Basic Calculus - 1

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Lecture 8 - Part 1

Limits at Infinity - Part 1

This is lecture 8 of basic Calculus 1. We recall what we did in the last lecture. We introduced the one-sided limits, that is, limit of a function as $x \to a+$ or as $x \to a-$. That means, instead of a neighborhood of the point a, where we are concerned about the limit, we are taking only right neighborhood or left neighborhood of a, where x will be varying. That is one-sided limit. Today, we will be talking about limits at infinity. It means, we will be taking the limit when x goes to $+\infty$ or x goes to $-\infty$. But there is a problem because ∞ and $-\infty$ are not real numbers.

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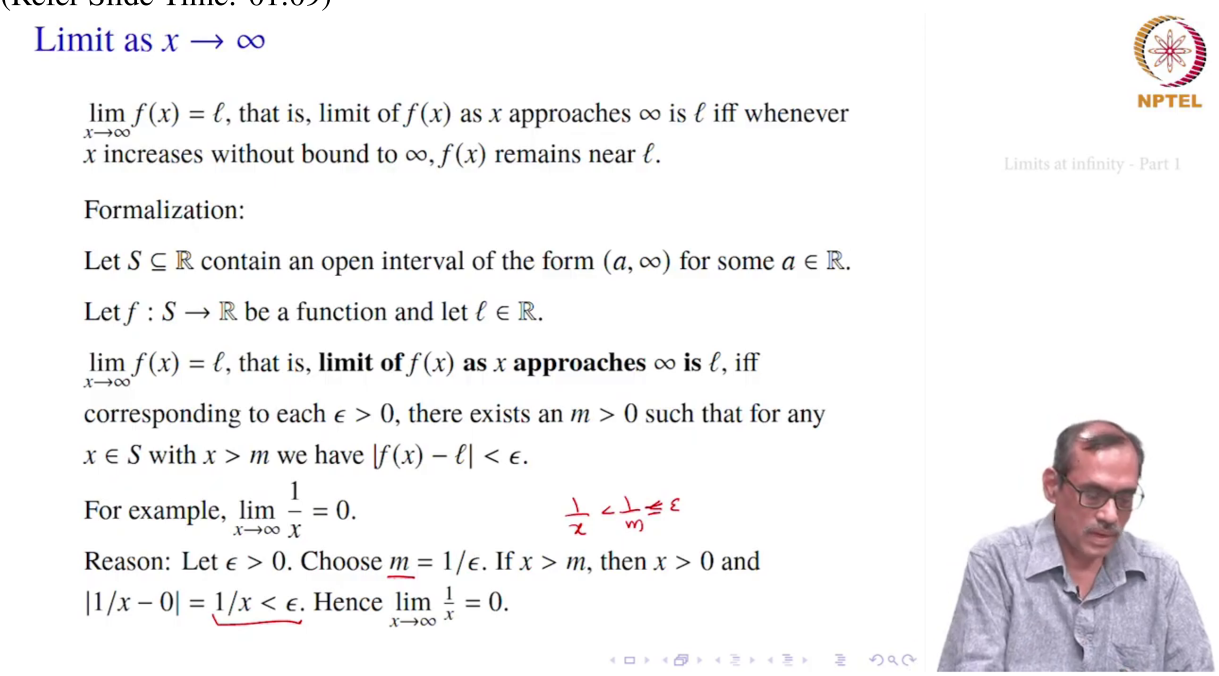

When we say that limit as x goes to ∞, we would be thinking that x should lie near infinity. But as infinity is not a real number, what is the meaning of near infinity? Intuitively, what it would mean is that x increases without bound. That is what we mean when x approaches infinity. That means, x takes positive values and it goes on increasing. That is the intuitive idea. We recall the one-sided limits. When we say limit as x goes to $c-$, x lies in a neighborhood of c, which is a left neighborhood such as $(c - \delta, c)$. Here, ∞ is thought of as if it is a very large one, it is not a number, but it is beyond all the large positive numbers.

So, what do you mean by a neighborhood of infinity? It would mean that x increases without bound; that means, we will be concerned about points x, which are to the right of any real number, so to say. So, any neighborhood of infinity will be an interval in the form (a, ∞), where a is any real number. We take the open interval; that means, all numbers greater than a.

