Basic Calculus - 1

Professor. Arindama Singh
Department of Mathematics
Indian Institute of Technology Madras

Lecture 6 - Part 1

Algebra of limits - Part 1

So, this is lecture 6 of basic calculus 1. In the last class, we had discussed about limits, the concept of limits; and then we had solved some examples using on the concept. We have discussed how to really execute the definition. We have seen that there is no way to find the limit. But if you guess something correct, then you can justify it using the definition. We will discuss today something related; it is called the algebra of limits, so that if you know some basic things, then you will be able to guess at least what the limits is. Most of the results we will be discussing today will not be proved. But they are obvious, because of this intuitive idea of nearness. We will see how to proceed.

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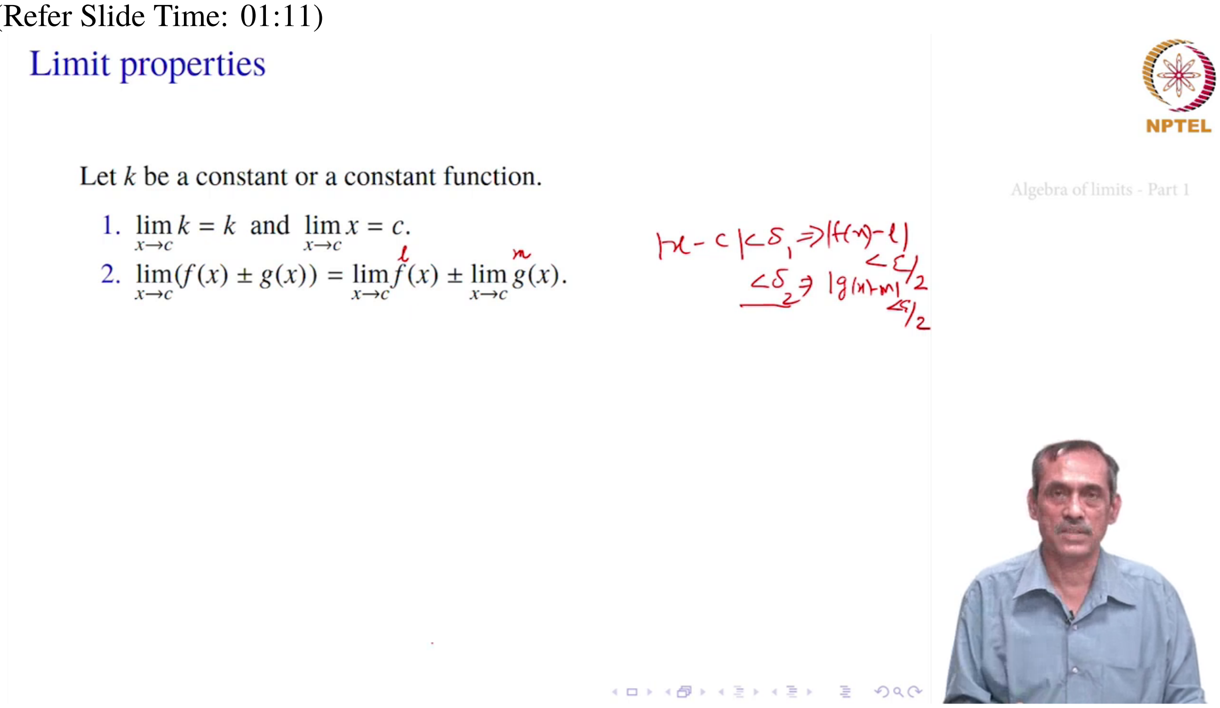

Let us start with the easiest function, which is a constant function, say, $f(x) = k$ for a fixed real number k and every x.The result says that the limit of this function, which is a constant function, will be equal to that constant; whatever this c may be. Of course, this should be such that the limit is well defined. Now, how do we justify this through our definition? Well, first we see that $|f(x) - k| < \epsilon$ whenever $|x - c| < \delta$ with $x \neq c$; of course, you can choose a δ for ϵ.

