Real Analysis - I
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Lecture – 22.3
Examples of Differentiation

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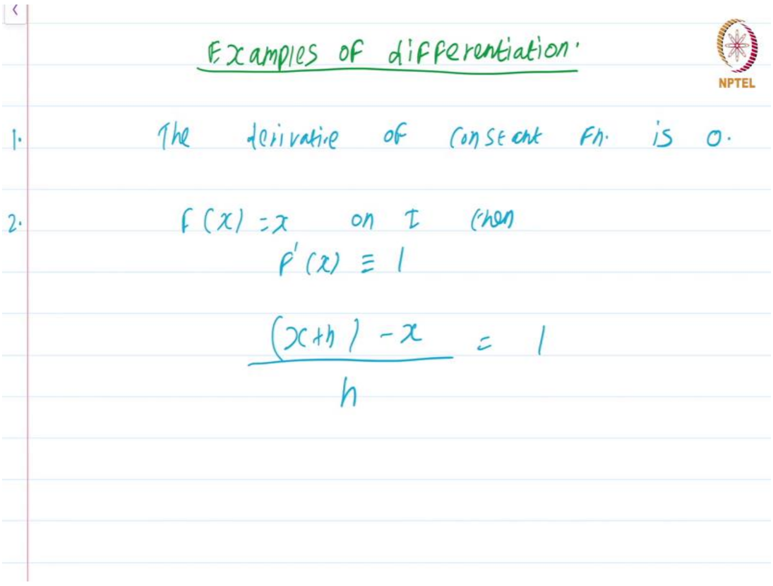

Let us look through some examples of differentiation. As you will see later when we come to integration, differentiating functions is not that hard compare to integration that is because no matter of what complicated function you are given by repeated application of the various laws of differentiation like sum of derivatives is derivatives of the sum, then product rule, Leibniz product rule and chain rule, you will be able to differentiate any complicated function provided you have the patience.

So, let us see first example. Well, the first example is the derivative of constant function is as you can expect 0. This is because the numerator in the Newton quotient will be identically 0 that is fairly easy. And next if you take $f(x) = x$ on the interval I, then $f'(x) = 1$, ok. Why is this? Well because at a given point to take the derivative, you will have to take $\frac{(x+h)-x}{h}$, h going to 0, this will just turn out to be 1.

So, because the Newton quotient will be identically 1, the limit will always be 1, and the derivative of the function $f(x) = x$ that is the identity function is as you can expect constant 1 ok.

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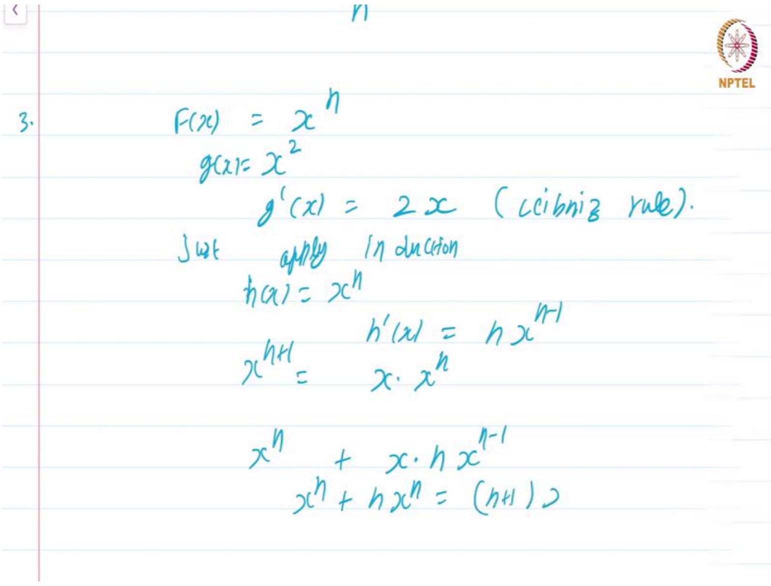

Third example, if you take $f(x) = x^n$, well, there are several ways to prove what the find out what the derivative is. My favorite way is to just repeatedly use the fact that if you take x^2, let us for instance take x^2, then let us call this $g(x) = x^2$, then $g'(x) = 2x$. This is just by Leibniz product rule, Leibniz rule. And how does this generalize to x^n?

