < c$, we have $[f(x) - f(c)]/(x - c) > 0$. It can be equal also, if $f(x)$ is a constant function. But for the constant function, it is easy to see that $f'(c) = 0$. So, we do not need equality here. Or, if you want to include it, then you can also write greater than or equal to 0.

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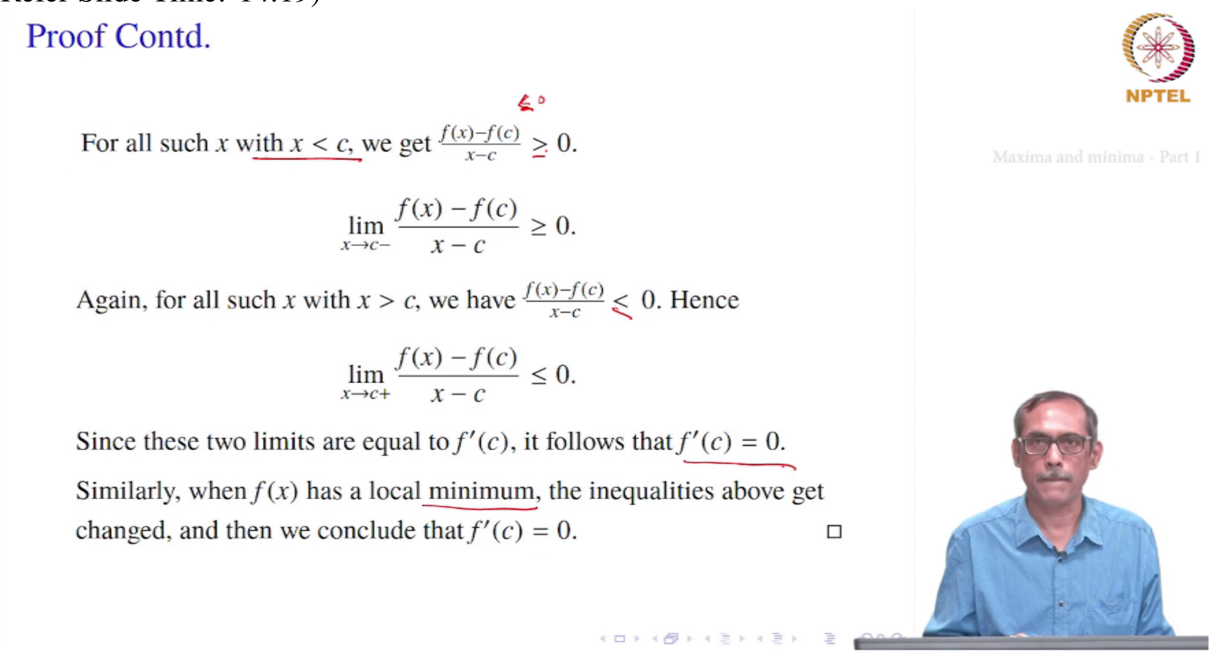

So, what do we get from this, even if you take greater than equal to 0? If you take the limit (which we know to exist), since all these values are greater than or equal to 0, the limit also must be greater or equal to 0. That is, the limit of $[f(x) - f(c)]/(x - c)$ as $x \to c-$ is greater than or equal to 0.

Similarly, you take the right neighborhood. That is, consider all x with $x > c$. Then again, the same inequality holds; $f(x) \leq f(c)$ because $f(c)$ is a local maximum. So, $f(x) - f(c)$ on the top is less than or equal to 0, but $x > c$, so, $x - c > 0$. So, you get here less than or equal to 0. When you take the limit, since it is less than or equal to, you get again limit as $x \to c+$ of $[f(x) - f(c)]/(x - c)$ to be less than or equal to 0.

Now, the thing is both the limits exits. So, one is greater than or equal to 0 another is less than or equal to 0; and they are equal to $f'(c)$. Hence, $f'(c)$ must be equal to 0. As both the limits must be equal, that must be equal to 0. That is the proof.

This happens at an interior point c. For the minimum case, all these inequalities will be reversed. You say it is less than or equal to 0; $[f(x) - f(c)]/(x - c) \leq 0$ for $x < c$ and this is greater than or equal to 0 for $x > c$. So, $f'(c)$ will become equal to 0.

