Basic Calculus - 1
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Lecture 7 - Part 1
One-sided Limits - Part 1

So, this is lecture 7 of Basic Calculus 1. In the last lecture, we had taken some examples from limits. And now, we consider one-sided limits. You will see that it is different; but not much different; it is the same process. However, it depends on the function and where it is defined. Sometimes even if the function is defined throughout an interval, we may consider one-sided limits. We will see slowly.

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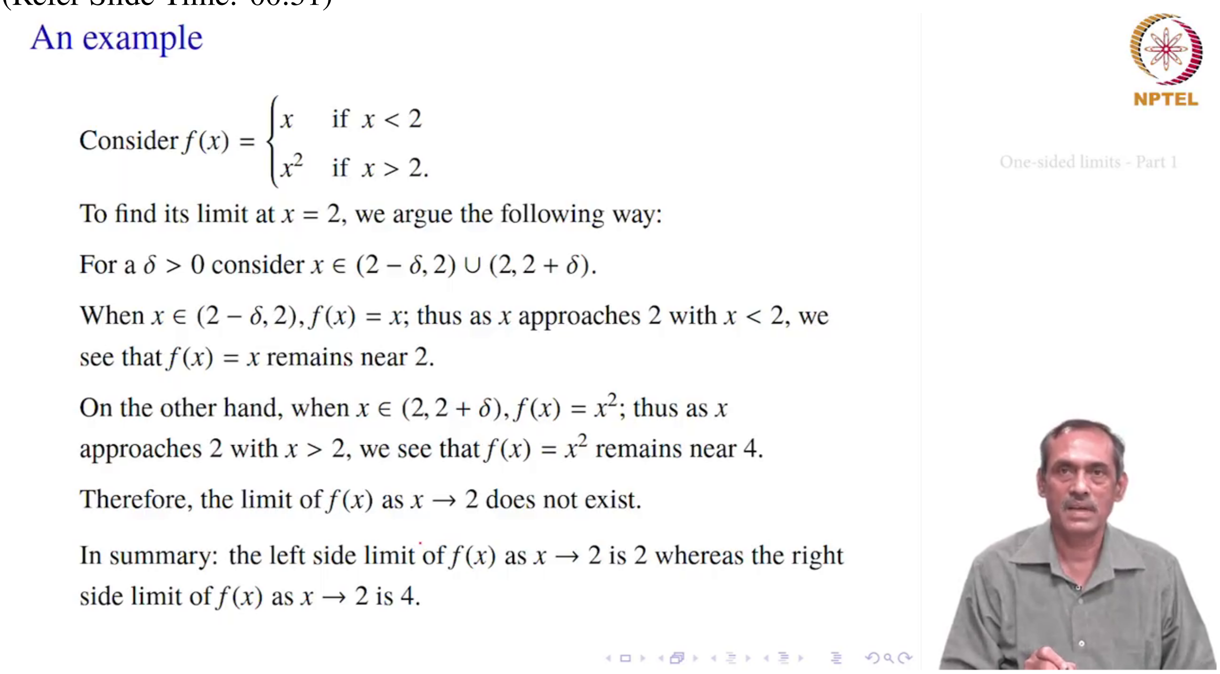

Let us start with this. Say, we have a function $f(x)$, which is defined at all real numbers except at the point 2. For $x < 2$, it is given as x; and for $x > 2$, it is given as x^2. We are required to find its limit at $x = 2$. Usually, when we go for the limit, we argue the following way.

We consider some interval around 2, leaving that 2. So, this is $2-\delta$ on the left-side, and $2+\delta$ on the right-side. When x belongs to the left neighborhood, left δ-eighborhood: $(2-\delta, 2)$, $f(x) = x$. That is how it is given. Thus when x is near 2, but remaining always less than 2, we see that $f(x)$ remains near 2. And on the right-side, we have $x \in (2, 2+\delta)$; there, $f(x) = x^2$. When x approaches 2, but remaining greater than 2, $f(x)$, which is x^2, remains near 4. So we conclude that the limit of $f(x)$ as x goes to 2 does not exist.