That is how we will be thinking about a neighborhood of infinity. Then our earlier definitions of limit would now work. To be exact, we will take the left side limit, because it is approaching infinity from the left. Now you may think of a neighborhood of infinity this way and go for a formalization. So, suppose S is a subset of $\mathbb{R}$; it should contain an open interval of the form (a, ∞). That means in the domain of definition of our function, a left neighborhood of infinity should be there. Of course, there is only one kind of neighborhoods here of infinity, the left one. There is no right one; since there is no real number greater than that. In fact, that itself is not a real number.

We will be thinking of functions defined from a subset S of $\mathbb{R}$ to $\mathbb{R}$, where S contains a neighborhood of infinity, an interval of the form (a, ∞). Then, we will be considering all those x bigger than a. Next, we should formalize the notion of nearness. We look at the definition of the left-side limit of $f(x)$ to be equal to ℓ as $x \to c-$. You will say "corresponding to each $\epsilon > 0$, there exists a left neighborhood of c such as $(c - \delta, c)$ such that when $x \in (c - \delta, c)$, $|f(x) - \ell| < \epsilon$". That would have been the definition when it is $c-$. Now, it is infinity; its neighborhood would be something of the form (a, ∞), or let us say, all $x > m$ for some m. So, the definition would be "corresponding to each $\epsilon > 0$, there should exist one a real number $m > 0$". Usually m denotes a natural number, but here it is taken as a real number. We are concerned about (a, ∞). So, we can still say m is bigger than 0; even bigger than 100 is all right; it would be the same thing. That m is bigger than 0 we take any $x \in S$ with $x > m$. Now since S contains (a, ∞), we may say this as $x > m$ for some m so that automatically it is all such x are in S. That is, m should be large so that this interval which is (a, ∞) is contained in S. That is we can choose our m to be bigger than a. So, corresponding to each $\epsilon > 0$, there should exist one $m > 0$ such that for any $x \in S$ with $x > m$, this condition of nearness of $f(x)$ to ℓ should be satisfied; that is, $|f(x) - \ell| < \epsilon$. This would be our formal definition of limit of $f(x)$ equal to ℓ as $x \to \infty$.

We repeat. The limit of $f(x)$ as $x \to \infty$ is equal to ℓ, or, $f(x)$ approaches ℓ as x approaches ∞, if corresponding to each $\epsilon > 0$, there is one $m > 0$ such that whenever $x > m$, we have $|f(x) - \ell| < \epsilon$. This is how we would be defining the limit at infinity.

For example, consider the function $1/x$, which is defined for all $x > 0$. So, the domain is $(0, \infty)$, the point 0 is excluded. Your S is that. Obviously, S contains any (a, ∞) where $a > 0$. To see this limit condition is satisfied, we guess that the limit should be 0. Because, once x becomes larger and larger, $1/x$ becomes smaller and smaller. So, you say that $1/x$ becomes 0; that is our guesswork. Let us see how to justify it.

That means, corresponding to each $\epsilon > 0$, we should have one m such that whenever $x > m$, $|1/x - 0| < \epsilon$. So, how do we find this m? We have to find one m corresponding to ϵ. So, suppose $\epsilon > 0$ is given, we choose m to be $1/\epsilon$. That is what $1/x$ suggests. Now, if $x > m$, then $1/x < 1/m$. We need $1/x < \epsilon$. So, we may take m so that $1/m < \epsilon$ or even $1/m = \epsilon$. So, we choose $m = 1\epsilon$. We should then verify. If $x > m$, then $x > 0$ and $|1/x - 0| = 1/x < 1/m = \epsilon$. So, $1/x < \epsilon$. Therefore, the limit of $1/x$ is equal to 0 as $x \to \infty$.

We are just checking whether our notion of nearness of $1/x$ to 0 as x becomes very large; and this is justified by our definition. All that you have to do is choose m appropriately. So, you choose

$m = 1/\epsilon$. Even if you choose m to be smaller than $1/\epsilon$, that will also work.

This is the notion of limit of a function as x goes to infinity formalizing the intuitive idea of x becoming larger and larger.

