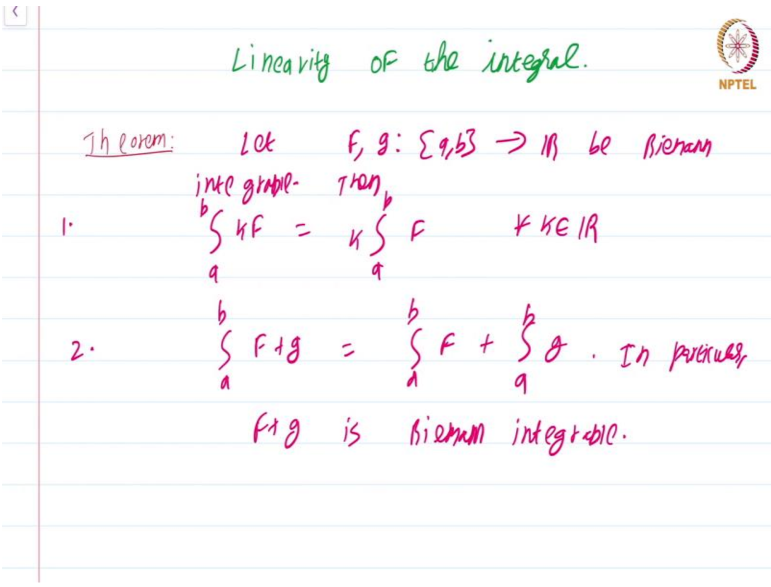
Real Analysis - I
Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 26.3
Linearity of Integral

(Refer Slide Time: 00:14)

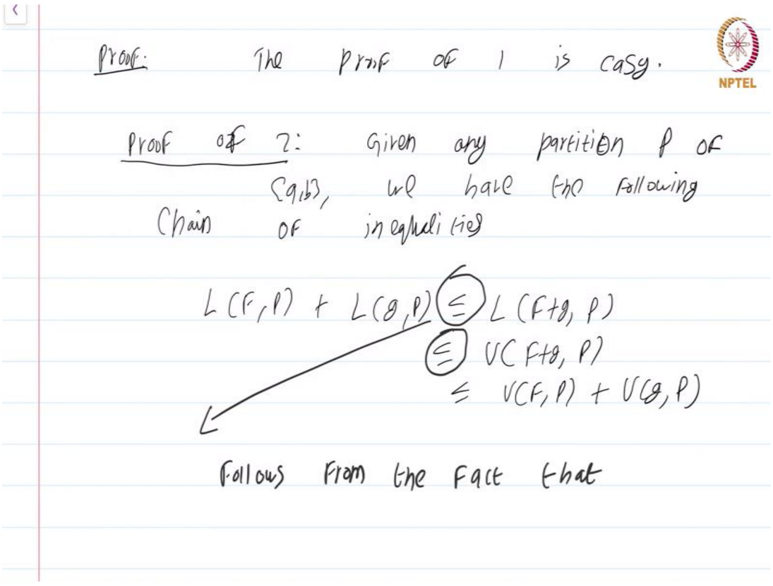


In this module we are going to prove a very important property of the integral called the linearity. If you are familiar with linear algebra the next theorem, why it is called linearity will make more sense. However, just with the statement even if you are not familiar with linear algebra, you will agree that calling this property linearity is a good idea. So, this is the theorem, this is the theorem. Let $f, g: [a, b] \to R$ be Riemann integrable.

So, you have given two Riemann integrable functions, then number (1) $\int_a^b k\, f = k \int_a^b f$, for all $k \in R$. If, you multiply the function by a constant k, then integrability is preserved, moreover the integral value is just $k \int_a^b f$. (2) $\int_a^b f + g = \int_a^b f + \int_a^b g$.

So, in particular $f + g$ is Riemann integrable. So, the sum of two Riemann integrable functions is indeed Riemann integrable and not only that the integral of the sum is the sum of the integrals as you would expect.