Look at the argument. We have not taken directly $(2-\delta, 2+\delta)$, but divided that into two sub-intervals. On the left-side, we are considering one, on the right-side we are considering the other. So we say that the left-side limit of $f(x)$ is 2, whereas the right-side limit of $f(x)$ is 4.

Now, we want to formalize this notion of left-side limit and right-side limit. They are called one-sided limits.

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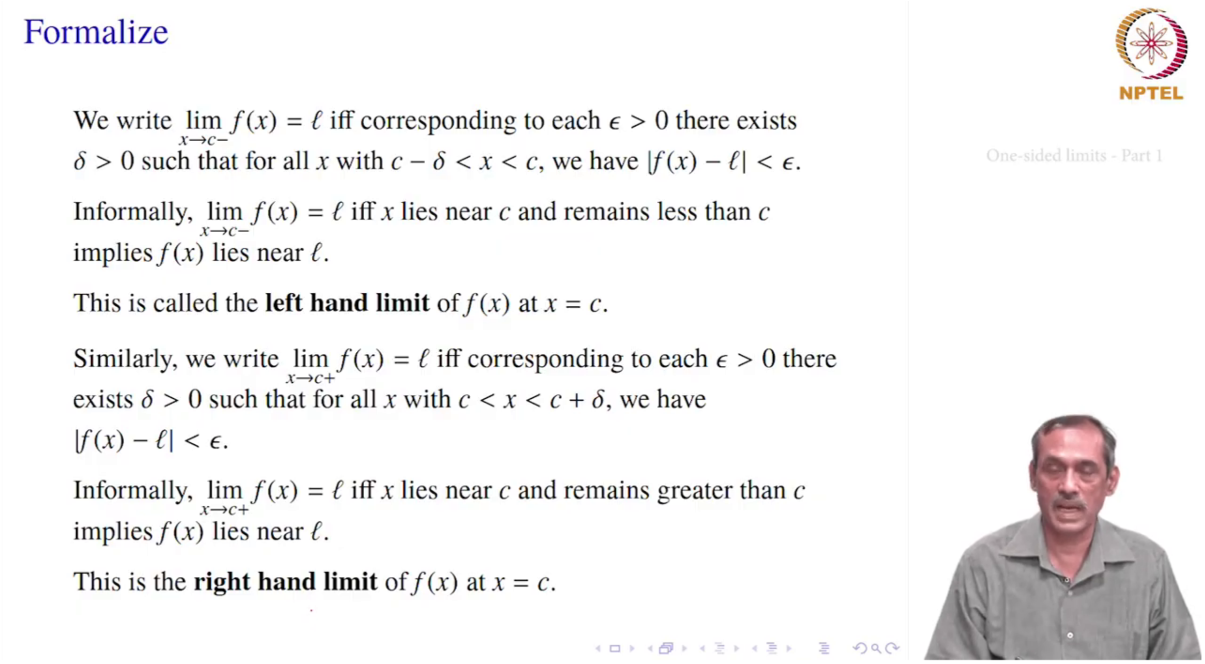

We give a notation for the left-side limit. Suppose f is a given function given and c is a point. You say x tends to $c-$, or, limit of $f(x)$ as x goes to $c-$. Here, $c-$ means the left-side limit. That is equal to ℓ if and only if corresponding to each $\epsilon > 0$, there exists a positive δ such that if x lies between $c - \delta$ c, then we have this condition $|f(x) - \ell| < \epsilon$.

Recall that in the definition of limit, we would have taken $c - \delta$ to $c + \delta$, leaving the point c. If condition $|f(x) - \ell| < \epsilon$ is satisfied, then we would have that the limit of $f(x)$ as x goes to c is equal to ℓ. Now, it is one-sided limit; so, we are only concerned about points which are near c, not equal to c, and to the left of c.

Similarly, you can define the right-side limit. For the right-side limit, we consider the interval $c, c + \delta)$. Let us see. We write this as the limit of $f(x)$ as x goes to $c+$, that is the right side limit. That limit is ℓ if and only if corresponding to each $\epsilon > 0$, there exists $\delta > 0$ such that whenever x lies between c to $c + \delta$, that is on the right δ-neighborhood, we have the same condition $|f(x) - \ell| < \epsilon$ satisfied.