So, suppose ϵ is given, an $\epsilon > 0$ is given. Let us say $\delta = \epsilon$. Now, when $|x - c| < \delta$, we have $|x - c| < \epsilon$. Now, consider $f(x) - f(c)$. We are suggesting the limit to be k, which is equal to $f(c)$. So, $|f(x) - k| = |k - k|$ since $f(x) = k$. Now, this is $k - k = 0$, and is always less than ϵ. So, that condition is satisfied. Hence, the limit of this constant function $f(x) = k$ is equal to k.

The second one is telling that limit of the identity function is equal to c when x goes to c. Notice that the function is $f(x) = x$ here. Let us choose for a given epsilon, $\delta = \epsilon$. We see that whenever $|x - c| < \delta = \epsilon$, as the functional value is x itself, $|f(x) - c| < \epsilon$. So, the limit of $f(x)$ is equal to c. We had seen this earlier; and also we had seen the same for x^2.

Now, instead of going for x^2, we will slightly generalize. The second result is telling this. Suppose you have a function $f(x)$ and you have a function $g(x)$. You take the new function as $f + g$, which is defined as $(f + g)(x) = f(x) + g(x)$. When you take limit of $f(x) + g(x)$ as x approaches c, the result is telling that you would get on the other side, the limit of $f(x)$ plus the limit of $g(x)$. Again, a slight work is required here. It is given that limit of $f(x)$ exists; that is assumed here implicitly.

Let us say that this limit is ℓ and this limit is m; limit of $g(x)$ is m. Now, it is given that for any $\epsilon > 0$, there is a δ, let us call it δ_1; so that whenever $|x-c| < \delta_1, x \neq c$, you would get $|f(x)-\ell| < \epsilon$. Similarly, for the function $g(x)$. Let us say, we want to prove that limit of $f(x) + g(x)$ is equal to $\ell + m$ given these two facts that limit of $f(x)$ is equal to ℓ and limit of $g(x)$ is equal to m.

So, let $\epsilon > 0$ be given. Now, let us consider $\epsilon/2$. For this $\epsilon/2$, there exists a δ_1 such that whenever $|x - c| < \delta_1$, you would get $|f(x) - \ell| < \epsilon/2$. Of course, $x \neq c$ on the left side. For take the other one: limit of $g(x)$ is equal to m. Let us take again $\epsilon/2$ instead of epsilon. For that $\epsilon/2$ in $f(x)$ there is a δ_1. Similarly, for the limit of $g(x)$ equal to m, we consider $\epsilon/2 > 0$ so that there exists δ_2 such that whenever $|x-c| < \delta_2$, you would get similar thing in g; that is, $|g(x)-m| < \epsilon/2$.

Now, you choose your δ to be something smaller than both δ_1 and δ_2, or even you say that it is the minimum of δ_1 and δ_2. When you take $|x - c| < \delta$, both the conditions $|x - c| < \delta_1$ and $|x - c| < \delta_2$ are satisfied. So, both the conclusions also hold. That is, $|f(x) - \ell| < \epsilon/2$ and $|g(x) - m| < \epsilon/2$. Then, you take $|f(x) + g(x) - (\ell + m)|$. By the triangle inequality, you get the addition of these two quantities, which gives you $\epsilon/2 + \epsilon/2 = \epsilon$. So, your limit condition is satisfied. Therefore, $\lim\limits_{x \to c} f(x) + g(x) = \ell + m$.

Similarly, you can do for the minus also. Now, if results for plus and minus hold, it also will hold multiplication. But let us see for multiplication by a constant first. Suppose k is a constant, a given real number, which is multiplied with $f(x)$. We will get a new function here, which is kf; $(kf)(x) = kf(x)$. Now, limit of this function constant times $f(x)$ is equal to that constant into limit of $f(x)$. The proof will be similar. So, instead of ϵ, you start with ϵ/k in the beginning and continue.

