Real Analysis - I
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Lecture – 25.1
Axiomatic Characterization of Area and the Riemann Integral

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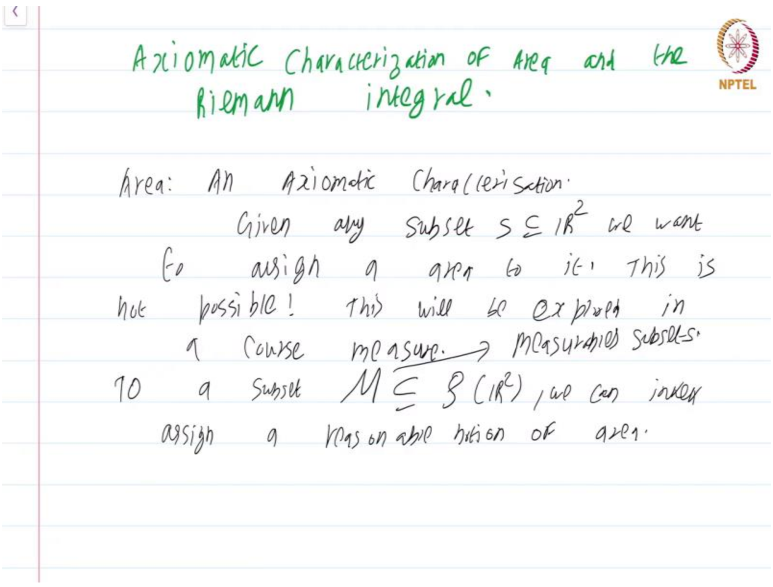

Much of the mathematics in this course is motivated by real world problems. By real world problems I actually mean problems from physics. The notion of derivative was solely designed to answer this question. Given a complicated function, can I approximate it reasonably well by a linear function? The linear function is much easier to work with than the complicated function and therefore, we often use the properties of the derivative to simplify the situation.

In a similar manner, the notion of integral also came to solve a physical problem, what is area right. Now, if you think about it in school you were just told that the area of a rectangle is the product of its sides. You might have derived the area of several other figures using this fact that the area of a rectangle is length into breadth. Like parallelograms or triangles etcetera, you might be able to find out the area formula for those using the area of a rectangle.

If you think about it more carefully from the perspective of an axiomatic approach that we are adopting in this course, the ideal way to proceed will be to characterize what area is through axioms and see that from these axioms the areas of all the commonly known figures are indeed what the formulas that you are familiar with say they are. So, what we will first do is before we get to the Riemann integral, let us talk a bit about area ok, an axiomatic characterization.

In an ideal world given any subset $S \subset R^2$, we want to assign an area to it. So, in an ideal world given any subset of R^2, you should be able to figure out what the area is, but things are never so ideal in mathematics. What happens is for deep set theoretic reasons this is not possible.

What I mean by this is not possible is of course, given any set S, I can always assign its area to be 0 and be done with it and go home. What I mean is you cannot assign a reasonable notion of area to all subsets of R^2 ok. So, this will be explored in a course on measure theory. It is actually a graduate topic, it is possible to do it at the undergraduate level, but it is a bit difficult, but nevertheless it is not within the scope of this course to see why this is not possible.

So, what we can do is, to a subset $M \subset \mathcal{P}(R^2)$, we can indeed we can indeed assign a reasonable notion of area. This M is called the collection of measurable subsets, ok.