Aagain informally, you say that this happens, that is, the limit of $f(x)$ as x goes to $c+$ is equal to ℓl, or, the right-side limit of $f(x)$ at c is ℓ, if whenever x lies near c remaining greater than c, $f(x)$ lies near ℓ. That is called the right-hand side or right limit of $f(x)$ at $x = c$. So, in the earlier example, what we have seen is that the left-side limit is 2, and the right-sided limit is 4. That conforms to this formal definition.

Something to note here. The left hand limit at $x = c$ is meaningful for a function f defined on a set containing an open interval of the type $(c - \delta, c)$. Recall that in the definition of limit, we said that the limit is defined only when there is a deleted neighborhood of c, which is $c - \delta$ to $c + \delta$

leaving that c, where the function is defined. Then only we can talk of limit. Similarly here, we would say that the function should be defined on $(c-\delta, c)$. Then only we can think of its left side limit. Similarly, the right-side limit is meaningful when the function is defined on a neighborhood of the type $(c, c+\delta)$; on the right-side of c, there is some neighborhood where f is defined.
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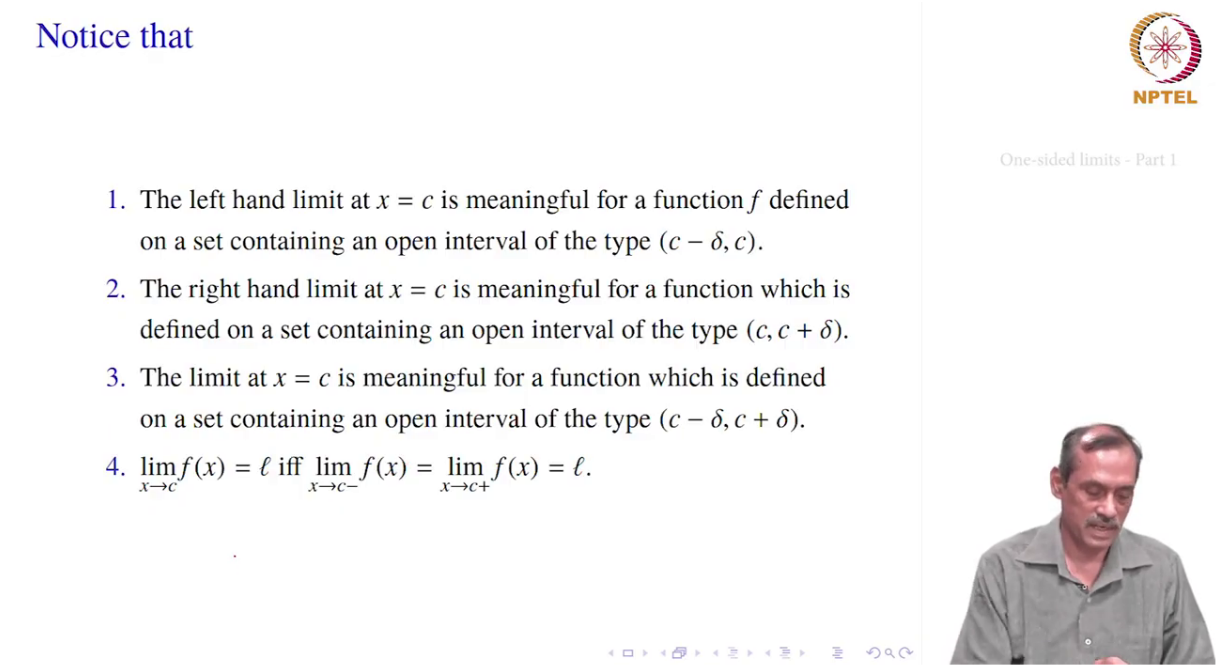

And for the limit, we know that it should be defined over $(c-\delta, c+\delta)$; perhaps, it is not defined at c. You may think of this as $(c-\delta, c+\delta) \setminus \{c\}$. If a function is defined on this, that is, on a deleted neighborhood of c, then you can think of its limit at c. And now you can see that if the limit is defined and the limit of $f(x)$ is equal to ℓ, then both the left-side limit and right-side limits are defined, and they are also equal to ℓ.