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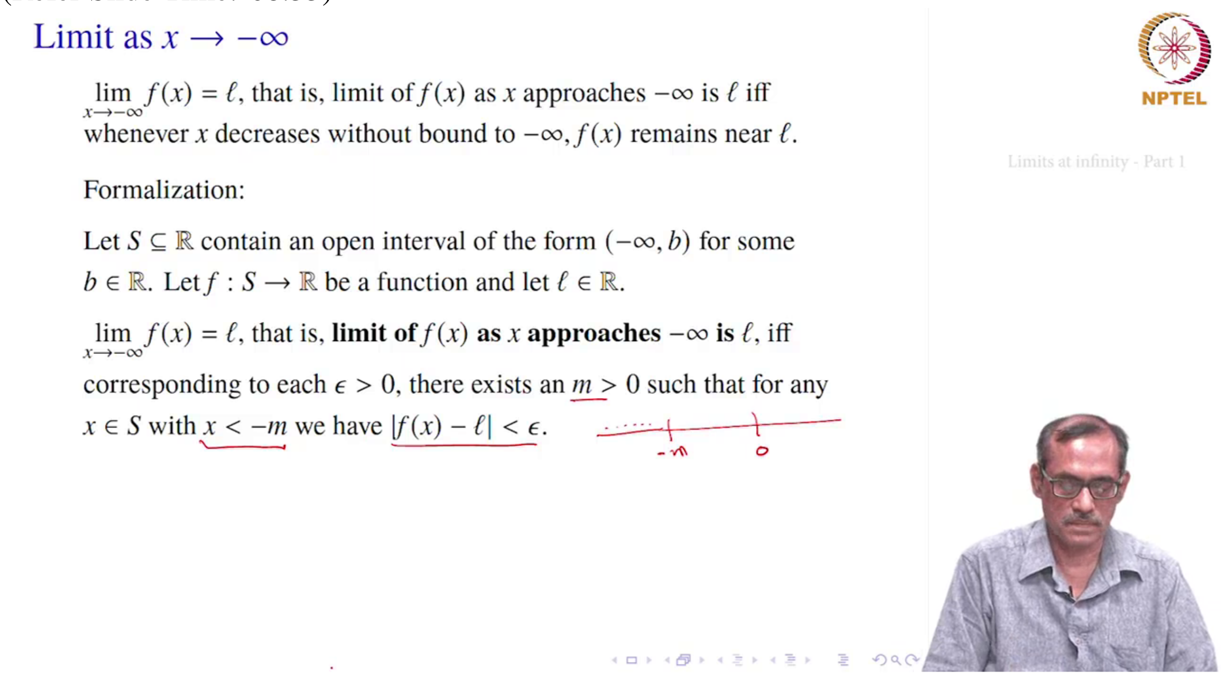

We will be concerned about the other infinity; that is, limit as x goes to $-\infty$. Again, the intuitive idea is that x approaches minus infinity means x becomes smaller and smaller without any bound even in the negative. To formalize, we would think a neighborhood of minus infinity. Our intuition is that infinity is to the right of every real number, and minus infinity is to the left of every real number. To the left means it will be negative; that is why it is minus infinity. But then, infinity and minus infinity are not real numbers. So, what is the notion of a neighborhood of minus infinity?

A neighborhood of ∞ is taken as an interval in the form (a, ∞). Similarly, we take a neighborhood of $-\infty$ as an interval in the form $(-\infty, b)$. You can take b to be negative here; that is okay. Also, you can take b as any real number. We may think of $(-\infty, b)$ as a neighborhood of $-\infty$. Now, the limit as $x \to -\infty$ will be really something like $x \to c+$; we are approaching $-\infty$ from the right side. That is the intuition.

For the limit, when x is near $-\infty$, we would see that $f(x)$ remains near ℓ. As earlier, when we say that the limit $x \to c+$, we would consider "for every $\epsilon > 0$, there is a δ such that if $x \in (c, c+\delta)$". This is suitably translated to $x \in (-\infty, b)$ For $x \in (-\infty, b)$, we should have $|f(x) - \ell| < \epsilon$. That is exactly the definition of limit. We would start with any subset S of $\mathbb{R}$, which should contain some interval of this form. Once it contains an interval of this form; $(-\infty, b)$, you see that if you take anything smaller than b, say $(-\infty, d)$, where $d < b$, then that $(-\infty, d)$ is also contained in S. That is a neighborhood of $-\infty$ that is contained in S. This is our precondition for defining the limit, where S is the domain of the function and ℓ is any real number. In such a case, we will define the limit of $f(x)$ is equal to ℓ as $x \to -\infty$. And that will hold if and only if corresponding to each