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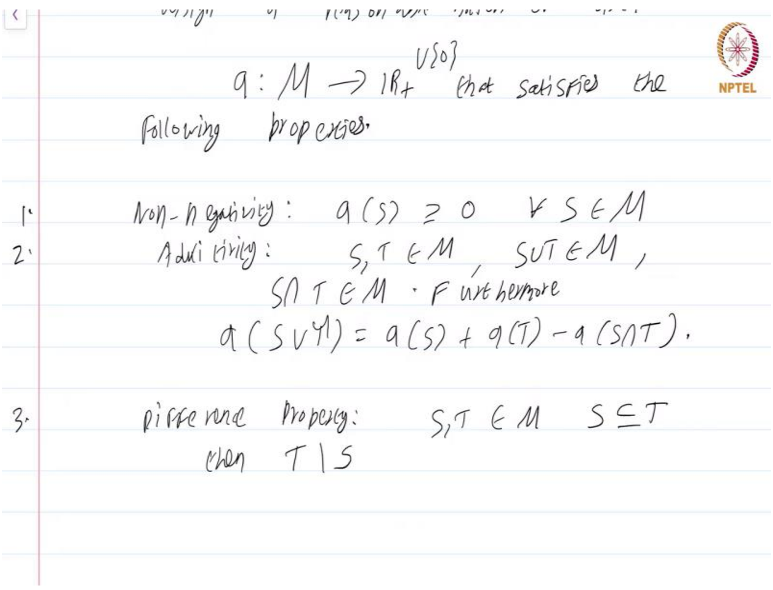

So, you have a function a from M to R^+ that satisfies the following properties, the following properties. Number 1 is actually built into the definition non negativity. So, to be 100 percent precise I must write$R^+ \cup \{0\}$. Non negativity: $a(S) \geq 0$, for all S coming from this collection of measurable subsets of R^2.

Number 2, additivity property: additivity. This just says that if $S,T \in M$, then $S \vee T \in M$ and $S \wedge T \in M$. furthermore $a(S \vee T) = a(S) + a(T) - a(S \wedge T)$. I hope you recall that this axiom is crucially used way back in ninth or tenth standard when you are figuring out the areas of common figures like quadrilaterals and so on.

You would have definitely used this axiom without actually thinking about it deeply. So, this is a property that you would definitely want areas to have. If you have two sets for which area makes sense, then area makes sense for the union and the area of the union is nothing but the sum of the areas minus the sum of the common part. The third axiom is equally very very intuitive.

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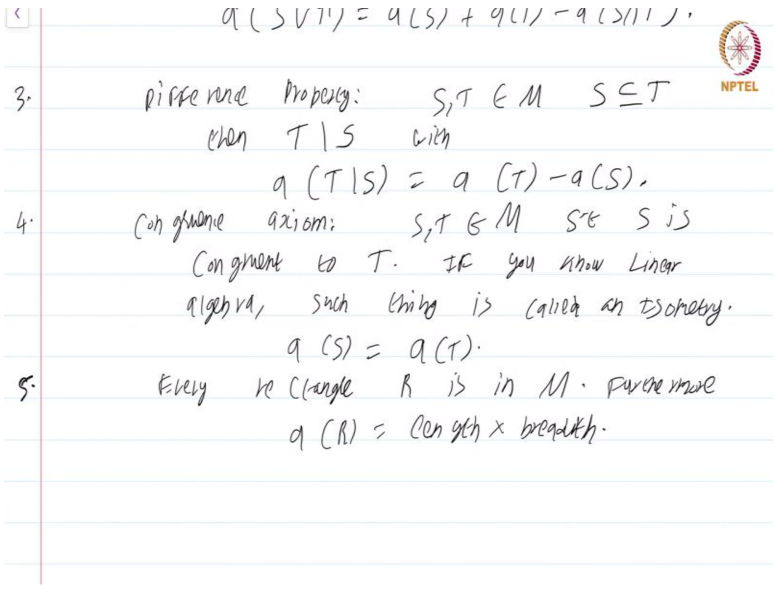

The difference property: this just says that if $S,T \in M$ with $S \subset T$ then so is T/S with $a(T/S) = a(T) - a(S)$, again a perfectly natural axiom. If you have two figures for which area makes sense and one figure is fully contained in the other then it makes sense to just excise that figure away from the larger figure and be left with the figure whose area is $a(T) - a(S)$. Again this is a perfectly reasonable perfectly intuitive axiom that you would definitely have, area to have.

Now, number 4 congruence axiom. Suppose $S,T \in M$ such that S is congruent to T. Now, what is S is congruent to T actually mean? Well, it means the following. I can take S, move it around, rotate it or reflect it about a line and somehow make S coincide perfectly with the set

T; that means, I can take S and somehow put it on top of T and I cannot do arbitrary stuff all I am allowed to do is move S, rotate S and reflect S on some line in R^2 ok.