This is the fourth statement. We say that limit of $f(x)$ is equal to ℓ if and only if both the left-side limit is equal to ℓ and the right-side limit is equal to ℓ. Also, if this happens, then we would say that the limit of $f(x)$ is equal to ℓ. Sometimes we will be able to see that it is easier to find the left-side limit or the right-side limit than the limit itself directly. In those cases, it will be helpful to find both the types of limits separately.

Let us take an example to see how it is applied. We have the function $f(x)$ which is equal to $x/|x|$; and we want to find its left hand and its right hand limits, and also if possible its limit at $x = 0$.

For the left hand limit, we should have a left neighborhood of 0, where the function should be defined. Of course, the function is defined everywhere except at $x = 0$. It is $x/|x|$. So, that condition is satisfied. Now we proceed to the left-side limit. For that, we should consider all $x < 0$, particularly in a neighborhood near 0. Look at how the function behaves when $x < 0$: $|x|$ becomes $-x$ when $x < 0$. So $x/|x| = x/(-x) = -1$; it is a constant. So, the limit of $f(x)$ is equal to -1. That is clear when x goes to $0-$. That is, the left-side limit of $f(x)$ at $x = 0$ is -1.

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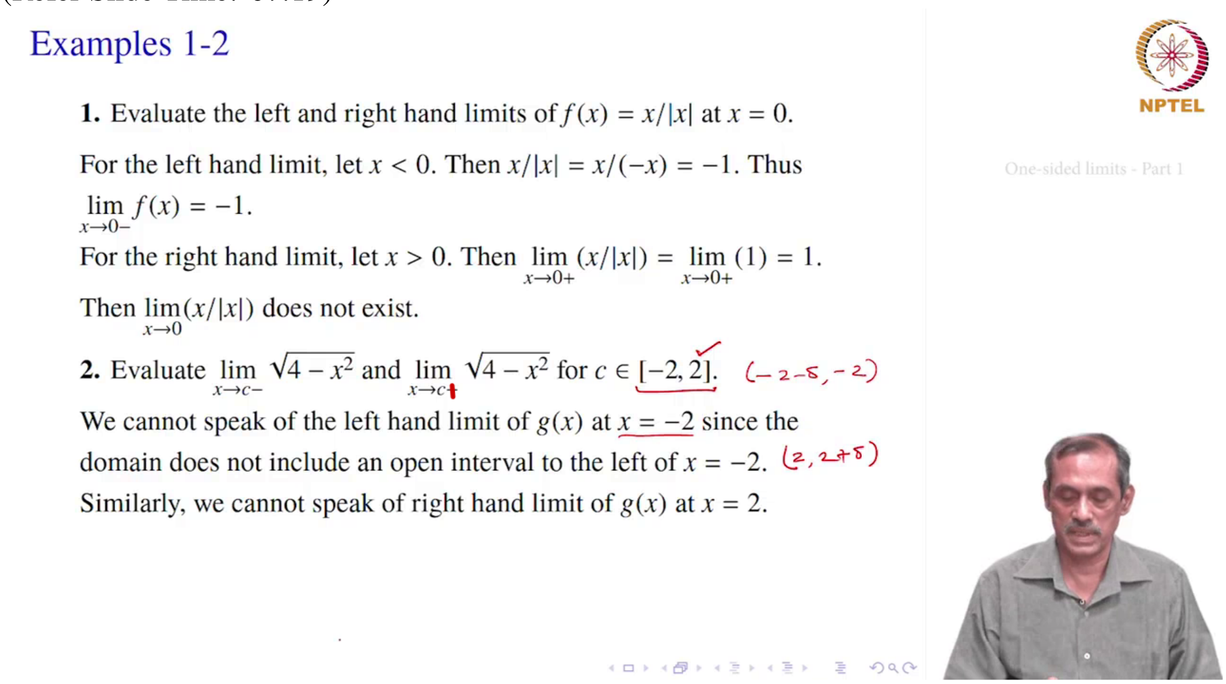

And what about the right-side limit? Let us take $x > 0$. Then, $|x|$ becomes x; so $x/|x| = a/x = 1$. Therefore, the right-side limit of $f(x)$ is equal to 1 at $x = 0$. Therefore, we may conclude that the limit as x goes to 0 of $x/|x|$ does not exist. It is because both the limits exist here and they are not equal; one is -1 and another is 1. It is similar to our first example, where 2 was the left-side limit, and 4 was the right-side limit.