Just apply induction now, that is assume that derivative of x^n if you take x^n, let us call this $h(x) = x^n$, then $h'(x) = nx^{n-1}$, then if you want to differentiate x^{n+1}, just apply this Leibniz product rule this is just xx^n.

So, when you take derivatives, you just get $x^n + x \cdot \mathrm{n}x^{n-1}$, which will just give you , $x^n + nx^n$,which is $(n+1)x^{n+1}$, which is what you want. There is another way to do it by using an algebraic identity, I leave it to you in the exercises ok.

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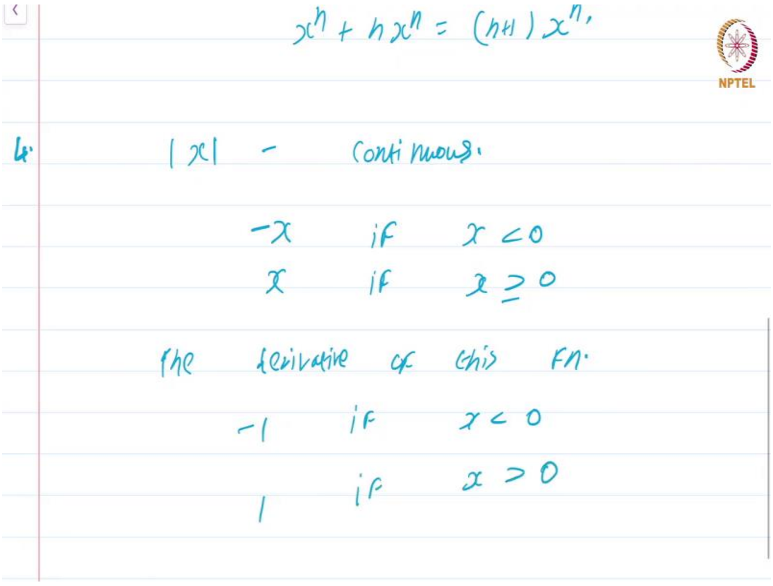

So, far three simple examples. Let us slightly modify this example at least in the case take $|x|$. This function as we all know is continuous; this function is certainly continuous. And this function is equal to $-x$ if $x < 0$; and x if $x \geq 0$, again in fact greater than or equal to 0. So, the derivative of this function - 1, if $x < 0$; and + 1 if $x > 0$. This much we know. What about at $x = 0$, what about at $x = 0$?

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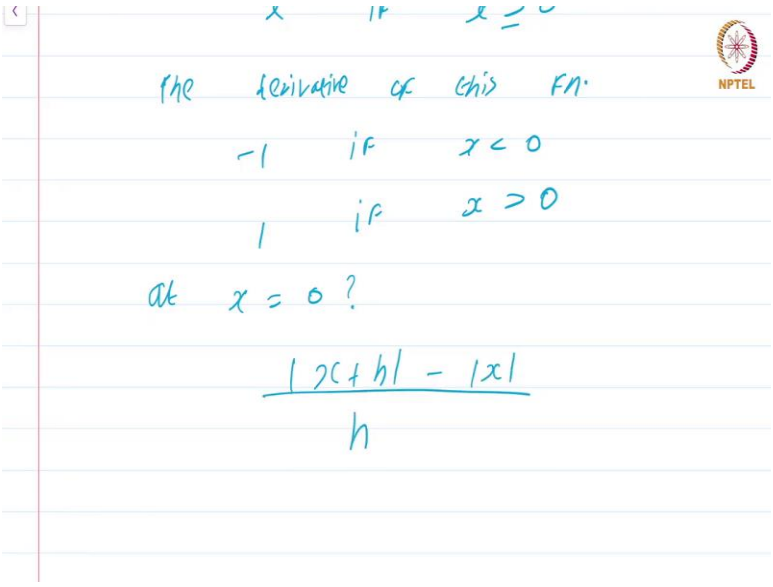

Well, let us check what happens, let us check what happens. You have to take $\frac{|x+h|-|x|}{h}$.