(Refer Slide Time: 01:52)



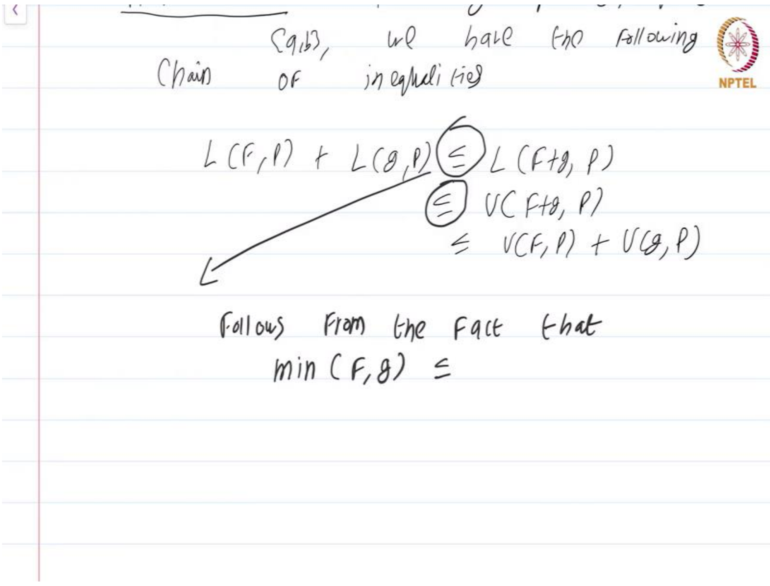
Now, let us go on to the proof, the proof of 1 is easy. So, I am going to leave it to you to figure out how to prove 1? Let us start by proving 2. So, proof of 2. So, now what we have to do is, we have to show that $f + g$ is Riemann integrable and not only that we have to show that this integral of the sum is the sum of the integrals.

So, what you do is given any partition P of $[a, b]$, we have the following chain of inequalities. So, this following chain of inequalities is the heart of the proof. We have $L(f, P) + L(g, P) \leq L(f + g, P) \leq U(f + g, P) \leq U(f, P) + U(g, P)$. We have this chain of inequalities.

Let us see, why we have this chain? Now, first of all certain inequalities here are straightforward this inequality $L(f + g, P) \leq U(f + g, P)$ just follows from one of the basic facts about upper sums and lower sums. Why is this first inequality true? Well, think about what is happening?

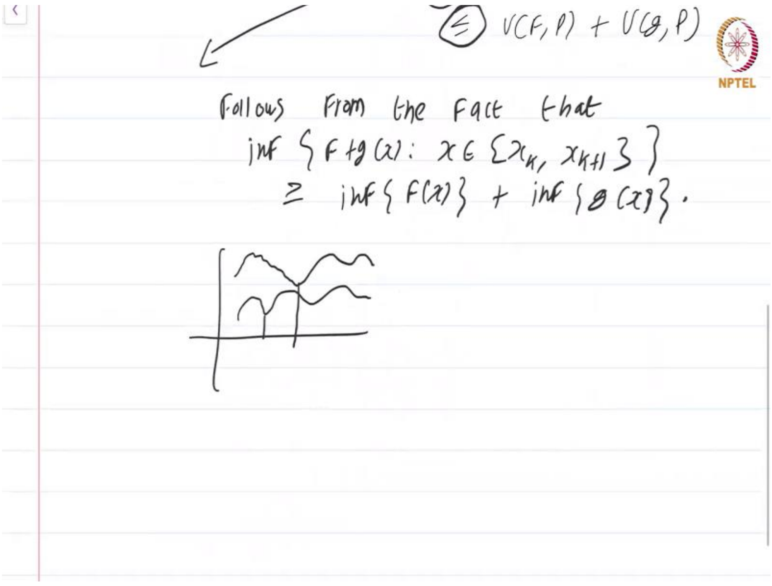
In each one of these partitions we are going to take the $\sum m_i \Delta x_i$ ok. Now, what will happen is when you take the minima of the functions f and g. Then in general the $\min(f + g) \geq \min(f) + \min(g)$ ok.