$\epsilon > 0$, there exists one number, instead of b, we are writing $m > 0$ such that whenever $x < m$; you are taking negative m always, now, because anyway, we are concerned with the neighborhood of $-\infty$, x should be less than $-m$; for all such x, we have $|f(x) - \ell| < \epsilon$. That means corresponding to each $\epsilon > 0$, you should get some m such that whenever x lies here, it is smaller than $-m$, m will be some bigger number, $-m$ will be smaller number. This is 0. Then, for all such x, $f(x)$ is lying ϵ-close to ℓ in that neighborhood. That is, $f(x)$ should lie between $\ell - \epsilon$ to $\ell + \epsilon$. We said that $f(x)$ remains near ℓ when x lies near $-\infty$. That is the notion which is formalized here.
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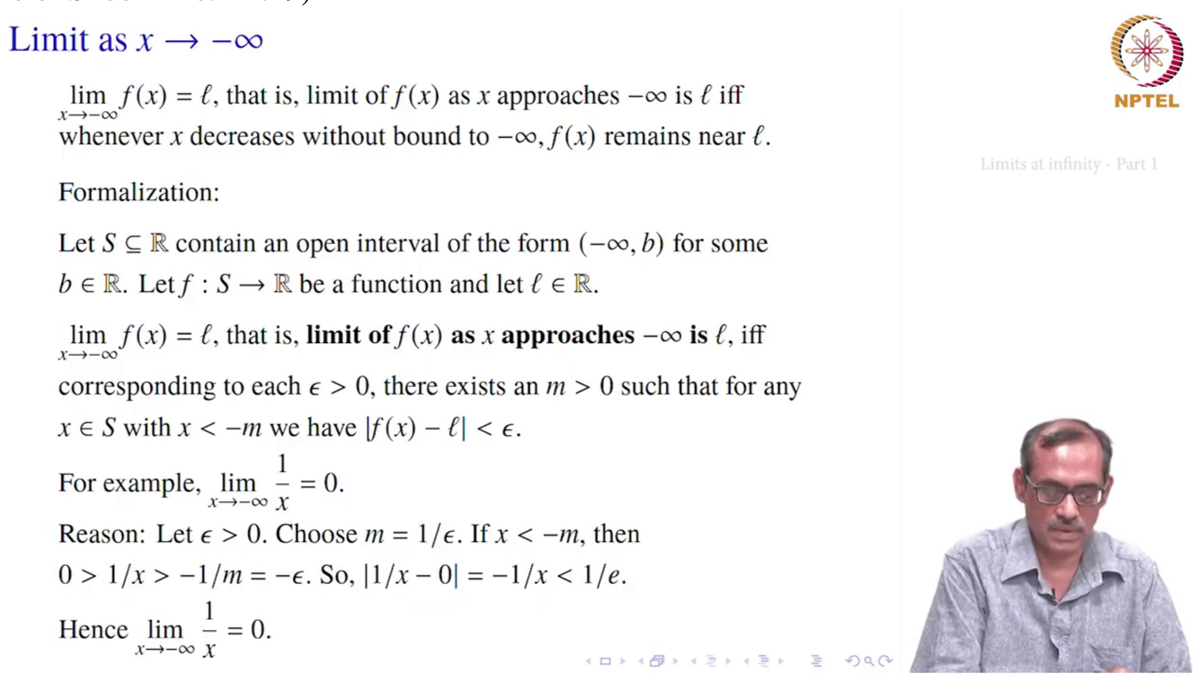

Let us see our earlier function $1/x$. Intuitively, it says that when x becomes smaller and smaller to ∞, $1/x$ becomes larger and larger. On the other hand, if you say x becomes bigger and bigger in absolute value, but remains negative, that is, x is chosen this way, then what happens to $1/x$? $1/x$ should be coming back here, and it should become closer to 0. That is what our intuition says.