Such a thing is called an isometry if you know linear algebra such a thing is called an isometry called an isometry ok. These are precisely the mappings from R^2 to R^2 that preserve lengths ok. So, congruence axiom just says that if two figures can be made to overlap each other by moving it around by applying what is known as a rigid motion then S and T must have the same area.

So, congruent figures have the same area, again intuitively perfectly obvious. And the fifth one so far the first 4 axioms are perfectly satisfied by the function a from power set of R^2 to R given by $a(S) = 0$, for all subsets right. So, this will give a completely nonsensical theory of area where every set has area 0. So, recall the remarks I made about area of rectangle. Well, you have to start somewhere. All theories must begin somewhere you cannot get something out of nothing. So, every rectangle $R \in M$.

Furthermore, area of this rectangle is length into breadth is length into breadth. So, this is sort of the starting point in some sense of this notion of area. You already know what the area of rectangles are.

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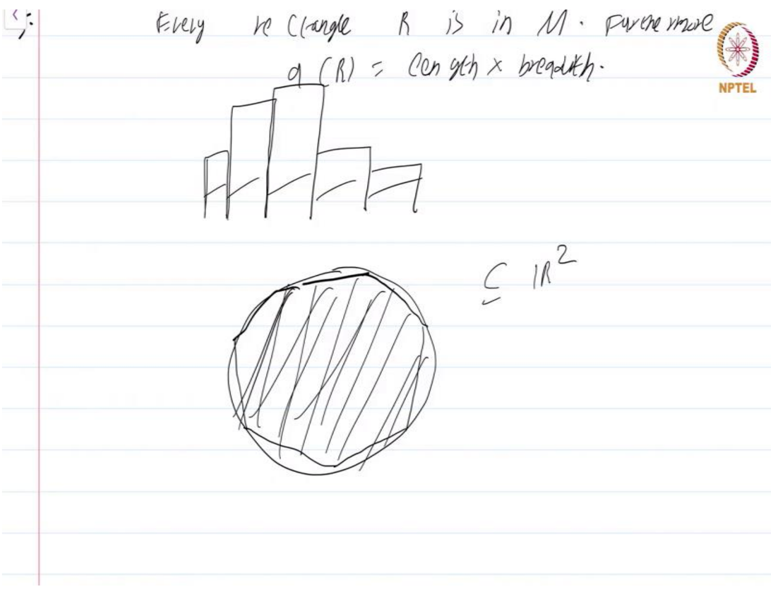

Now, notice that just with these axioms you can assign areas to figures like this, a bunch of adjacent rectangles that is going to be just the sum of these areas, ok.

So, you can find out areas of more complicated figures by using the first 5 axioms, but to really get the full force and have a notion of area that can deal with many many complicated figures, I need to give you an example of a complicated figure and it will turn out to be not so complicated at all its just a circle.

Now, if you ponder for a few hours in fact, you will be able to realize that there is absolutely no way just by using these first 5 axioms, we can conclude that this set this circle, when I mean circle I mean with the interior with the interior which is contained in R^2 somewhere.

There is no way to conclude that first of all this thing has a well defined area second of all that the area of that is πr^2 neither of that will be possible just from the 5 axioms ok. However, we already know there is one way to find out an approximation of the area of this circle that is you just inscribe a polygon in it.

You just inscribe a polygon. This might be familiar to you from your high school studies or you can circumscribe a polygon, I am not even going to draw and ruin what is already quite a ugly figure. So, you can inscribe polygons and you can circumscribe polygons also and it will be easy to see that for these polygons just from the fact that rectangles have areas and unions, intersections all those have areas, you can show that polygons will have well defined areas.

You can approximate this circle to as good a degree as you desire by polygons that are inscribed inside the circle as well as polygons that are circumscribed in the circle. And these approximations by taking some sort of limit you will be able to conclude what the area of the circle is. Now, we want to incorporate this idea of exhaustion from inside and outside as an axiom.