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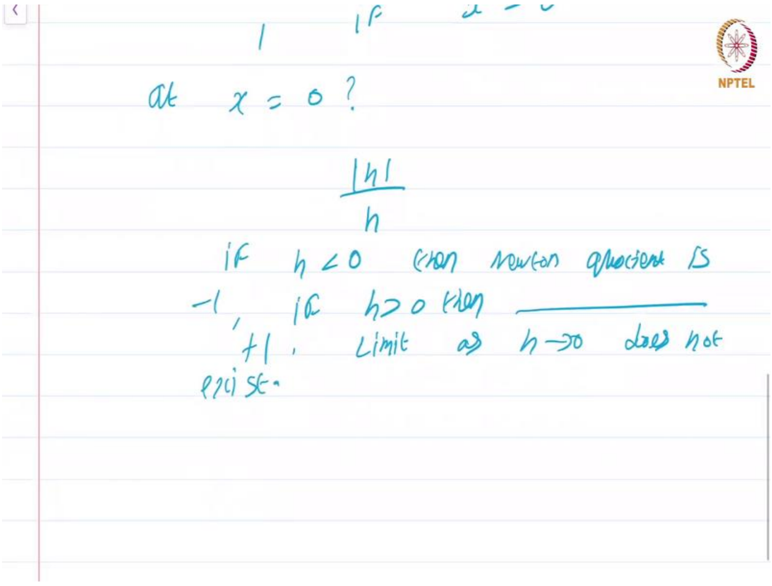

And here $x = 0$. So, you just have to take $\frac{|h|}{h}$. This is the result of the Newton quotient. If $h < 0$, then Newton quotient is - 1. If $h > 0$, then Newton quotient is + 1, limit as h approaches 0 does not exist. So, this function is not differentiable at the point $x = 0$, which is sort of intuitive because the picture of this function looks like a V ok.

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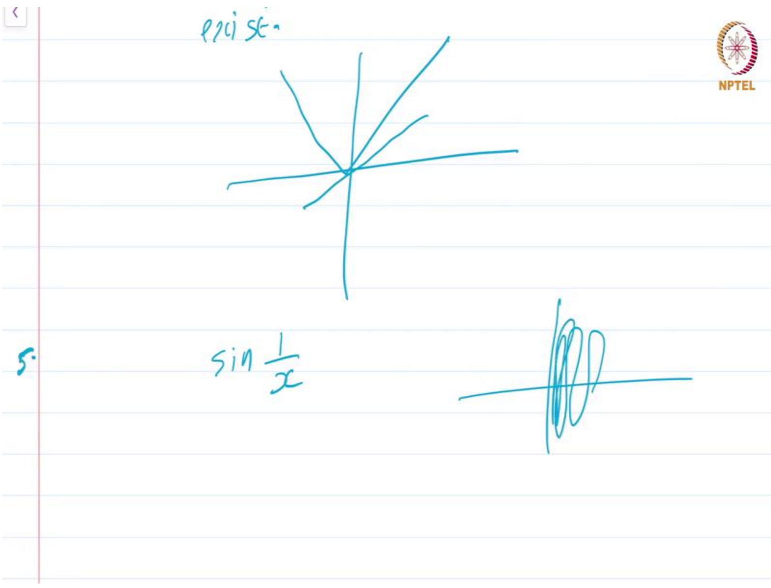

Now, the interpretation is that the slope of the tangent at this particular point does not make any sense. There is a sharp bend, this curve the graph is not turning smoothly at the point zero, it seems like an abrupt break. So, you are not able to define the slope of the tangent at that particular point ok. So, the derivative here is not defined at $x = 0$, ok.

Let us take a more complicated example where things start to go wrong at the origin. Recall the function $\sin\left(\frac{1}{x}\right)$, this is the function that I dread drawing every time because it is insanely difficult to draw. It is just oscillates wildly as you approach the origin. However, even though it oscillates widely it oscillates between + 1 and - 1 that is a basic property of the sin function.

