

Basic Calculus - 1
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Lecture 9 - Part 1
Infinite Limits - Part 1

This is Lecture 9 of basic calculus 1. Recall that in the last lecture, we had talked about limits of functions at infinity. Which means if you have a function $f(x)$ defined at least on one interval of the type (a, ∞) , then you can talk of the limit of $f(x)$ as x goes to ∞ . Similarly, when $f(x)$ is defined on an interval of the type $(-\infty, b)$, you can talk of limit of $f(x)$ as x goes to $-\infty$. The basic idea was that a neighborhood of ∞ is of the form (a, ∞) , and a neighborhood of $-\infty$ is an interval of the type $(-\infty, b)$. We will explore that idea; but today we will be talking about infinite limits, that is, when the limit of the function will be equal to infinity. As you guess the function must be growing to become unbounded; it cannot remain near a real number, and it surpasses all real numbers, so to say. We will talk about it soon.

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Limit = ∞

1. Let the domain of $f(x)$ contain an interval of the form (c, b) . Then $\lim_{x \rightarrow c^+} f(x) = \infty$ iff corresponding to each $m > 0$, there exists a $\delta > 0$ such that for every x with $c < x < c + \delta$ we have $f(x) > m$.
 Informally, $\lim_{x \rightarrow c^+} f(x) = \infty$ iff as x decreases to c , $f(x)$ increases to ∞ without bound.
2. Let the domain of $f(x)$ contain an open interval of the form (a, c) . Then $\lim_{x \rightarrow c^-} f(x) = \infty$ iff corresponding to each $m > 0$, there exists a $\delta > 0$ such that for every x with $c - \delta < x < c$ we have $f(x) > m$.
 Informally, $\lim_{x \rightarrow c^-} f(x) = \infty$ iff as x increases to c , $f(x)$ increases without bound to ∞ .
3. $\lim_{x \rightarrow c} f(x) = \infty$ iff $\lim_{x \rightarrow c^+} f(x) = \infty = \lim_{x \rightarrow c^-} f(x)$.

$c \in (a, b) \setminus \{c\}$



Infinite limits - Part 1



Let us look at this. You have a function $f(x)$ whose domain contains an interval of the form (c, b) , where c, b are real numbers, where of course, c must be less than b . In this case, you may think of this (c, b) to contain a neighborhood of the form $(c, c + \delta)$ for some δ . In that case you can think of the right side limit at c . You think of the limit of $f(x)$ as x goes to $c+$. Now, we are talking about when this limit will be equal to ∞ . That means, it would grow to infinity or becomes unbounded, surpasses all the numbers, or positive real numbers.

We wish to say when the limit of $f(x)$ is equal to ∞ as x goes to $c+$. But, infinity is not a real number; that is why we are not able to use the earlier definition. We have to really redefine it. This means if you have some real number $m > 0$, then you can choose your x near c in such a way that

$f(x)$ should be bigger than m . That is how this idea of unboundedness idea is expressed. That is what we will write, but which neighborhood of c ? Since it is x goes to $c+$, that is only the right side limit, we would limit our search for x inside an interval of the form $(c, c + \delta)$. So, we define this way: corresponding to each $m > 0$, there exists a $\delta > 0$ such that if you take any x between c to $c + \delta$, then $f(x)$ must be greater than m . In that case, you would say that the limit of $f(x)$ as x goes to $c+$ is equal to ∞ .

Again, it is just a symbol ' $= \infty$ ', because ∞ is not a number. This limit is 'equal to infinity' is only a symbol, which gives the idea that the function increases, surpasses all the real numbers. It is very informal that way. Actually, what this definition tells is that for any $m > 0$, you will find one x near c and to the right of c so that $f(x) > m$. That is what you say informally, that $\lim f(x) = \infty$ if x decreases to c , because it is from the side, then $f(x)$ increases to infinity without bound.