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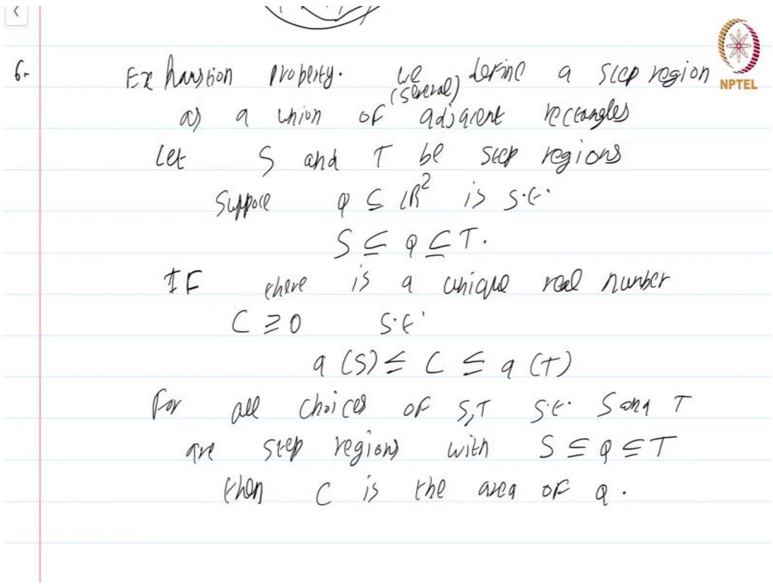

And that is finally, the exhaustion property. Without this it will be impossible to assign areas to figures like ellipses and circles and so on. Exhaustion property ok; we define we define a step region as a union of adjacent rectangles ok. This is not just union of several adjacent rectangles that is what a step region is because it looks like a staircase ok.

So, let S and T be step regions. from the previous axioms both S and T will be measurable and you can assign areas to S and T. Suppose, Q subset of R^2 is such that is such that $S \subset Q \subset T$ ok.

So, you can squeeze Q between the subsets. If there is a unique real number $c > 0$ such that $a(S) \leq c \leq a(T)$ for all choices of S, T such that S and T are step regions, S and T are step regions with $S \subset Q \subset T$, then c is the area of Q, ok.

So, this sort of precisely says that this exhaustion procedure that we used to determine the area of the circle; there we use polygons here we are allowing only step regions for our purposes this will actually be more than enough. Just by step regions you can show that this polygonal approximation is also valid. You will be able to compute the area of many many figures by approximating from the inside and the outside by the step regions.

So, this exhaustion property makes this precise ok. So, now, I have written 6 properties that we would like areas to have ok. So, I have already told you that it is impossible to find a function a from R^2 not R^2 from power set of R^2 to R, you cannot find a function that satisfies all these

6 axioms that is for deep set theoretic reasons which you will definitely visit in a course on measure theory.

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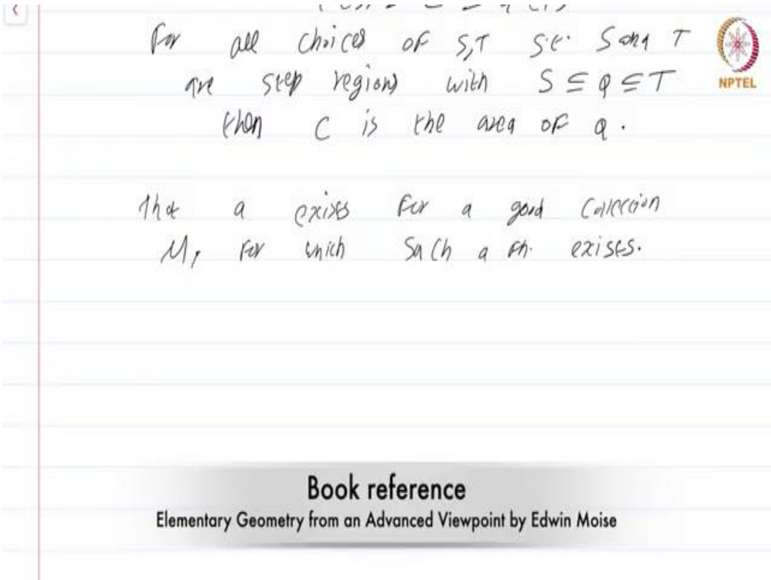

On the other hand; on the other hand, that a exists for a good collection M for a good collection M. By good collection I just mean that any reasonable set that you are likely to encounter in any physical application of engineering or physics or chemistry or whatever.