So, this really asks us to give some name to such points, wherever the tangent is horizontal at an interior point. But we include some more points along with this. We say that an interior point c is called a critical point if $f'(c) = 0$ or $f(x)$ is not differentiable at $x = c$. That is also included, because we assumed there that $f(x)$ must be differentiable at c. If f is not differentiable at c, then also we call c as a critical point. So, we say that an interior point c is called a critical point, if either $f(x)$ is not differentiable at c or once it is differentiable, its derivative is equal to 0 at c. Such a point is a critical point. And it is defined only for the interior points.

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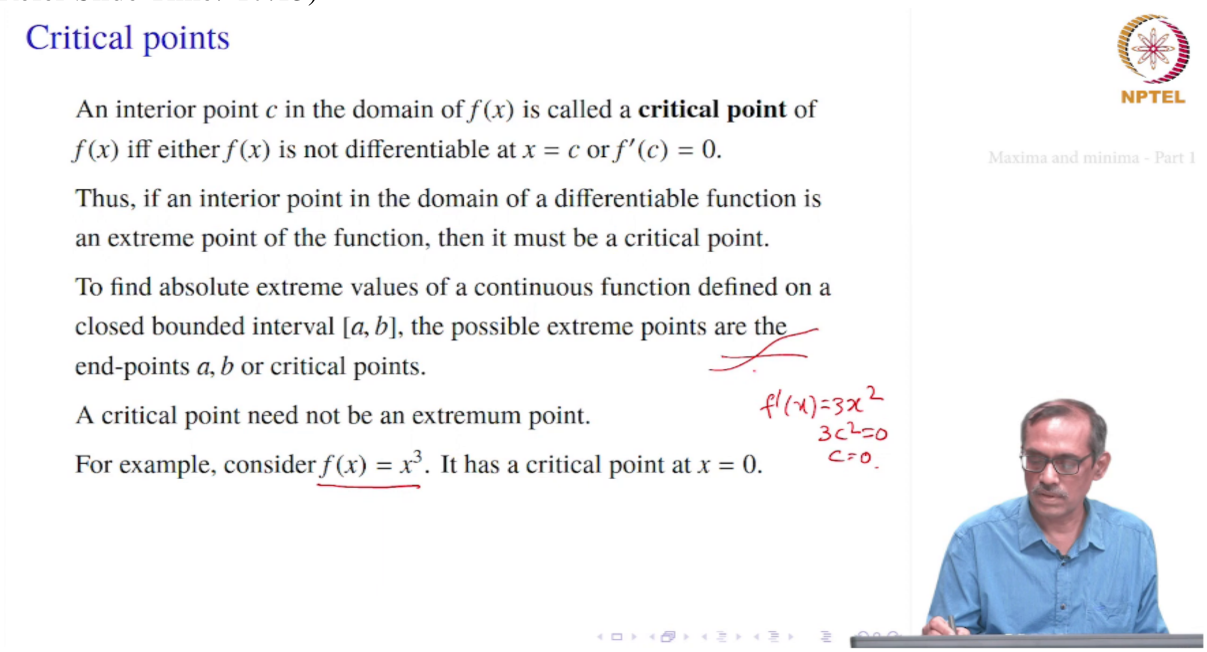

Our earlier theorem says that if $f(x)$ has a point of extremum at some interior point c, then c must be a critical point. If $f(x)$ is differentiable at c, then $f'(c) = 0$; if it is not differentiable, then also it is a critical point. Anyway it is a critical point. That is what our earlier theorem now says. If an interior point in the domain of a differentiable function is an extreme point of the function, then it must be a critical point. If it is not differentiable at that point, then also it is a critical point.