Let us go to second problem. It asks for finding out the left-side limit of the function $\sqrt{4-x^2}$; and the right-side limit, (it is not minus, it should be plus) of the same function. The function is defined from -2 to 2. And these limits are defined for this. We want to find out the limits at every point inside the interval $[-2, 2]$. The point c is not given exactly, but c is between -2 to 2.

Naturally, we will consider cases like $c = -2$, $c = 2$, and $-2 < c < 2$. Let us see. Here, the function is $\sqrt{4-x^2}$. We see that $4 - x^2 \geq 0$, that is, $x^2 \leq 4$. So, x must be lying also between -2 to 2. That is the domain of the function.

Now, at $x = -2$, for the left-side limit, we need a neighborhood of the form $(-2-\delta, -2)$. Such a neighborhood should be inside the domain of the function. But the function is not defined for this. The function is defined only after -2. So, the left hand limit of $g(x)$, this function, we call it $g(x) = \sqrt{4-x^2}$, is not meaningful really. So, we cannot speak of that limit.

Similarly, we cannot speak of the right hand limit at $x = 2$, because in that case, we would need an interval of the form $(2, 2+\delta)$, where the function should be defined. So, we cannot even speak of that. When $c = -2$, we cannot speak up the left-side limit, and when $c = 2$, we cannot speak up the right-side limit. But when $c = -2$, we can think of the right-side limit, and when $c = 2$, we can think of the left-side limit.

Let us take the case of the right-side limit at -2. There, $\sqrt{4-x^2}$ remains near 0. Therefore, the limit is 0. And when $c = 2$, we are considering the left-side limit. There also, $\sqrt{4-x^2}$ remains near 0. So, the left-side limit at $c = 2$ is 0.

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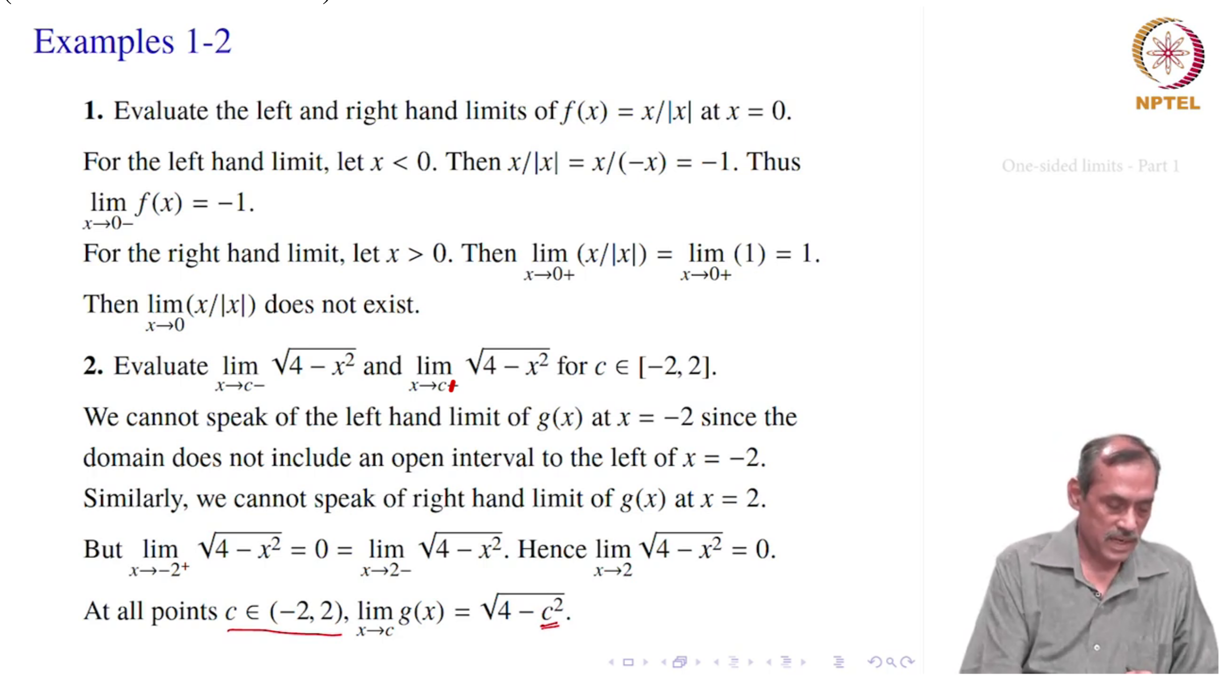