But if k is 0, then you can separate that case. You can see that, on the left side it is the 0 function, which is a constant function. So, it goes back to property one. On the right side, you will always get 0. So, for multiplication also a similar thing holds. That is, the limit as x goes to c of $f(x)g(x)$ is equal to the limit of $f(x)$ into the limit of $g(x)$. In all these things, our assumption is that all the limits really exist. At least on the right-hand side if you have limit of $f(x)$ into limit of $g(x)$, that means limit of $f(x)$ must exist and limit of $g(x)$ also must exist. Then limit as x goes to c of $f(x)g(x)$ will be equal to their product. That is how you have to read the statements here.

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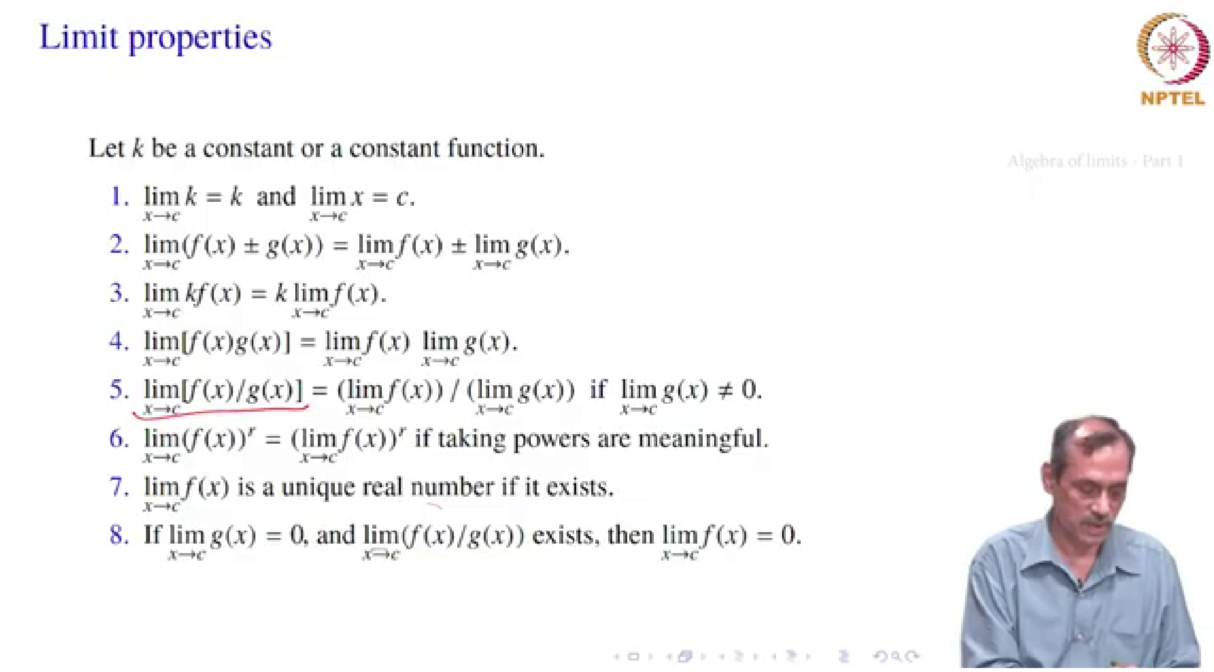

A similar thing also holds when you divide instead of multiplying, but you have to be a bit careful here. On the left side, when you say limit as x goes to c of $f(x)/g(x)$, our assumption is that $f(x)/g(x)$ is well defined. That means $g(x)$ is never equal to 0 at least in a neighborhood of c, or rather in a deleted neighborhood of c. At c we do not know what happens. But to the left of it and to the right of it, on these open intervals say, $(a, c) \cup (c, b)$ for some a and b, we should have $g(x) \neq 0$. If $f(x)/g(x)$ is well defined and the limit as x goes to c of $f(x)/g(x)$ is well defined, then its limit will be equal to this provided these two limits exist and the limit of $g(x)$ is not equal to 0. Because once the limit of $g(x)$ is equal to 0, this division itself will not be meaningful. We are not going to prove them but each one involves a bit of trick like you choose $\epsilon/2$ or ϵ/k or $\delta = \epsilon$ and so on. Some little trick will be required to prove these things.