So, we are going to check limit as x goes to $-\infty$, $1/x$ should be equal to 0. How do we check from the formal definition? That means, given $\epsilon > 0$, we should be able to find one $m > 0$ such that whenever $x < -m$, $|1/x|$ should be less than ϵ. As earlier we will be choosing our m. So, suppose $\epsilon > 0$. Again, we choose $m = 1/\epsilon$. Whenever $x < -m$, we would find that $1/x$ be compared with $-1/m$. So, that is the idea of choosing this m this way. So, let us choose this m, which is $1/\epsilon$. Now, suppose $x < -m$. Then we find that $1/x > -1/m$. And, since x is negative, $1/x$ is also negative. But $x < -m$. So, what happens to $1/x$? That is still negative, so $1/x < 0$. That is what we write here. Then $0 > 1/x$; $1/x$ is negative. And it is bigger than $-1/m$. But $-1/m = -\epsilon$. So, we get $1/x > -\epsilon$; so, $|1/x - 0|$, which is $|1/x|$; since x is negative, it becomes $-1/x$. Since $1/x > -\epsilon$, you would say $-1/x < 1/\epsilon$. And that is what we see, this $1/(1/\epsilon)$ would give you only ϵ. So $-1/x > -\epsilon$. So, $1/x < \epsilon$. That is what it is. And that condition is satisfied, that is, for any $\epsilon > 0$ given, we choose $m = 1/\epsilon$; we find that if $x < -m$, then $|1/x - 0| < \epsilon$. Therefore, the limit condition is satisfied.

And our claim is proved. So, the limit of $1/x$ as x goes to $-\infty$ is equal to 0. We also had the limit of $1/x$ as x goes to ∞ is equal to 0. In both the cases it goes to 0. That is what we have seen. But it may not happen always; here this $1/x$ is looking very symmetric, for positive or negative x.

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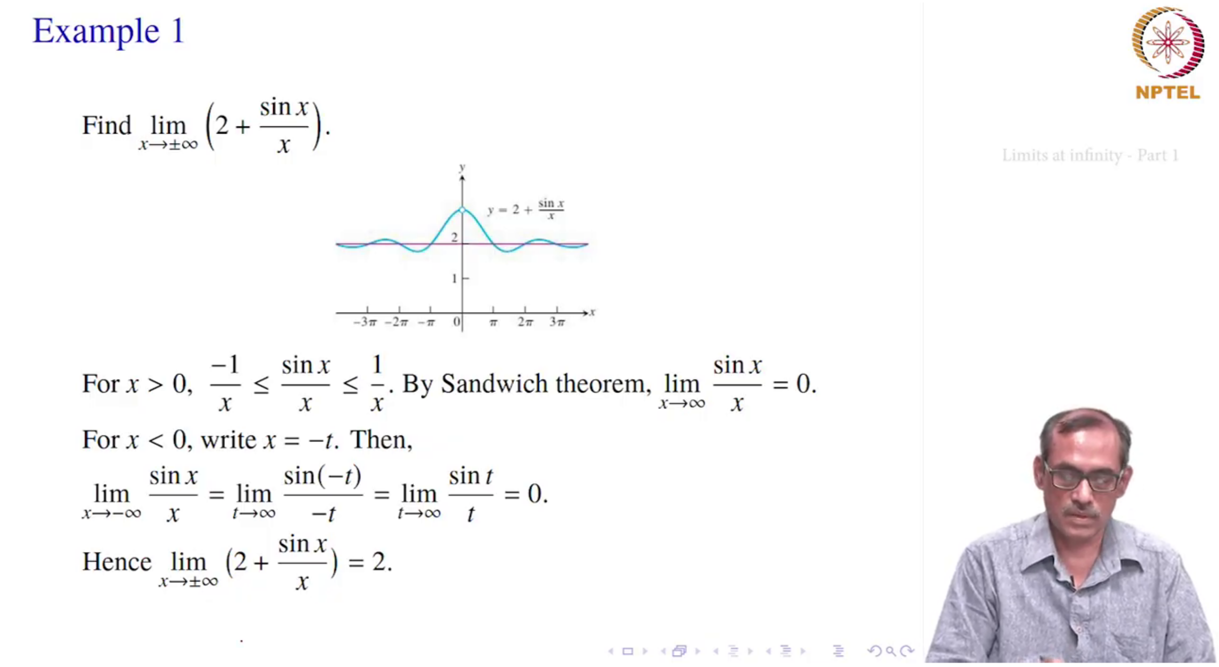

Let us take some more examples and see what happens. Suppose I have the function as $2 + (\sin x)/x$. This is the function. We are required to find its limit as x goes to $+\infty$, or x goes to $-\infty$. Basically, we will be concerned about finding the limit of $(\sin x)/x$. How does the function look like? It is just shifted by 2; it is $2 + (\sin x)/x$. This, the curve of $(\sin x)/x$, starting from 0, so shifted 2 units, so the curve looks like this, the blue one. This is $y = 2 + (\sin x)/x$. As we see, when x goes to ∞, this oscillation becomes smaller and smaller; it becomes 0. Similarly, on the left side, this 2 remains; so the limits would be 2. That is our guess. Let us find out how does it happen.