We emphasize. at $c = 2$, the right-side limit is not to be talked about, and at $c = -2$, the left side limit is not to be talked about. So, this one, which we write here, we cannot really write it, because we cannot speak of even the limit when x is near 2 and bigger than 2. Similarly, when x is near -2 and smaller than -2, So, we can only think of $-2+$ and $2-$.

And, what about c lying in the open interval $(-2, 2)$? So, c is now any point between -2 and 2, but not equal to -2 or 2. It is easy to see that both left and right side limits are defined and are equal to $\sqrt{4 - c^2}$. So, as x goes to c, when c lies between the points -2 and 2 but not equal to either of them, the limit of $\sqrt{4 - x^2}$ is $\sqrt{4 - c^2}$.

Let us take another example. This limit is really very helpful in computing many of the trigonometric limits. We wish to find the limit of $(\sin t)/t$ as t goes to 0. What happens to $\sin t$ as t goes to 0? We know that the limit of $\sin t$ is 0. There is a possibility that the limit exists, or a limit it does not exist; but we cannot discard it as it is, because the top one has the limit 0, bottom one also has the limit 0. We have to really go back to our trigonometric estimations. Let us see how it goes.

Anyway, t is lying near 0. Let us consider first the case $t \to 0+$. That is, $t > 0$ and t is in a neighborhood of 0. So, even for 0 to $\pi/2$, we can have some derivation and then proceed. It is enough if you take 0 to 0.1; say, t varies between 0 to 0.1. That is also enough. But let us see what would be easier. Suppose t is between 0 to $\pi/2$. This angle is 0 to $\pi/2$ and that is our t. now. Then we take a circle of radius 1. On both the sides: it is 1, that is, from 0 to I, it is 1, and on the other side the radius is 1; and OQ is also 1, that is, the unit length. Since this is the angle t, $\sin t$ will be equal to the perpendicular which is in red now, divided by OQ, which is 1. Therefore, this length of the red line here is $\sin t$. And, $\cos t$ is the base divided by hypotenuse, which is 1. Let us call this point as A. Then, the length OA is $\cos t$ and QA is $\sin t$.

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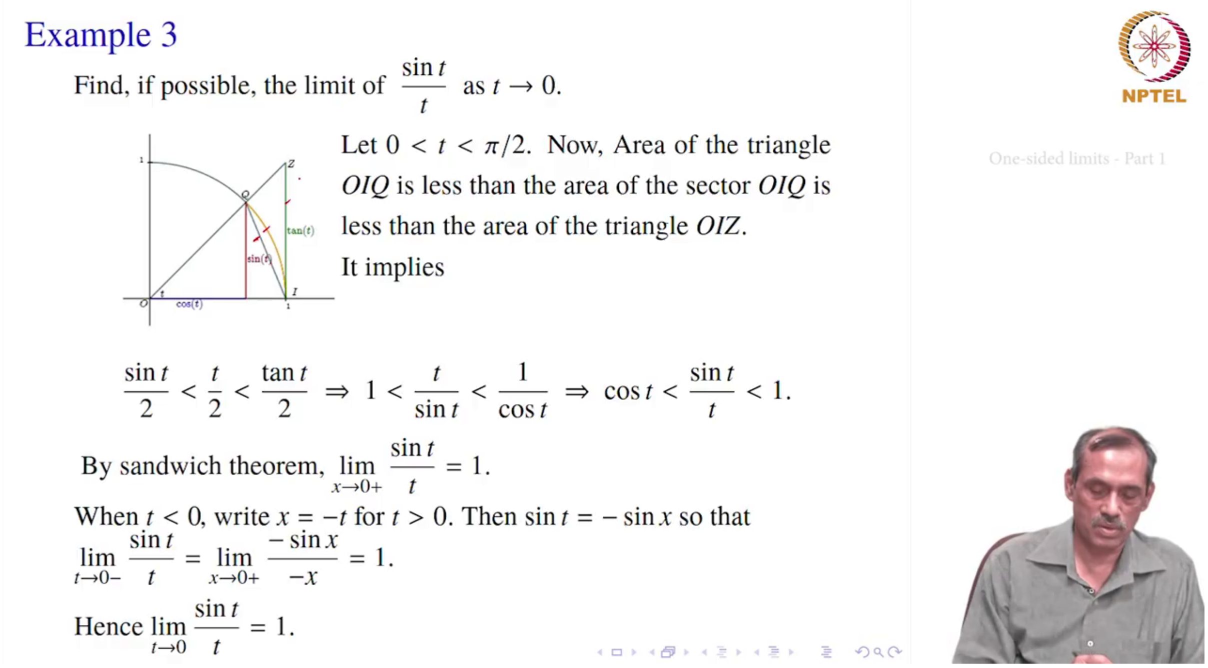