Similarly, you can define the limit of $f(x)$ is equal infinity when x goes to $c-$. It means there is a left neighborhood of c contained inside the domain of definition of f . First of all, the domain of $f(x)$ must contain an open interval of the form (a, c) , a left neighborhood. In that case, we would say that this limit is equal to infinity, if it surpasses again any m . That is, if corresponding to each $m > 0$, there exists a positive δ such that if x is any number between $c - \delta$ to c , then its functional value $f(x)$ will be larger than m . Again informally, we would say that the limit of $f(x)$ is ∞ as x goes to $c-$ if as x increases to c , $f(x)$ increases without bound to infinity.

These are all the possibilities when limit is equal to ∞ . Similar things happen when the limit can be equal to $-\infty$. In that case, instead of $f(x)$ is bigger than m , we would take $f(x)$ is smaller than $-m$. That is the only change which would give limit is equal to $-\infty$.

Let us reformulate again. But before that we should give a comment. When do you say that the limit as x goes to c ? As earlier, we will take the limit as x goes to $c-$, as x goes to $c+$ and require that both the limits are same. In this case, both of them should be infinity. We should remember that this limit of $f(x)$ as x goes to c will be defined only when there is one open interval of the type (a, b) around c ; that is, c must belong to this and the domain of f must include such an interval, so that on both the sides you have a neighborhood.

This is actually a neighborhood of c . Of course, at c we do not need $f(x)$ to be defined. All that we need is that the function should be well defined on a deleted neighborhood of c , we do not require it to be defined at c . If it is defined, fine, but it is not required. So, if an interval such as $(a, c) \cup (c, b)$ should be inside the domain of definition of the function f , then only we can talk of the limit of $f(x)$ as x goes to c , and that limit is equal to ∞ if both these limits are equal to ∞ .

Let us go to $-\infty$ as we thought. Let the domain of $f(x)$ contain an open interval of the form (c, b) . Since we are talking about c, b , it is the right side neighborhood. We will be thinking of limit as x goes to $c+$. We said that the limit of $f(x)$ as x goes to $c+$ is $-\infty$ if corresponding to any $m > 0$, any positive m , we will be able to find one $\delta > 0$ such that if x is any number between c to $c + \delta$, because it is $c+$, what we have is $f(x) < -m$. It will be decreasing to $-\infty$ without any bound. Again informally, we would say this limit is equal to $-\infty$ if and only if as x decreases to c , $f(x)$ decreases without bound to $-\infty$.

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Limit = $-\infty$

4. Let the domain of $f(x)$ contain an open interval of the form (c, b) . Then $\lim_{x \rightarrow c^+} f(x) = -\infty$ iff corresponding to each $m > 0$, there exists a $\delta > 0$ such that for every x with $c < x < c + \delta$ we have $f(x) < -m$.
Informally, $\lim_{x \rightarrow c^+} f(x) = -\infty$ iff as x decreases to c , $f(x)$ decreases without bound to $-\infty$.
5. Let the domain of $f(x)$ contain an open interval of the form (a, c) . Then $\lim_{x \rightarrow c^-} f(x) = -\infty$ iff corresponding to each $m > 0$, there exists a $\delta > 0$ such that for every x with $c - \delta < x < c$ we have $f(x) < -m$.
Informally, $\lim_{x \rightarrow c^-} f(x) = -\infty$ iff as x increases to c , $f(x)$ decreases without bound to $-\infty$.
6. $\lim_{x \rightarrow c} f(x) = -\infty$ iff $\lim_{x \rightarrow c^+} f(x) = -\infty = \lim_{x \rightarrow c^-} f(x)$.

$(a, c) \cup (c, b)$



Infinite limits - Part 1



For x goes to minus $c-$, a similar thing happens. But now, we will require one neighborhood $(c - \delta, c)$. The domain of f must contain an open interval of the type (a, c) . In this case we would say that the limit of $f(x)$ as x goes to $c-$ is equal to $-\infty$ whenever for any $m > 0$, there exists a positive δ such that if $x \in (c - \delta, c)$, then $f(x) < -m$. Informally, we would say that this limit is equal to $-\infty$ if whenever x increases to c , $f(x)$ decreases without bound to $-\infty$. It is increasing because it is on the left side neighborhood.