Then similarly, we have for the power function. Suppose, r is a real number such that $f(x)$ to the power r is well defined as a function. You know that power function is not always defined, for example, negative to the power something. We have to be cautious, like $(-1)^{1/2}$ is not defined and so on. So, we have a constraint here that taking powers are meaningful. In this case, usually we assume that $f(x) \geq 0$ and also this limit is greater than or equal to 0. Then this will be meaningful, obviously, for any r. So, in this case, we will say that the new function here is $f(x)$ to the power r. That limit, when x goes to c, will be equal to limit of $f(x)$ whole to the power r. We have to read it carefully: this whole to the power r. This is so provided that the limit exists.

That is, you can take the limits in an obvious way with plus-minus, constant multiple, multiplication, division, and with power; that is what these results are telling.

Now, the other one says that the limit whenever exists is really unique. We cannot have two limits of the same function as x goes to c. It does not say that limit always exists. But if the limit exists, then there is a unique number to which the limit will be equal, it cannot be two different numbers. Of course, we always assumed that this is so, and you say 'the limit'. But it has to proved and the proof is not very difficult here. If you take two different limits, then for any epsilon, you would get $|f(x) - \ell| < \epsilon/2$ with some delta there. And, similarly, $|f(x) - m| < \epsilon/2$. So you add them to get for some x at least, $|\ell - m| < \epsilon$ for every epsilon. Now, recall how to prove equalities using inequalities. Since for every $\epsilon > 0$, $|\ell - m| < \epsilon$, ℓ must be equal to m. That is how the proof will go. But we are not worried about the proof now. We should see that the limit must be unique. That is why you can say 'the limit' as x goes to c.

So far, it says there is some constraint about this division. Suppose limit of $g(x)$ is equal to 0. And limit of $f(x)/g(x)$ exists. It will be equal to 0 or anything, but it is a real number. It exists means it is a real number. If this is a real number, and limit of $g(x)$ is equal to 0, then the limit of $f(x)$ must be equal to 0. So, if you look at 5 and 4, it will be coming from there immediately. Because the limit of $g(x)$ can be written as limit of $f(x)/g(x)$ multiplied by the limit of $g(x)$. Since limit of $g(x)$ is equal to 0, the limit of $f(x)$ must be equal to 0.

But we will look at it another way; it gives some more information. Though it is very obvious, it says this. Suppose the limit of $f(x)/g(x)$ exists and the limit of $g(x)$ is equal to 0, then it is not possible that the limit of $f(x)$ can be a non-zero number. On the other side, if the limit of $f(x)$ is a non-zero number, and limit of $g(x)$ is equal to 0, then the limit of $f(x)/g(x)$ cannot exist; it will not be a real number. As you see, it is 0. That means $g(x)$ becomes very small. So, $f(x)/g(x)$ becomes very large, either it goes to plus infinity or to minus infinity. That is why it does not exist. That is what it says.

In this form it is easier to remember and use that. If the limit of $f(x)/g(x)$ is a real number and the limit of $g(x)$ is equal to 0, then the limit of $f(x)$ must be 0 also. And it is easy to see this from the properties 4 and 5 also.

Let us go to the other two very special properties, which are very helpful in most of the theoretical problems and even computation of limits in very particular cases, where you do not have direct ways. The first one says something about domination. Suppose $f(x) \leq g(x)$ for all x around c. Here, 'around c' means some open interval (a, c) which is to the left of c union some open interval (c, b) to the write of c. We could have taken an open interval (a, b) containing c, but we do not need c to be included here.