Now, what about this green line? You can take the green line. This is t. Consider the bigger triangle OZI. Here, I have t. Then, $\tan t$ is equal to the perpendicular divided by base. Now, the perpendicular is ZI and base is OI, which is of unit length. Therefore, this is $\tan t$. Now we are going to compare the area of the triangle OIQ and the sector OIQ, and the triangle OZI. Let us compare the area of this.

As the geometry shows, the area of the triangle OIQ is less than the area of the sector OIQ, and that is less than again the area of the triangle OZI. Now, you can go for the formulas of the area. So, triangle OIQ is half, base is 1, and perpendicular is $\sin t$; so it is $(\sin t)/2$. That is, $(\sin t)/2$ is the area of the triangle OIQ.

And since this is t, and if you had measured this in radians, that is what take, we get the area of the sector equal to $t/2$. Since this length is t, we get this as $t/2$.

And the area of the triangle OZI is again $\tan t$ multiplied by base, which is 1, divided by 2; so it is $(\tan t)/2$. That is how you get the inequality:

$$\frac{\sin t}{2} < \frac{t}{2} < \frac{\tan t}{2}.$$

We cancel 2 and divide $\sin t$ because $\sin t$ is nonzero. Then, you get

$$1 < \frac{t}{\sin t} < \frac{1}{\cos t}.$$

This is what we get for all t between 0 to $\pi/2$.

We take the limit as t goes to 0. By Sandwich theorem, the limit of $\cos t$ as $t \to 0+$ is 1. As t goes to 0 plus, its limit is 1, and the right side is a constant function 1. Therefore, the limit of $(\sin t)/t$ as t goes to 0+ is equal to 1.

This is for the right-side limit. What happens for the left-side limit? Well, what happens here is instead of t let us take $-t$. When $t < 0$, write $x = -t$ for $x > 0$. Now, $\sin(-t) = -\sin t$, implies $(\sin t)/t = (\sin(-t))/(-t) = (\sin x)/x$. As $t \to 0-$, $x \to 0+$. Then, this limit as $x \to 0+$ gives us 1. Therefore, the left-side limit, the right-side limit, both are 1. Hence the limit is equal to 1.

Okay? I think this is pretty easy; we are only comparing the areas. There is another way of comparing the lengths of this cord and this and this, but this looks easier.