(Refer Slide Time: 04:37)



So, this inequality follows from the fact that minimum of f , g is less than or equal to or rather here it was the infimum though it turns out to be the minimum also.

(Refer Slide Time: 04:40)



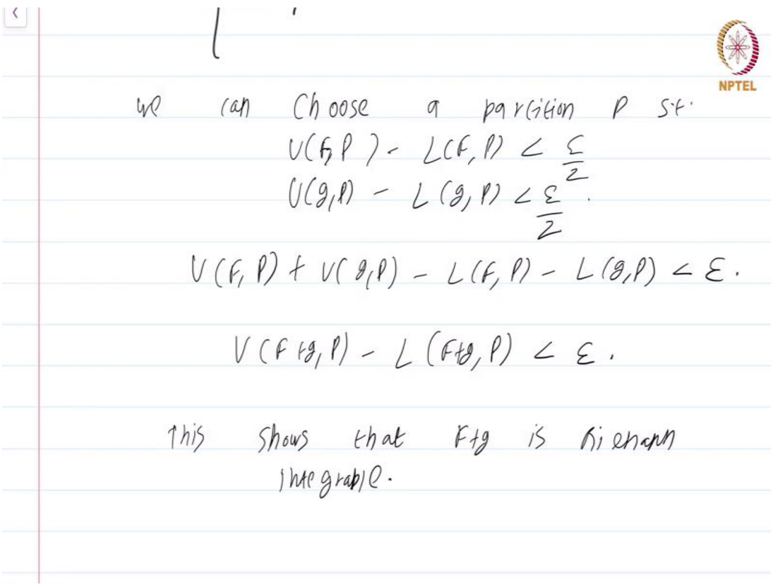
$\inf\{(f+g)(x) x \in [x_k, x_{k+1}]\} \geq \inf\{f(x)\} + \inf\{g(x)\}$, as x runs through the same set ok. So, what is happening is when you take the sum of these two functions. So, a graph will illustrate what is happening, ok; if, I take this to be the graph of f and this to be the graph of g.

Now, there is the minima of the function f is clearly here whereas, the minima is here. And, they are not coinciding. Unless they coincide it will not happen that $\inf\{f+g\} = \inf\{f\} + \inf\{g\}$. So, because of this when you take the lower sum on each sub interval $[x_k, x_{k+1}]$, the terms coming here will dominate the sum of these two terms; the corresponding sum of these two terms.

So, because of this first inequality is true. In an entirely analogous way we have this inequality, that $U(f+g,P) \leq U(f,P) + U(g,P)$. Again when you take the sum of the maxima or supremum of $f + g$ on a particular interval $[x_k, x_{k+1}]$, it is highly unlikely that the points at which f attains supremum and g attains supremum.

It is highly unlikely that they will coincide. Because of this it will turn out that the $\sup\{f + g(x)\}$: as x runs through that interval is in general less than or equal to the $\sup(f) + \sup(g)$, ok. So, we get this chain of inequalities. Now, how does this chain of inequalities help? Well, we can we have full flexibility over the choice of partition.

(Refer Slide Time: 06:53)



We can choose, we can choose a partition P such that $U(f,P) - L(f,P) < \frac{\varepsilon}{2}$. And, similarly we can choose such that $U(g,P) - L(g,P) < \frac{\varepsilon}{2}$.

Note, I have said you can choose a single partition such that both of these inequalities are simultaneously satisfied. The this is coming from the criterion for Riemann integrability, we know that f and g are Riemann integrable. Therefore, we can find a partition such that $U(f,P) - L(f,P) < \frac{\varepsilon}{2}$.