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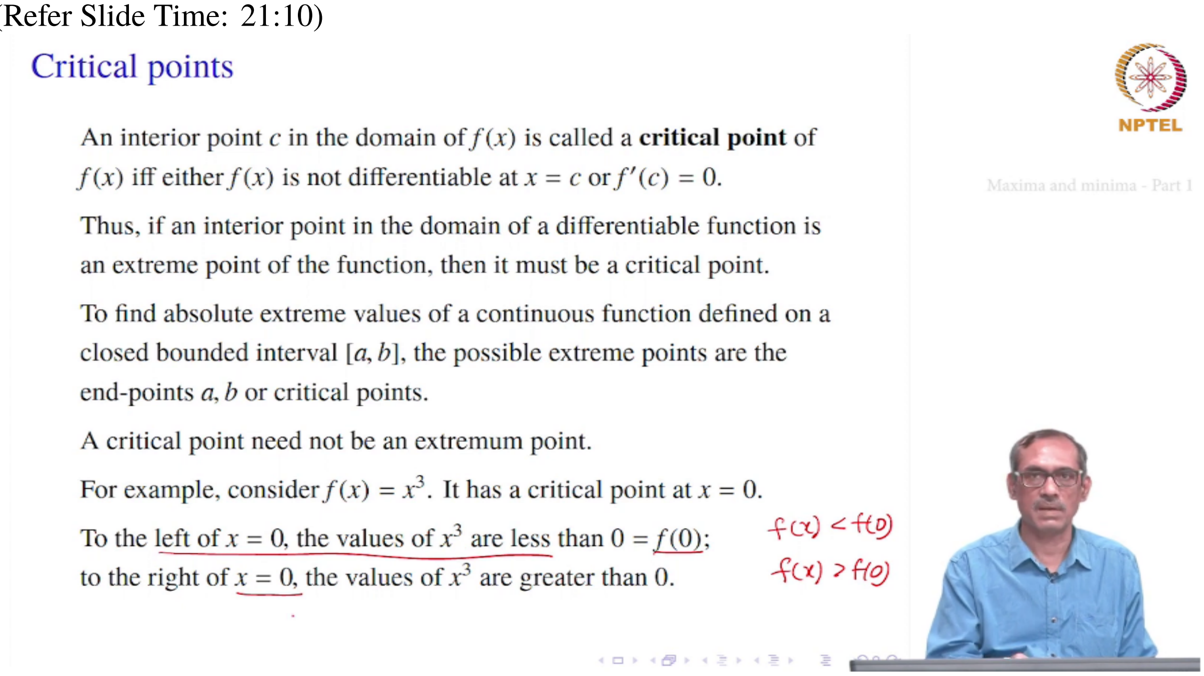

Now then, where to search for these extremum points? Suppose there is a maximum. Then, if it is an interior point, it has to be a critical point. It has to be a local maximum first; then it is a critical point. But c need not be an interior point, like the function is defined over a closed interval, where c is an endpoint. All that we have to search for absolute maximum are critical points and the endpoints. These are the points where the function may achieve its maximum values.

The same comment goes also for minimum values. And once $f(x)$ is bounded, our Extreme Value Theorem says that for this function, maximum and minimum values are attained. Where they will be attained? They will be either the endpoints or the critical points. That is how our search becomes limited. We will see how to apply it. But first, we should note that a critical point may not be an extreme point. It can be a false find for the candidate of maximum or minimum, like it can be local maximum, for example. But it need not be a local maximum, as we have seen from the graph. It can also happen that it is neither, it is not even a local extreme point. We will see such examples.

Well let us see one. Consider $f(x) = x^3$. It is defined all over $\mathbb{R}$. So, every point is an interior point. Now we differentiate, say, at any point c. You get $f'(x) = 3x^2$. At $x = c$, its value is $3c^2$. It has a critical point at $x = 0$ because $f'(c) = 0$ would give $c = 0$. So, $c = 0$ is a critical point of this. But $c = 0$ is neither a maximum nor a minimum. You can see the same thing from its graph also. Is that fine?

So, what does it say, why it is not so? Because we can find that to the left of $x = 0$, the values of x^3 are negative. They are less than 0, that is they are less than $f(0)$. We see that to the left, every $f(x)$ is less than $f(0)$. And to the right, x is positive, so that x^3 is positive. Then, you see that $f(x)$ is bigger than $f(0)$. That means $f(0)$ is neither a maximum, because there are values to the right of it which are bigger; and it is nor a minimum, because there are values to the left of this, which are less than that. So, it is neither; $x = 0$ is not an extreme point. But 0 is a critical point. So that can happen; our search can go in vain; but our search will be limited. That will help us in finding out the absolute maximum and absolute minimum.