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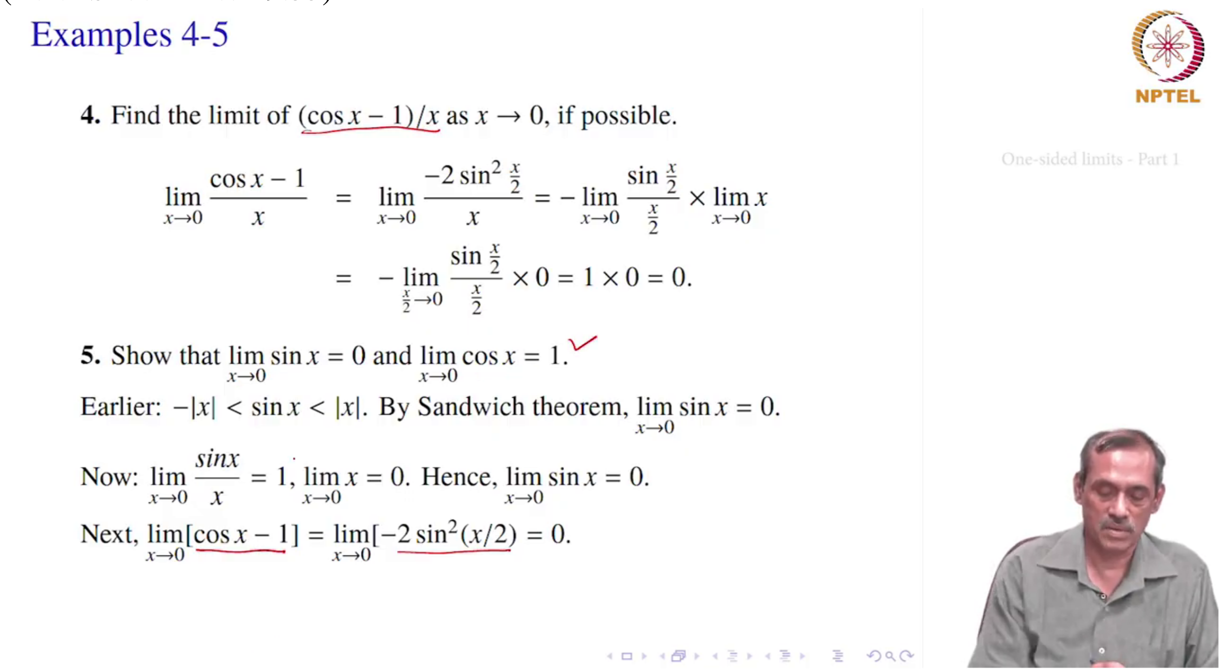

So, once we get this limit, we can apply it on other limits. Let us see how. Say, we are required to find the limit of $(\cos x - 1)/x$ as x goes to 0. We do not know whether its limit exists or not. But let us see what happens for this. For $\cos x - 1$, we can use the half sine formula. It is $(-2\sin^2(x/2)/x$. Then we can use $(\sin x)/x$ formula.

So, it is minus, the minus sign goes out, the limit $(\sin^2(x/2))/x$ as x goes to 0. We multiply $x/2$, bring this 2 down. So, it is $x/2$ times limit of x as x goes to 0. Now this limit, we know to be 1, and this limit is 0. Therefore, the total limit is 0. So the limit of $(\cos x - 1)/x$ exists and is equal to 0. That is what it says.

Let us take another example. These are the limits. You see here limit of $\sin x$ and limit of $\cos x$. We know this to be 0. How did we do for $\sin x$ earlier? If you remember, $\sin x$ is bounded by $-|x|$ and $|x$. Then, when x goes to 0, by Sandwich theorem, this goes to 0, this also goes to 0. Therefore, by Sandwich theorem, limit of $\sin x$ is equal to 0.

However, now we can use $(\sin x)/x$. You see that the limit of $(\sin x)/x$ is equal to 1. This is $\sin x$ and limit of x, that is the denominator, that is equal to 0. As our earlier result says, if the denominator limit 0 and the limit of the ratio exists, then the limit of the numerator must also be 0 at that point. So, we can conclude directly that limit of $\sin x$ is equal to 0.

Similarly, for $\cos x$. Earlier we had a different way of doing it: $\cos x - 1$ or $1 - \cos x$, we can bound it from the left and right. But now we can use $(\sin x)/x$, which says $\cos x - 1$ is $-2\sin^2(x/2)$. So, that goes to 0 as you see from the earlier one. Also, you can use the fourth one. This limit we know to exist and the of the denominator goes to 0. So the limit of the numerator must also be 0. As we have done here, the same way you can conclude that the limit as x goes to 0 of $\cos x - 1$ is equal to 0. And that gives the limit of $\cos x$ equal to 1. So, this is another way of doing the same old problems.