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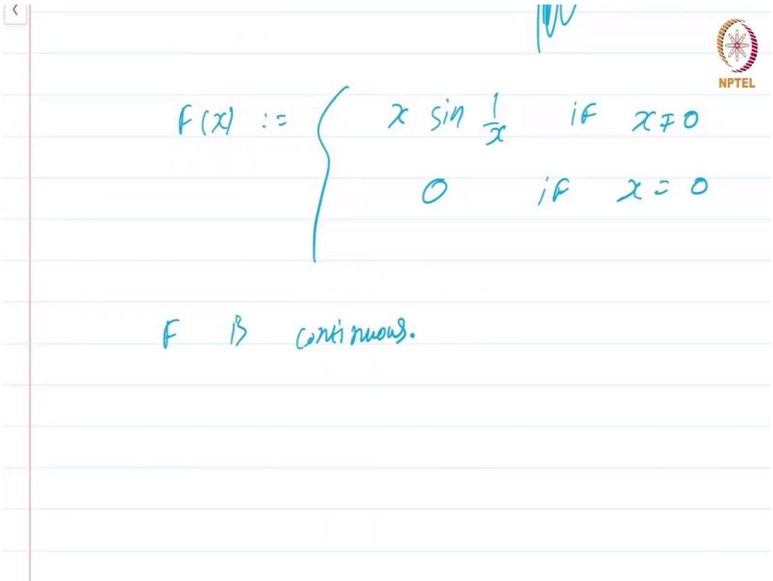

So, let us try to temper these oscillations by multiplying by x. So, consider the function $f(x) = x\sin\left(\frac{1}{x}\right)$ if $x \neq 0$; and 0 if $x = 0$. We already know that this f is continuous. This we have seen in the example section of continuity.

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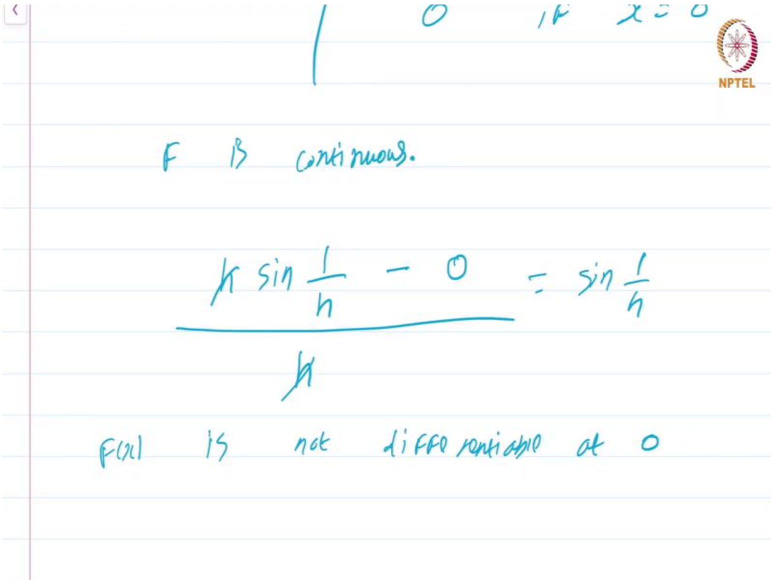

Now, let us try to take the derivative at the origin. So, what you have to do is take $\frac{h\sin\left(\frac{1}{h}\right)-0}{h} = \sin\left(\frac{1}{h}\right)$. And therefore, this function is not differentiable, $f(x)$ is not differentiable at 0, because you end up with the function $\sin\left(\frac{1}{h}\right)$, which is not, when as h goes to 0, this widely oscillates, the limit does not exist. So, $f(x)$ is not differentiable at 0 ok.