Suppose $x > 0$. We are trying to find the limit as x goes to ∞ now, we will tackle $-\infty$ separately. For that, we can assume $x > 0$, even we can take $x > 100$ because x would be lying closer to infinity. So, suppose $x > 0$. For this x, this types of x, we have this inequality satisfied, because $\sin x$ lies between -1 to 1, and x is positive. So, $-1/x \le (\sin x)/x \le 1/x$. Why are you doing it? Obviously, we will be using Sandwich theorem because we already know the limit of $-1/x$ and of $1/x$ as x goes to ∞. We know that limit of $1/x$ as x goes to ∞ is 0. What about $-1/x$? Well, it is just a minus sign, so that is also going to 0. By Sandwich theorem, we say that the limit of $(\sin x)/x$ is 0; because this goes to 0 as $x \to \infty$, this goes to 0 as $x \to \infty$.

Remember that we have proved or mentioned Sandwich theorem only for the limits when $x \to c$. But that also holds for $x \to \infty$, because it is the same neighborhood idea. By using the neighborhood idea again here, Sandwich theorem also holds for the case $x \to \infty$. With that in mind, we have done this $-1/x$ to $1/x$ estimates and that gives us the limit as x goes to infinity of

$(\sin x)/x$ as 0.

Now, what about the limit as x goes to $-\infty$? Well, for that, imagine $x < 0$. Then, instead of doing it fresh again, because we have done already $(\sin x)/x$, we will substitute x with $-t$. We are going to do some limiting or limit calculation here. The limit as x goes to $-\infty$ is same thing as the limit as t goes to ∞, because $t = -x$, or $x = -t$. And what happens to the $\sin x$ in terms of t? It is $[\sin(-t)]/(-t)$. But $\sin(-t) = -\sin t$. So, these two minus signs cancel and you get the limit as t goes to ∞ as it is; the ratio becomes $(\sin t)/t$. That is same as $(\sin x)/x$ as $x \to \infty$. The variable is x or t, it does not matter, because it is the same neighborhood idea. So, this is equal to 0; that means the limit of $(\sin x)/x$, whether x goes to infinity or whether x goes to minus infinity, is equal to 0. That conforms to our intuitive idea that we explained earlier.

Once this limit is equal to 0, you can go to $2 + (\sin x)/x$. That should be equal to 2 plus 0, right, that is 2. That is what we see. Hence the limit as x goes to $+\infty$, and as x goes to $-\infty$, both the things we wrote here, of the function $2 + (\sin x)/x$ should be equal to 2. See how the estimates are helpful.

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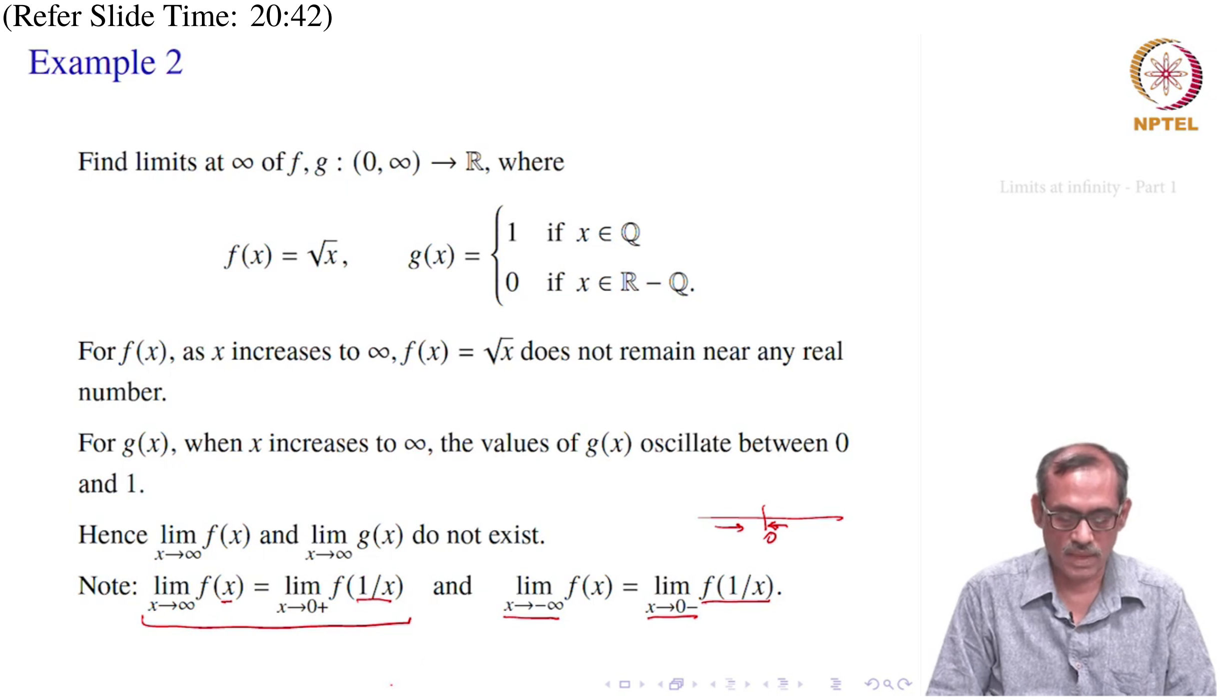