There is a good collection M for which such a function exists. So, there is a reasonably big collection for which you can find an area function.

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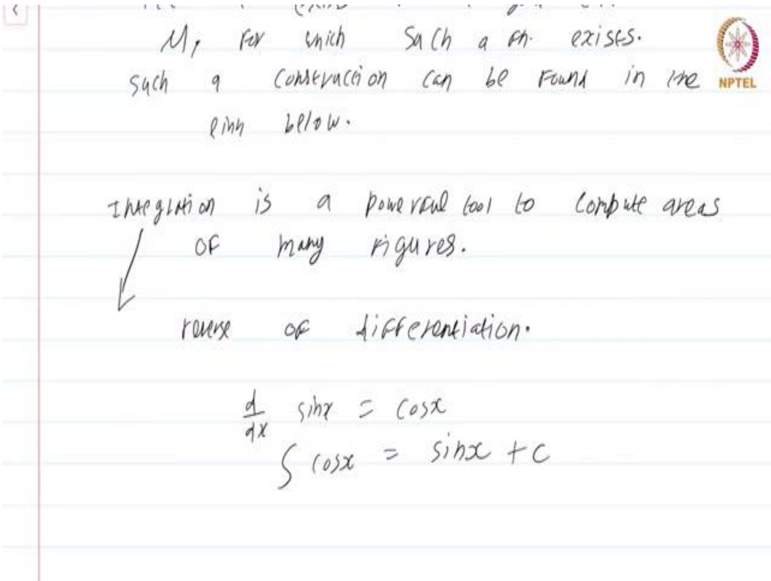

Such a construction can be found in the link below. Rather it is just a reference it is not really a link. So, anyway there is a textbook you can go through it how this is constructed in quite detail.

Now, this is about general area. There is a function that for most reasonable subsets of R^2 you can assign an area. The next question is there is a function area is very different from? I give you this figure tell me what its area is. Just knowing that there is such a function is of no practical use, unless there is some way to compute right.

So, how do you compute areas? Well, integration is a tool in fact, a powerful tool to compute areas of many figures. So, in the coming modules, we will go through the construction of the Riemann integral in quite detail. We will do it in quite detail.

The question also arises that this integration is supposed to be reverse of differentiation. In fact, when you learn integration in school, at least when I learnt integration in school I was not defined the Riemann integral at first. I was taught that $\frac{d}{dx}\sin x = \cos x$. Therefore, integral of $\cos x$ is $\sin x + c$; otherwise you will lose -1 or some such stupidity.

But, this is the way I was taught that integration is the reverse of differentiation, but that completely hides the motivation behind integration. The motivation behind integration was to find areas. The motivation behind differentiation was to find slopes of the tangent at a particular point.

It turns out that integration and differentiation are sort of reverse of each other and that is called the fundamental theorem of calculus. It is called the fundamental theorem of calculus simply because it is the most important theorem from calculus except maybe the mean value theorem.

Now, since we are taking the more conceptually correct approach of first talking of Riemann integrals as areas, the question might arise what is all this have anything to do this notion of area and Riemann integral, why would you even suspect that it is in fact, related to differentiation in some way. Well, it is because of the following theorem.

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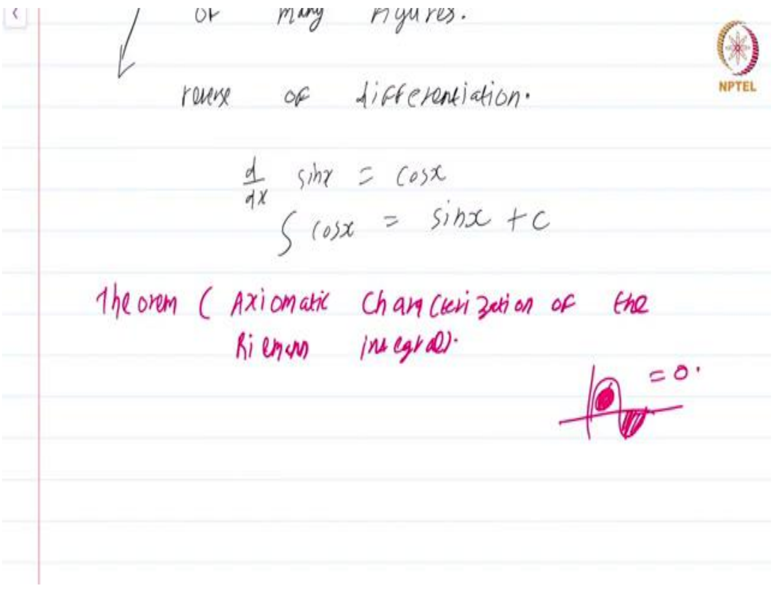