These are about the limit being equal to infinity, or equal to minus infinity at a real number c . We can combine both the limits and say that the limit of $f(x)$ as x goes to c is equal to $-\infty$ if both the left side limit and the right side limit are equal to $-\infty$. In that case again we should have one neighborhood around c which is contained the domain of definition of f . Of course, we are talking about a deleted neighborhood; that is, of the type $(a, c) \cup (c, b)$, which should be a subset of the domain of f . In that case only we can talk of the limit of $f(x)$ as x goes to c . These are about a point c , which is a real number; around that c , we are finding the limit, and the limit can become equal to infinity or minus infinity. That is why they are called infinite limits.

But it can happen that we are able to combine both the things; and talk about limit at infinity becoming equal to infinity. That means you should consider also infinite limits at ∞ and at $-\infty$. Let us consider first the limit at ∞ . Our idea of the neighborhoods of infinity will be helpful. We are talking about limits at infinity. Let us think about the limit of $f(x)$ as x goes to ∞ is equal to ∞ . When does it happen? It means, we should consider a neighborhood of ∞ as earlier we took a neighborhood of c since it is $x \rightarrow \infty$. It is only the left side limit to infinity, imagining ∞ as a point there on the extreme right; so it is a left side limit.

In that case what will happen? First of all we should have a left side neighborhood of ∞ which should be contained inside the domain of definition of f . That gives rise to formally an interval of the type (a, ∞) ; and it must be a subset of the domain of f . So, the function f is defined over

such an interval. Then we will say that the limit of $f(x)$ as x goes to ∞ is equal to ∞ if and only if corresponding to each $m > 0$, there exists a $\delta > 0$ such that for every x with x belonging to a neighborhood of ∞ , that is, $x > \delta$ now, we should have $f(x) > m$.

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Infinite limits at ∞

7. Let the domain of $f(x)$ contain an open interval of the form (a, ∞) . Then $\lim_{x \rightarrow \infty} f(x) = \infty$ iff corresponding to each $m > 0$, there exists a $\delta > 0$ such that for every x with $x > \delta$ we have $f(x) > m$.
Informally, $\lim_{x \rightarrow \infty} f(x) = \infty$ iff as x increases without bound to ∞ , $f(x)$ also increases without bound to ∞ .
Notice that $\lim_{x \rightarrow \infty} f(x) = \infty$ iff $\lim_{x \rightarrow 0^+} f(1/x) = \infty$.
8. Let the domain of $f(x)$ contain an open interval of the form (a, ∞) . Then $\lim_{x \rightarrow \infty} f(x) = -\infty$ iff corresponding to each $m > 0$, there exists a $\delta > 0$ such that for every x with $x > \delta$ we have $f(x) < -m$.
Informally, $\lim_{x \rightarrow \infty} f(x) = -\infty$ iff as x increases without bound to ∞ , $f(x)$ decreases without bound to $-\infty$.
Again, $\lim_{x \rightarrow \infty} f(x) = -\infty$ iff $\lim_{x \rightarrow 0^+} f(1/x) = -\infty$.



Infinite limits - Part 1



It is in the neighborhood of infinity on the other side. That is what we write: corresponding to each $m > 0$, there exists a $\delta > 0$ such that for every x with $x > \delta$, we have $f(x) > m$. If such a thing is satisfied, then we would say that the limit of $f(x)$ is equal to ∞ as x goes to ∞ . Again informally, you can say that as x increases without bound to ∞ , $f(x)$ also increases without bound to ∞ .

You can think of this limit of $f(x)$ equal to ∞ as x goes to ∞ by making a substitution of x with $1/x$. You say that the limit of $f(1/x)$ is equal to ∞ as x goes to 0^+ . It is now reversed. As x goes to 0^+ , $1/x$ goes to ∞ . So, the limit of $f(1/x)$ is equal to ∞ as x goes to 0^+ is same as $1/x$ goes to ∞ . Sometimes this substitution helps because at infinity we may not be able to imagine correctly, but 0 can be easier. Of course, it depends on the problem; we will see.