So even if c is not included, it is fine. That means $f(c)$ can be equal to $g(c)$ provided they are defined; that is allowed. Even greater than that is also allowed; it does not matter for us. So, suppose $f(x) \leq g(x)$. Then it says that the limit of $f(x)$ will be less than or equal to the limit of $g(x)$. The other conditions are just preliminary things, which are in the background, such as f and g are functions whose domains include these. That is around some c, it is well defined. Then both the limits should exist, then after that the conclusion follows.

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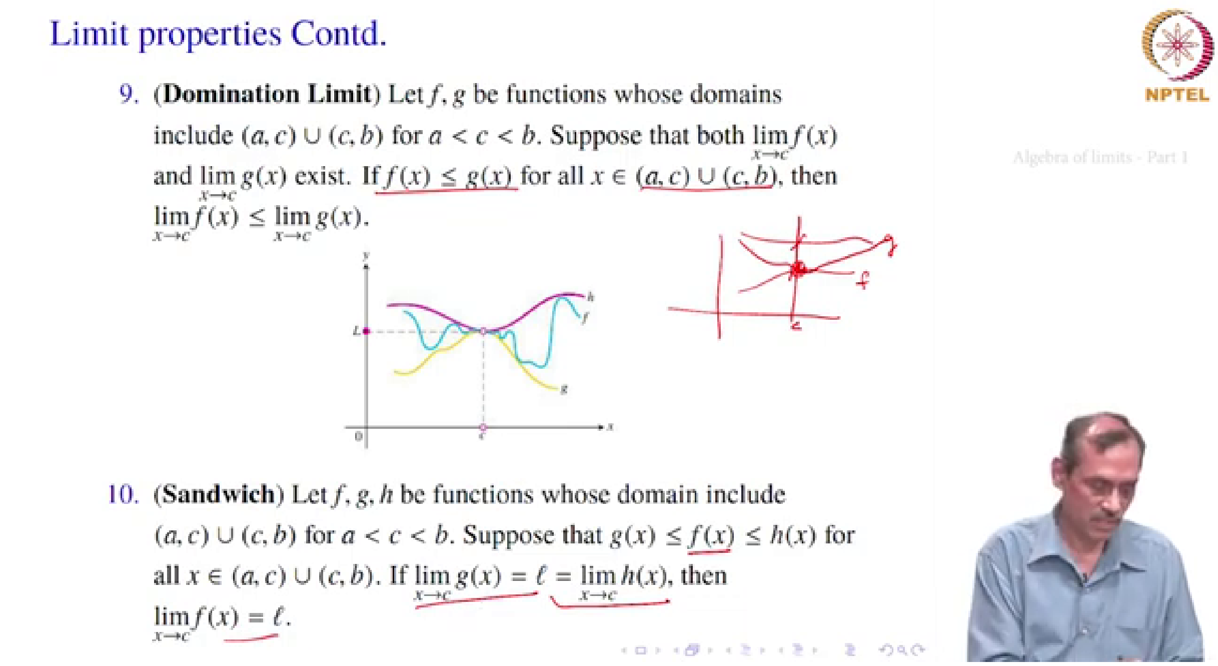

Suppose, this is $f(x)$ and this is $g(x)$ with $f(x) \le g(x)$ everywhere, for every x around c. Then it says that the limit of $f(x)$ as x goes to c and the limit of $g(x)$ as x goes to c are comparable. In the sense that the same inequality holds. From $f(x) \le g(x)$ we not conclude that the limit of $f(x)$ is less than the limit of $g(x)$. We will see an example for that. It does not say that. It only says that, if $f(x) < g(x)$, forget less than or equal to, suppose it is strictly less than, then also in the limit, it is less than or equal to.

In that case, if $f(x) < g(x)$ for all x around c, in a neighbourhood of c, then the limit of $f(x)$ is also less than or equal to the limit of $g(x)$. Even if it is less than this can be less than or equal to; we will see an example shortly. So, do not conclude that 'is less than' implies it will be strictly less than, it can be equal also. But once it 'less than or equal to', this is also 'less than or equal to'.