Theorem; this is the axiomatic characterization of the Riemann integral of the Riemann integral. What does it say? Now, the Riemann integral is not very useful in computing the area of some arbitrary curve or arbitrary region in R^2, it is completely useless.

It is more useful to compute areas of figures that can be broken down into subfigures each of which is some graph ok, each of which is actually the region under a graph. So, this Riemann integral is used to find out the area under the graph of a function and it is a peculiar type of area, it is not exactly the same area as what we have been talking about up until now.

It is sort of a signed area which means, if you take this figure if you take this figure this area will not be twice this area, it will the total area will not be twice this area rather it will be 0 because this figure underneath that is supposed to be negative.

So, why we are considering areas to be negative when you come to Riemann integrals will become very clear, it makes the theory lot simpler and anyway you can make it compute the actual areas by doing simple things ok. So, this Riemann integral is supposed to measure the area under the graph of a function. So, this is our setup.

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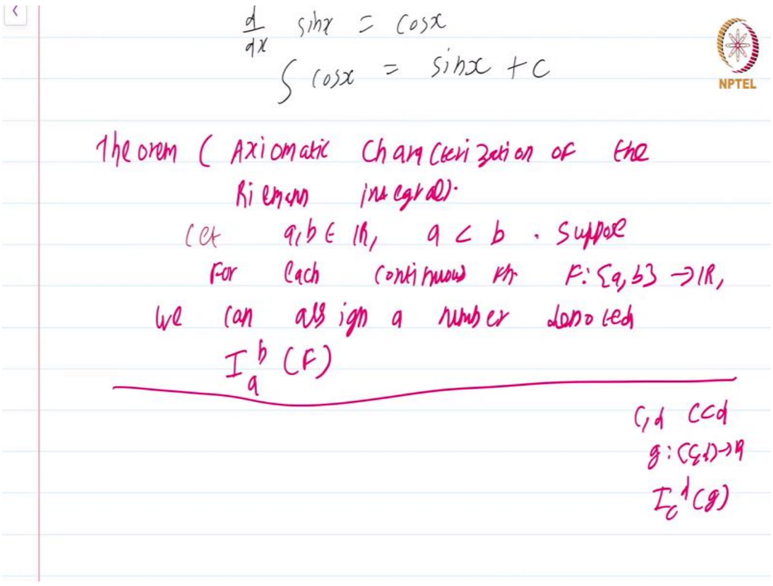

Let $a, b \in R$, $a < b$, ok. Suppose, for each continuous function $f: [a, b] \to R$ we can assign a number denoted $I_a^b(f)$. This is just going to stand for integral I from a I integral of f from a to b ok, but I am going to denote it by $I_a^b(f)$. Note, what this hypothesis is so far saying is that given any $a, b \in R$ and each continuous function you can do this.

In particular if you take some other c, d and $c < d$ and some other function $g: [c, d] \to R$, you can talk about $I_c^d(g)$ also. So, whatever closed interval you give, whatever continuous function on that closed interval you take there is a way to assign this number which we have denoted $I_a^b(f)$ ok. So, that is the preliminary remarks about the setup. This I_a^b satisfying denoted I_a^b satisfying, so, this is going to be an axiomatic characterization.