Next, we will be talking about the limit is equal to $-\infty$ as x goes to ∞ . Informally, it should say that this is equal to $-\infty$ whenever if x increases without bound to ∞ , then $f(x)$ should be decreasing without bound to $-\infty$. That is what we want to formalize. Well, the formalization will be as earlier. If you take any $m > 0$, then our $f(x)$ should be less than $-m$, but for which x ? For all those x , which are greater than δ for some $\delta > 0$. You would say that corresponding to each $m > 0$, there exists a $\delta > 0$ such that if we take any $x > \delta$, x goes to ∞ , so, $x > \delta$, then $f(x)$ would be less than $-m$ so that it decreases to $-\infty$. So, these are our definitions. You need not really mug them up; they can be obtained from our usual definition by remembering the neighborhood concept of ∞ and of $-\infty$ as we have discussed earlier.

So, this is about the limit as x goes to infinity, when they become ∞ , or when they become $-\infty$. Similarly we have the notions of the limit of $f(x)$ equal to ∞ or $-\infty$ as x goes to $-\infty$.

(Refer Slide Time: 14:46)

Infinite limits at $-\infty$

9. Let the domain of $f(x)$ contain an open interval of the form $(-\infty, a)$. Then $\lim_{x \rightarrow -\infty} f(x) = \infty$ iff corresponding to each $m > 0$, there exists a $\delta > 0$ such that for every x with $x < -\delta$ we have $f(x) > m$.
- Informally, $\lim_{x \rightarrow -\infty} f(x) = \infty$ iff as x decreases without bound to $-\infty$, $f(x)$ increases without bound to ∞ .
- Here, $\lim_{x \rightarrow -\infty} f(x) = \infty$ iff $\lim_{x \rightarrow 0^-} f(1/x) = \infty$.
10. Let the domain of $f(x)$ contain an open interval of the form $(-\infty, a)$. Then $\lim_{x \rightarrow -\infty} f(x) = -\infty$ iff corresponding to each $m > 0$, there exists a $\delta > 0$ such that for every x with $x < -\delta$ we have $f(x) < -m$.
- Informally, $\lim_{x \rightarrow -\infty} f(x) = -\infty$ iff as x decreases without bound to $-\infty$, $f(x)$ also decreases without bound to $-\infty$.
- Once more, $\lim_{x \rightarrow -\infty} f(x) = -\infty$ iff $\lim_{x \rightarrow 0^-} f(1/x) = -\infty$.



Infinite limits - Part 1



Let us consider those two cases. What happens there? We want the limit at $-\infty$, and also infinite limits. The limits can be $\pm\infty$. How to define those cases? When you say that the limit as x goes to $-\infty$, we should have some interval of the type $(-\infty, a)$ inside the domain of f . Then only we can define it. Assume that it happens for this function f . Then we say that the limit of $f(x)$ is equal to ∞ as x goes to $-\infty$ if corresponding to each $m > 0$; our $f(x)$ should be bigger than m , but for which x ? For all those x which are in a neighborhood of $-\infty$, that is, $x < -\delta$ for some δ .

So, the formal definition would look like this: corresponding to each $m > 0$, there exists a $\delta > 0$ such that for each x with $x < \delta$ or $x < -\delta$ because it is $-\infty$, we have $f(x) > m$. Informally we would say that this happens when x decreases without bound to $-\infty$, $f(x)$ increases without bound to ∞ .

These are all the cases from playing with $\pm\infty$. We have one more cases, the case that when limit of $f(x)$ is equal to $-\infty$ for x going to $-\infty$. In that case we should again have $(-\infty, a)$ to be contained inside the domain of f for some a , and then we would be able to talk of the limit of $f(x)$ is $-\infty$ as x goes to $-\infty$. The condition is: if corresponding to each $m > 0$, there exists a δ such that for every x with $x < -\delta$, its corresponding value $f(x)$ is less than $-m$. Both are now minus. Again if you want to substitute, you can substitute this way. As x goes to 0^- , the limit of $f(1/x)$ is equal to $-\infty$; that will be the substitution. Informally it would say that as x decreases without bound to $-\infty$, its functional value also decreases without bound to $-\infty$.

These are all the notions connected with the limits at infinity and limits becoming infinity. We should see some examples. These are the cases which are really clumsier a bit, when both the things are infinity: x as well as limit of $f(x)$.