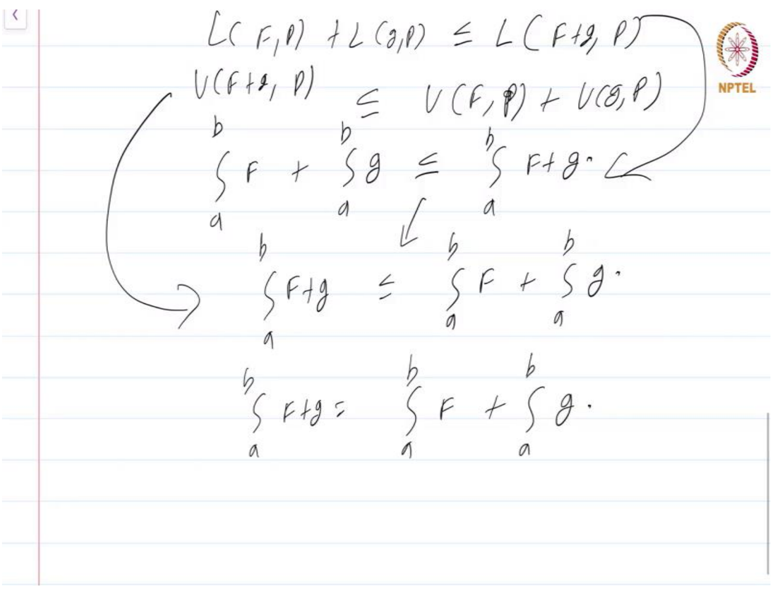
Similarly, you can find another partition such that $U(g,Q) - L(g,Q) < \frac{\varepsilon}{2}$. But, I have written the same partition, because we can take the common refinement $P \cup Q$ and this will still be true, that common refinement I am continuing to call P ok.

So, we can choose a partition such that $U(f,P) - L(f,P) < \frac{\varepsilon}{2}$ and $U(g,P) - L(g,P) < \frac{\varepsilon}{2}$. What this gives is $U(f,P) + U(g,P) - L(f,P), -L(g,P) < \varepsilon$ ok.

So, excellent what does this give? Let us go back little bit up what do we have. We have $L(f,P) + L(g,P)$ as the right extreme and $U(f,P) + U(g,P)$ as the right extreme. What this says is that we immediately get $U(f+g,P) - L(f+g,P) < \varepsilon$ ok. Because this inequality is true for the left and right extremes it has to be true for anything that is sandwiched in between as well ok.

So, this will show that the function that $f + g$ is Riemann integrable, that much is clear. Now, what remains to be shown is that the integral of the sum is the sum of the integral. Why does that follow? Well let us think for a moment. We know that the integrals exist therefore, this integral value has to be the supremum of $L(f+g,P)$ or the infimum of $U(f+g,P)$ as you run through all partitions right.

(Refer Slide Time: 09:51)



But, as you run through all partitions we still have the inequalities, $L(f,P) + L(g,P) \leq L(f+g,P) \leq U(f,P) + U(g,P)$. I am just taking part of this inequalities. So, we are going to take the infimum as you run through all partitions of the quantity $L(f+g,P)$ that is what we are going to do.