Intuitively it is obvious because if $g(x)$ is lying always above this, then it can happen that this is also possible. Like here, this is also possible at c; but we do not need it. So, in that case limit can be equal or limit will be less.

Now look at the Sandwich theorem; the next one is the picture is for that. There are three functions: f is the function, which is the blue one, g is the one which is yellow, and h is the one which is in magenta. What is given is, $g(x)$ and $h(x)$ really sandwich this $f(x)$ on either side. That is, $g(x) \le f(x) \le h(x)$ at any point x in their domains. But we do not need the whole domain. We need this to happen at least in an open interval or a deleted neighbourhood of c. If this condition is satisfied around c, then we have two other conditions also, that the limit of $g(x)$ is equal to ℓ and the limit of $h(x)$ is also equal to ℓ. This is how it is picturized. Suppose both the limits are same and $g(x) \le f(x) \le h(x)$. Then it follows that the limit of $f(x)$ is equal to the same ℓ. That is what we mean by sandwich theorem; that if f is sandwiched between g and h, and both have the same limit, then f also will have that limit as x goes to c.

In fact, we are not assuming that the limit of $f(x)$ exists. All that we are assuming is that the limit of $g(x)$ exists, the limit of $h(x)$ exists, and both of them are same, let us say that it is ℓ. Then the limit of $f(x)$ exists and it is equal to that ℓ. This is how this theorem will be useful, and we will be using it shortly.

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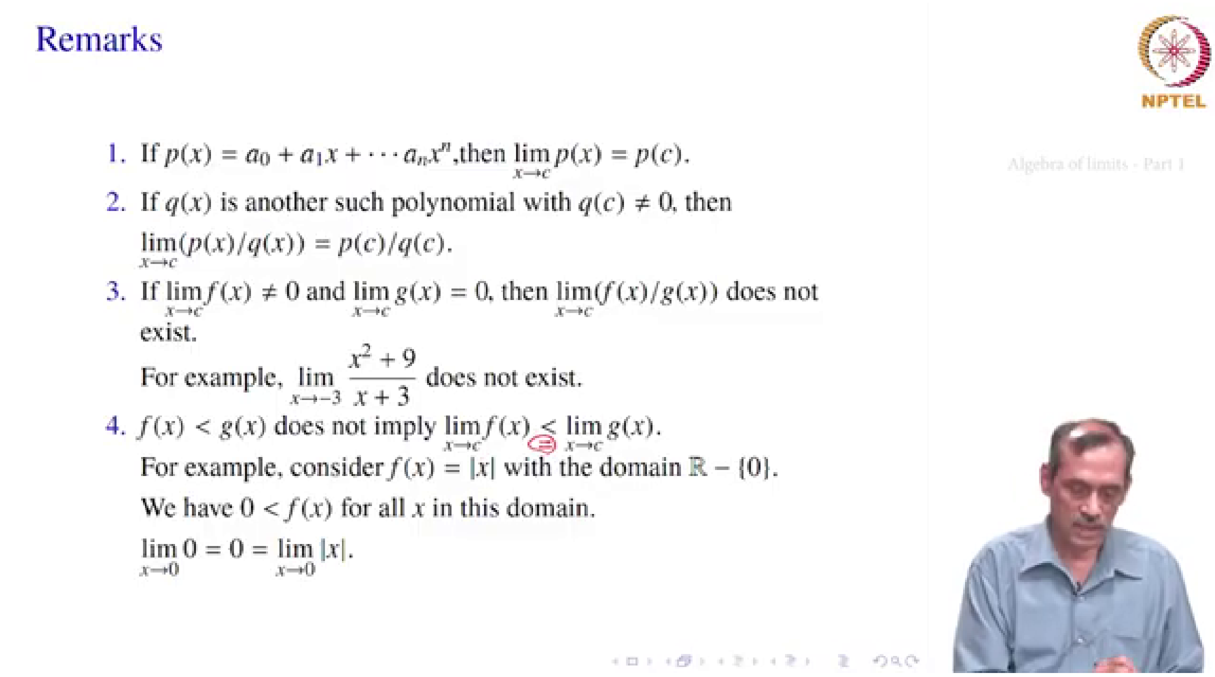