We will take one more example. Let us say we are having two functions here; we have to find the limits at infinity, that is, limit as x goes to infinity of $f(x)$ and of $g(x)$, where $f(x)$ is given as $\sqrt{x}$, and $g(x)$ is defined conditionally this way: it is 1 when x is a rational number and it is 0 if x is irrational. Of course, the domain is $(0, \infty)$. When you say $x \in \mathbb{Q}$, here it means $x \in \mathbb{Q}$ and x remains positive. And the second one is x is irrational and also x is positive; it is a positive irrational. For those x, $g(x)$ is defined this way. It is 1 for positive rationals and 0 for positive irrationals. Only for positives, it is defined here.

We want to find out the limit of $f(x)$ as x goes to infinity. Intuitively what it says is that as x increases to infinity, $f(x) = \sqrt{x}$ also increases to infinity. That means it will not remain near any real number. Suppose I say 100, now, will square root of x remain near 100 for every x? No? If I take, say, $x = 1000^2$, then its square root will become 1000. So, it does not remain near 100. For every ϵ, I should get one δ so that I should have $|f(x) - 100| < \epsilon$. It is not so, because it is becoming 1000 here. If I choose $\epsilon = 1$, then it does not. That is the argument that $\sqrt{x}$ does not remain near any real number, it becomes larger. So, that is the intuition. But you can justify it through the formal definition also. It means $f(x) = \sqrt{x}$ will not have a limit as $x \to \infty$.

What about $g(x)$? As x increases to infinity, it will take rational values as well as irrational values. We have the denseness of rationals that between any two real numbers there is a rational number, and there is also an irrational number. That means, as x becoming larger and larger, x goes through real numbers. Whenever it is rational, $g(x)$ becomes 1, and whenever it is irrational, $g(x)$ becomes 0. So, it does not remain near either of them, it becomes 1 sometimes, and 0 sometimes. It oscillates between these two two values. Really, becomes 1 or 0, 1 or 0 at almost every point. When it remains in the neighborhood of infinity, it assumes both the values, so to say; it does not remain near any one of them. Therefore, that limit also should not exist.

That is why we say that the limit of $f(x)$ does not exist, and the limit of $g(x)$ does not exist as x goes to infinity.

Let us do some more problems basing on this notion. There is a note here, which says that you can convert x goes to infinity to x goes to 0+ by substituting x with $1/x$. So, here it is t, $t = 1/x$, so, x goes to ∞ is same thing as t goes to 0+. You would say that the limit of $f(x)$ as $x \to infty$ is same as the limit of $f(1/x)$ as x goes to 0+.

Similarly, when we take $f(1/x)$, then as x goes to $-\infty$, would mean as $x \to 0-$. That means, from the negative side of 0; so, you will be taking this way, and it is a new function: $f(1/x)$. On the other hand, if x goes to infinity, we will say $f(1/x)$ as $x \to 0+$. It is in the neighborhood of 0 but remaining positive. Sometimes, this substitution is helpful because it can simplify some notation and some expressions. You may not have occasion immediately, we will see its use later.