Let us see an example. Suppose we have a polynomial $p(x) = a_0 + a_1x + \dots + a_nx^n$. Since it is of degree n we would assume that $a_n \neq 0$. If $a_n = 0$, then its degree can become smaller. As $p(x)$ is a polynomial of degree n , $a_n \neq 0$. And, suppose I choose a number b , which is bigger than 1.

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Examples

Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$, where $a_n \neq 0$. Let $b > 1$.

$$1. \lim_{x \rightarrow \infty} \frac{p(x)}{x^n} = \lim_{x \rightarrow \infty} \left(\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + \frac{a_{n-1}}{x} + a_n \right) = a_n.$$

2. Using (1),

$$\lim_{x \rightarrow \infty} p(x) = \lim_{x \rightarrow \infty} \frac{p(x)}{x^n} \lim_{x \rightarrow \infty} x^n = a_n \times \infty = \begin{cases} \infty & \text{if } a_n > 0 \\ -\infty & \text{if } a_n < 0. \end{cases}$$

We say that as x approaches ∞ , $p(x)$ behaves like its leading term.



Infinite limits - Part 1



The first problem asks to find the limit of $p(x)/x^n$. That is pretty easy. We just divide x^n throughout. You get

$$\frac{a_0}{x^n} + \frac{a_1}{x^{n-1}} + \cdots + \frac{a_{n-1}}{x} + a_n.$$

As $x \rightarrow \infty$, $1/x \rightarrow 0$. So, this term a_0/x^n goes to 0; this also goes to 0, everything goes to 0, except the last term. Therefore, you get the limit as a_n .

A formal justification will be required. That can be done by using the fact that when x goes to ∞ , $1/x$ goes to 0, which we have seen already while considering limit at infinity. So, this is how we proceed to obtain the limit of $p(x)/x^n$ as a_n as x goes to ∞ .

We will go to the next problem. It asks for finding the limit of $p(x)$ as x goes to ∞ . Here, it is $p(x)$ itself and not $p(x)/x^n$. As we have computed the limit for $p(x)/x^n$, we can use that. That says, you can break it into this provided both the limits exist. We know this one; we do not know the right one; but it is not the existence of limit now. What it says; limit of $p(x)/x^n$ as x goes to ∞ is something which is nonzero. So, when you multiply that with something which is growing to infinity (unbounded), then that number multiplied by that unbounded thing will become an unbounded quantity, the same way.

So, suppose $a_n > 0$. Then this limit will be a_n and something which is bigger than m in our definition. That becomes again bigger than m ? Not exactly. We can choose m/a_n . Then, a_n will cancel, and it will be bigger than m . That is how the formal definition is. For any given m , you first think of m/a_n . Suppose $a_n > 0$. Then, $m/a_n > 0$. Then, there exists a δ such that whenever $x > \delta$, we have $f(x) > m/a_n$. Now, that will be multiplied with a_n . That will confirm: for all those x , we have $p(x) > m$. Therefore, it will be infinite.

We write that in a shorter form: $a_n \times \infty$. This does not make any sense, really. Its formal sense comes from the definition. That gives us $a_n \times \infty$, as we write informally, which will be really ∞ provided $a_n > 0$.

Similarly, if $a_n < 0$, then you choose one $m > 0$ and consider $m/(-a_n)$, because a_n is negative, so that $m/(-a_n)$ is positive. So it goes to $-\infty$ provided you have a $\delta > 0$ such that if $x > \delta$, then this is less than $-m/(-a_n)m$, which is m/a_n . Again, a_n is multiplied; so, that makes $f(x) < -m$. Then, it will be $-\infty$. So, we write $a_n \times (-\infty) = -a_n \times \infty = -\infty$. In this case, a negative number times some positive m will give you $-m$. That is what it means formally. Informally, we write $a_n \times \infty = -\infty$ when a_n is negative.

Here, that means the limit of $p(x)$ is equal to ∞ as x goes to infinity if the leading coefficient of $p(x)$ is bigger than 0, and it is $-\infty$ if the leading coefficient of $p(x)$ is smaller than 0. So, $p(x)$ behaves almost like its leading term, that is what we say, but only in sign. Answer is infinity anyway, whether plus or minus.