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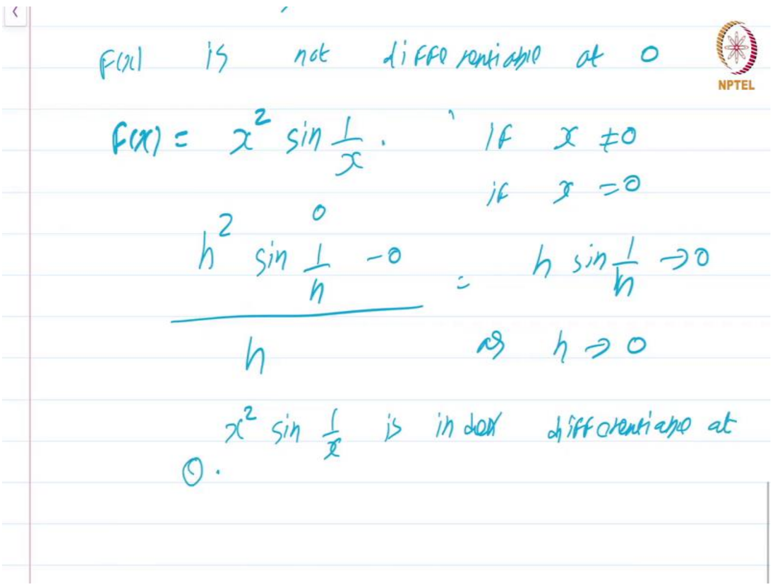

So, tempering with the x did not help. Well, let us temper with x^2 maybe that has a better effect to temper down the wild oscillations of $\sin\left(\frac{1}{x}\right)$. Remember as x approaches 0, x^2 sort of goes to 0 faster than x. So, that is the idea, maybe this will help. Well, again we have to take, so again let me define the function $f(x) = x^2 \sin\left(\frac{1}{x}\right)$ if $x \neq 0$; and 0 if $x = 0$. So, let us take the Newton quotient. So, it will be $\frac{h^2 \sin\left(\frac{1}{h}\right) - 0}{h} = h \sin\left(\frac{1}{h}\right)$.

And this approaches 0 as h goes to 0 because the oscillations of $\sin\left(\frac{1}{h}\right)$ are bounded between + and - 1, and the factor h will ensure that the product is 0. So, the function $x^2 \sin\left(\frac{1}{x}\right)$ is indeed differentiable at 0 ok; and the derivative is 0 ok.

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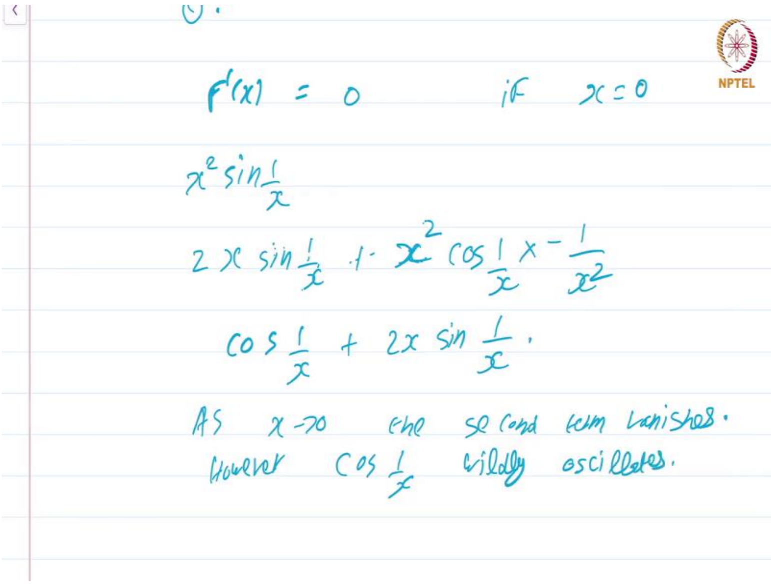

Now, let us try to figure out what the derivatives behavior is. We already know that $f'(x) = 0$, if $x = 0$. What about other points. Well, we can do use the product rule, chain rule and so on. So, $x^2 \sin\left(\frac{1}{x}\right)$, when you differentiate, you will get $2x \sin\left(\frac{1}{x}\right) + x^2 \cdot$ the derivative of $\sin\left(\frac{1}{x}\right) = 2x \sin\left(\frac{1}{x}\right) + x^2 \cdot \cos\left(\frac{1}{x}\right)\frac{-1}{x^2} =$, ok. This just comes from the quotient rule.