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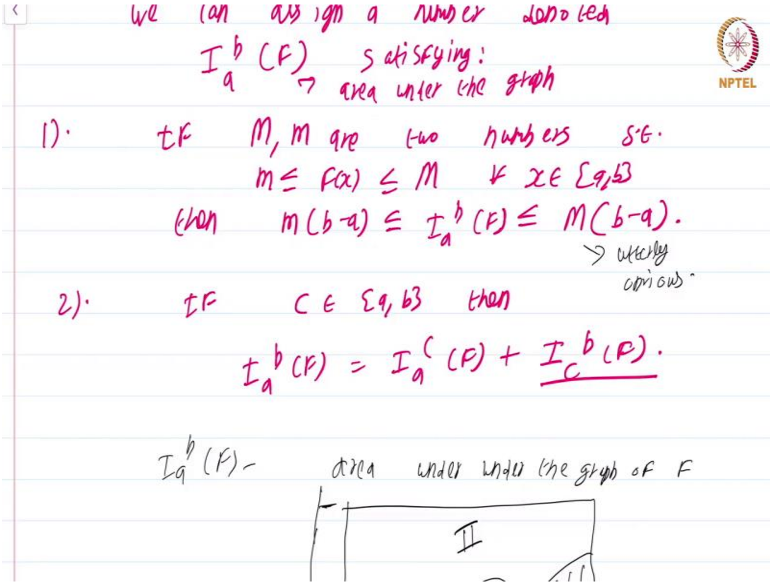

So, I will have to characterize right, I will have to tell you what it satisfies. The first property is that if m, M are two numbers such that $m \leq f(x) \leq M$, for all $x \in [a, b]$ then $m(b-a) \leq I_a^b(f) \leq M(b-a)$. 2: If $c \in [a, b]$ then $I_a^b(f) = I_a^c(f) + I_c^b(f)$ ok, excellent.

So, these are the two things that I_a^b satisfies. I repeat again, the way it is written it might think as it might look as if a and b are fixed, they are not. Given any two numbers a, b and any continuous function $f: [a, b] \to R$, these two properties must hold. So, this is sort of an hypothesis on not just continuous functions defined on this given interval $[a, b]$ to R, but continuous functions defined on any closed interval in R ok.

So, please do not make the mistake of thinking that this function is defined only for this particular pair. If that were so, this will not even make sense. These two quantities will not even make sense ok.

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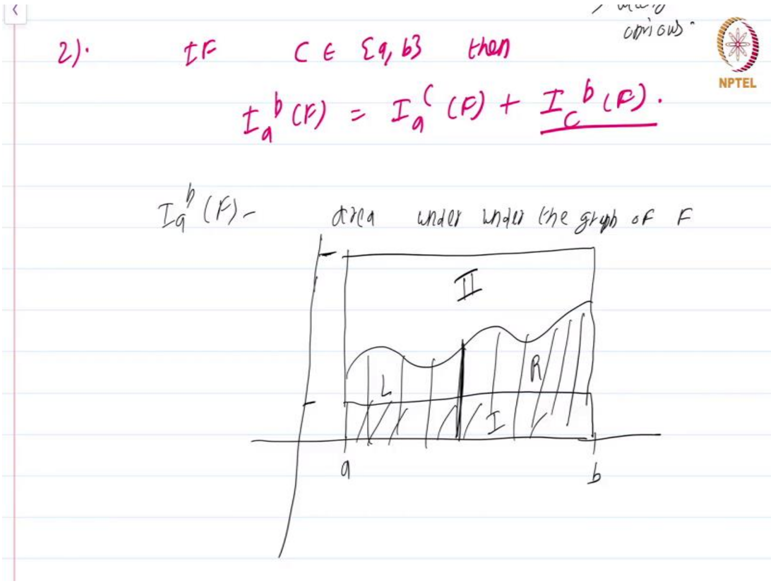

So, before I write down the conclusion, let me tell you what these two axioms are trying to say. This $I_a^b(f)$ is supposed to be area under the graph. This is supposed to be area under the graph under the graph of f ok. What this is saying is the following.

Suppose you have the x and y axis and just for concreteness and not confusing you, let us just take a completely positive function and a nice looking one in that. Then what you are doing is you are choosing these numbers small m and capital M such that small $m \leq f(x) \leq M$, for all $x \in [a, b]$. So, small m could be somewhere like here and capital M is somewhere like here ok. So, let us draw these figures from a to b ok.