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Example 3

Find the infinite limits and limits at infinity for $f(x) = \frac{x+3}{x+2}$.

$$f(x) = 1 + \frac{1}{x+2}. \text{ Thus, } \lim_{x \rightarrow -2^-} f(x) = -\infty \text{ and } \lim_{x \rightarrow -2^+} f(x) = \infty.$$

$x < -2 \implies x+2 < 0$ $x > -2 \implies x+2 > 0$



Infinite limits - Part 1



Let us go to the next problem. In this example, we will find the infinite limits, that is, when the limit become infinite, and limits at infinity of this function $f(x)$. All the four things are there, when the limits become $\pm\infty$, what are the limits at $\pm\infty$. The function is given as $f(x) = (x+3)/(x+2)$.

It will be convenient to rewrite it. We will write it as this way. It is correct, $x+2+1 = x+3$ in the numerator. So, $f(x) = 1 + 1/(x+2)$. First, let us find limits at ∞ . If you take limits at infinity that will be pretty easier here. Because it would say that $1/(x+2)$ as $x \rightarrow \infty$ is 0. So, the answer should be 1, even at $-\infty$, the same thing happens.

And, when does the limit become infinity? As we feel, if x is closer to -2 , then this expression $x+2$ becomes closer to 0. So, $1/(x+2)$ will be increasing or decreasing without bound to ∞ or to $-\infty$. Let us check.

I take $x < -2$ and x is near -2 . Then, $x+2 < 0$; it is negative, $x+2$ is negative. When you take $1/(x+2)$, it should go to $-\infty$. This limit, when x goes to -2 from the negative side, from the left side, it becomes $-\infty$. This one really does not matter because it is already $-\infty$. That gets absorbed. Now, let us take $x > -2$ and x is near -2 . Then, $x+2 > 0$, and $1/(x+2)$ goes to $+\infty$. This one gets

absorbed, so you get the limit as ∞ as x goes to $-2+$.

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Example 3

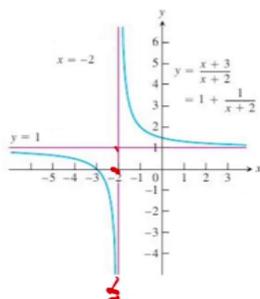
Find the infinite limits and limits at infinity for $f(x) = \frac{x+3}{x+2}$.

$f(x) = 1 + \frac{1}{x+2}$. Thus, $\lim_{x \rightarrow -2^-} f(x) = -\infty$ and $\lim_{x \rightarrow -2^+} f(x) = \infty$.

Also, for no other point c , limit of $f(x)$ is $\pm\infty$ as x approaches c .

So, $x = -2$ is the only point where the limit of $f(x)$ is $\pm\infty$.

Next, $\lim_{x \rightarrow \infty} f(x) = 1$. And, $\lim_{x \rightarrow -\infty} f(x) = 1$.



Infinite limits - Part 1



If you take any other point c that is, $c \neq -2$, then this remains bounded. So, it is never $\pm\infty$ as x approaches c . Of course, the limit exists. We could say that this blows up at $x = -2$ and everywhere else, it approaches a real number. If you take any other point, any point other than -2 , then it will have some limit, not $\pm\infty$.

Next, when you compute the limit at ∞ , as we commented, $f(x)$ will approach 1. In fact, at $-\infty$ also the limit is 1. Its plot looks like this. This is the function $(x+3)/(x+2)$. On both sides of the line $x = -2$, it looks like this. It says that as x approaches -2 from the left side, so this one is appropriate, $f(x)$ goes to $-\infty$. On the other side, it says that as x approaches -2 from the right, $f(x)$ approaches ∞ . Sometimes this line is called an asymptote. Anyway we are not going to discuss that in detail.

Now, when you find that the limit of $f(x)$ as x goes to ∞ is equal to 1, what does that mean in the graph? It means that y becomes 1 when x goes to ∞ . When x increases to ∞ , y is remaining near 1. So, the line $y = 1$ is also an asymptote here. Similarly, on the other side, the limit of $y = f(x)$ as x goes to $-\infty$ is also equal to 1. On this side also, it reaches 1, that is, as if it is touching the line $y = 1$ at infinity. That is what we plot in the picture.