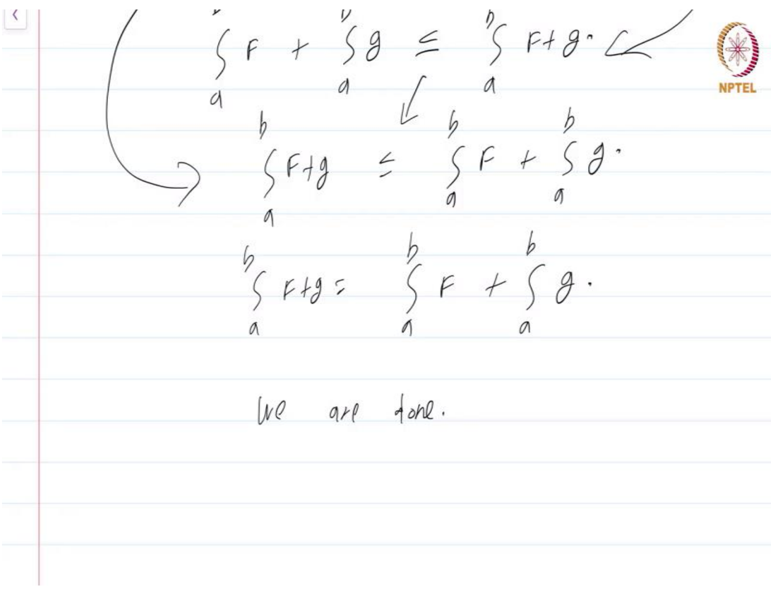
So, clearly what we will get is $\int_a^b f + \int_a^b g$ is less than, or sorry we are not going to take the supremum, scratch that, we are going to take the supremum, sorry about that the integral is obtained by taking the supremum of the lower sums or the infimum of the upper sums ok.

So, we get $\int_a^b f + \int_a^b g \leq \int_a^b (f+g)$ ok. That is just coming from the first inequality wait a second I should write the other inequality also that is $U(f+g,P) \leq U(f,P) + U(g,P)$.

Now, this is what we obtain from the first inequality, from the second inequality taking infimum on both sides, you will get $\int_a^b f + g \leq \int_a^b f + \int_a^b g$ ok.

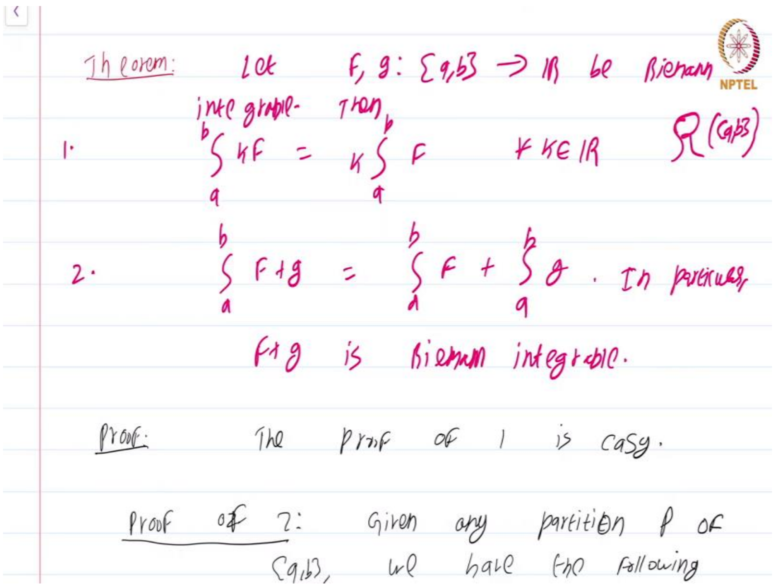
Putting these two together we get the desired result $\int_a^b f + g = \int_a^b f + \int_a^b g$ ok. So, this concludes the proof, the sum, the integral of the sum is the sum of the integral we are done, we are done ok.

(Refer Slide Time: 12:02)



So, this property that we have done that $\int_a^b k\,f = k\int_a^b f$ and $\int_a^b f + g = \int_a^b f + \int_a^b g$ is called linearity. So, essentially what will happen is if you consider the collection of all Riemann integrable functions that is usually denoted by the script R.

(Refer Slide Time: 12:34)



The collection of all Riemann integrable functions is usually denoted by script $\mathcal{R}([a, b])$, this is a collection of functions which can be made into a vector space over the real numbers ok. And, you have a linear transformation that is the integral acting on $\mathcal{R}([a, b])$, it is actually a linear functional. This allows us to study integration from the perspective of linear algebra. Now, this is not part of this course, but it is worth knowing ok.

So, if you are familiar with linear algebra I urge you to explore the properties of Riemann integral as a linear functional. This is a course on Real Analysis and you have just watched the module on the linearity of the Riemann integral.