I would like to give some comments here. Suppose you have a function which is a polynomial: $f(x)$ or here say, $p(x) = a_0 + a_1x + \cdots + a_nx^n$ for some particular n. Then you can use our earlier properties. The limit of $p(x)$ will be equal to the limit of $a_0 + a_1x + \cdots + a_nx^n$. Now, the limit of a_0 is a_0 of course. The limit of a_1x will be a_1c as the limit of x as x goes to c, is c. And what is the limit of x^n? Well, $f(x)g(x)$ has the limit as the limit of $f(x)$ into the limit of $g(x)$. Using it n times we see that the limit should be equal to c^n. That means the limit of $p(x)$ as x goes to c is equal to $p(c)$.

So, for polynomial functions, it is nicely working, because that polynomial is well defined at c, and everywhere it is defined; so you get this one. Similarly, if you take the ratio of two polynomials, then also it is applicable. Because we have the limit of $f(x)/g(x)$. But we have a condition that $q(c)$ should not be equal to 0. And of course, in a neighbourhood of this c, $q(x)$ should not be equal to 0. So, under those conditions, we will get the limit of $p(x)/q(x)$ equal to $p(c)/q(c)$. So, rational functions can be handled.

We had seen earlier that if the limit of $f(x)/g(x)$ exists and the limit of $g(x)$ is equal to 0, then the limit of $f(x)$ is also equal to 0. We have put the same thing here in a different way. If the limit of $f(x)$ is not 0, but the limit of $g(x)$ is equal to 0, then the limit of their ratio does not exist. As we felt, it might be blowing up; it can be very large or very small going near minus infinity. So, it can be reduced or it can be increased as we please. Given any number, we can make it bigger than that by choosing some x. So, in this form also it is useful.

Let us see take x^2+9 as our $f(x)$; this is f and this is our g. Now, in any neighbourhood of -3, $x+3$ is never 0. At -3, of course, it is 0, but we are not worried about that, when we take the limit. So, $f(x)/g(x)$ is well defined. Now, the limit of $f(x)$ on the top as x goes to -3 is $(-3)^2+9$, that is equal to 18. This is not 0 but limit of $g(x)$ is equal to 0. Therefore, we immediately conclude that this limit does not exist, that is, the limit of $(x^2+9)/(x+3)$ does not exist. But had it been something like x^2-9, then the limit of $f(x)$ is equal to 0, and we cancel the factor $x+3$ to get $x-3$, whose limit will is -6.

This is what we were talking earlier. If $f(x) < g(x)$ around c, it does not necessarily imply that their limits will keep that 'less than' relation. It can be 'less than or equal to'; 'equal to' is also a possibility. So, we have given a nice example here, very easy example.

Let us consider $f(x) = |x|$. We take the domain as $\mathbb{R}-\{0\}$; 0 is excluded. We do not need 0 and we want to take the limit when x goes to 0. We have $0 < f(x)$ for all $x \in \mathbb{R}-\{0\}$. Why? Here, 0 is also taken as a constant function. Only at 0, $f(x)$ is 0. Here, for all x in the domain of $f(x)$, $f(x)$ is positive. That is, $0 < f(x)$.

Now you consider 0 as the constant function and $f(x)$ as another function. When you take the limit as x goes to 0, this gives 0. On the right side also the limit of $|x|$ is equal to 0. That means even if $f(x) < g(x)$ around c, the limits may not satisfy 'less than', it can be equal to also. That is what it says.

So, let us solve some examples. We will see how to use these results in solving the examples.