So, this is a and this is b. So, small m is here. So, consider this rectangle and consider this larger rectangle ok. Now, look at this area, look at this area. Now, clearly this smaller rectangle which I call 1 is fully contained in the region under the graph and this region 2 fully contains the region under the graph. Therefore, if you look carefully at the axioms that characterize area this is nothing this is just this is just utterly obvious. This is just utterly obvious.

If this $I_a^b(f)$ indeed is measuring the area under the graph this should be utterly obvious ok. So, now, let us look at the second property that is also going to say the same thing. That is just saying if you break up this area into two pieces into two pieces let us say L and R then the area under the graph is nothing but the area $L + R$. Again this also follows directly from the various axioms of area that we have written ok.

So, if you notice these two axioms for the Riemann integral are just coming from some of the properties that characterize area ok. This is not an entire characterization of area that we are using here just these two simple properties.

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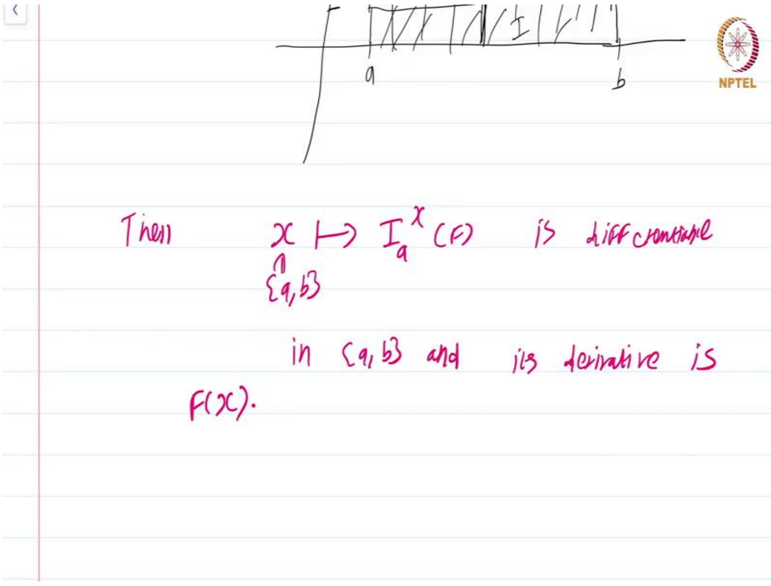

Then the conclusion. So, let me move back to red because we are at the conclusion stage. Then the conclusion of this theorem is follows is as follows. Look at the function $x \mapsto \mathrm{I}_{\mathrm{a}}^{\mathrm{x}}(f)$, $x \in [a, b]$. So, what you are doing is you are taking the given point x which is there in the interval $[a, b]$ and mapping it to the area under the graph of f, but only till x not all the way till b.

Then this function is differentiable in a, b and its derivative is as you can guess nothing but f of x nothing but f of x. So, this big theorem called the axiomatic characterization of the Riemann integral just says that if you take the most primitive properties that you would expect a function that takes a given function to the area under the graph of the function.

If you take the very most basic properties that you would expect that is properties 1 and 2 that I have written here, it turns out that something like the fundamental theorem of calculus is true. This is just if you look back at your NCERT textbooks, you will notice that this is just the disguise form of fundament, it is not even that disguise, it is a poorly disgraced form of the fundamental theorem of calculus. So, what this axiomatic characterization is saying is that areas and Riemann integral sorry; areas and differentiation are very tightly knit together.

If you take the very basic properties of area under the graph the fundamental theorem of calculus has to drop out as a simple consequence. Now, as it is this module is gone for quite some length, I will leave it you to process whatever I have told. I have told a lot, please process this. In the next module we will see a proof of this characterization and then onwards to the construction of the Riemann integral.

Note, just because you say that if there is a function that satisfies something then it is got to be it has to satisfy fundamental theorem of calculus, does not tell you that there is such a assignment. We have not told you how $I_a^b(f)$ is to be computed that will be done by a precise construction which will be done in a later module.

This is a course on real analysis and you have just watched the module on the axiomatic characterization of the area and the Riemann integral.