So, what you get is $\cos\left(\frac{1}{x}\right) + 2x\sin\left(\frac{1}{x}\right)$, ok. Now, let us take limit x going to 0 of this or rather let me write it better as x goes to 0, the second term vanishes ok. However, $\cos\left(\frac{1}{x}\right)$, widely oscillates.

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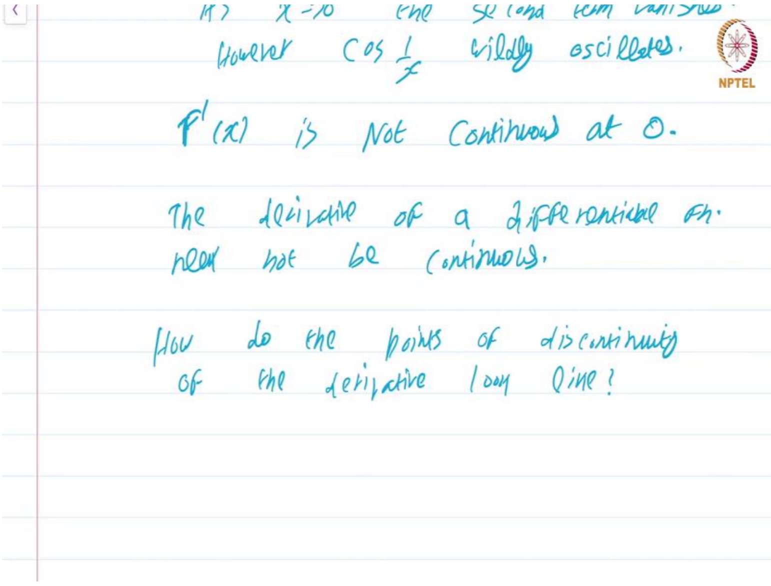

Which means this function the derivative $f'(x)$ is not continuous at 0. So, conclusion is the derivative of a differentiable function need not be continuous. There could be discontinuities of differentiable functions. The next natural question to ask is how do the points of discontinuity of the derivative look like? This is the next question that should be on your mind.

Here in this case we got this function $\cos\left(\frac{1}{x}\right)$ as one of the terms in the derivative and that oscillates widely. So, the singularity or rather the point of discontinuity at the origin is very badly behaved, there is a wild oscillation. The simpler form of discontinuity is that we have studied jump discontinuity is not occurring here ok.

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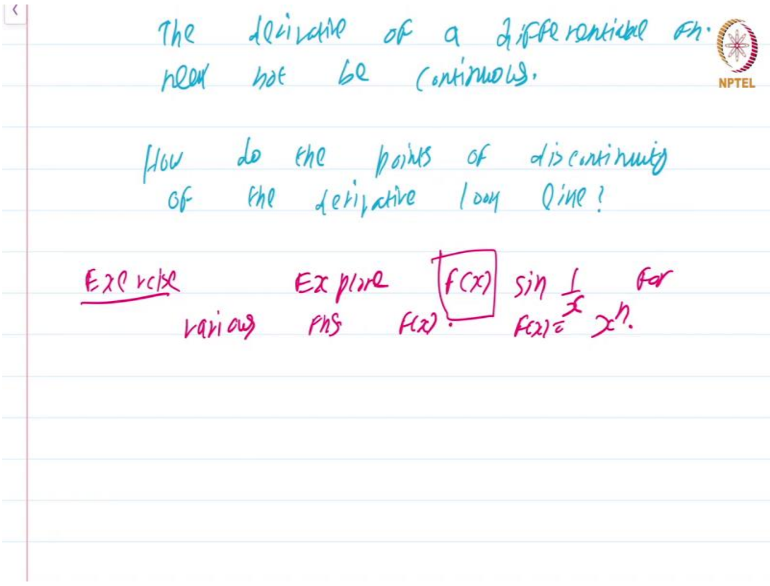

So, I am going to leave you with an exercise. Explore, see this is an exploratory exercise, so it is up to you how much time you want to spend on this. You can spend the next 10 years doing a Ph.D, on this or you can spend the next maybe an hour or so. It is up to you to how much you want to explore. Explore $f(x)\sin\left(\frac{1}{x}\right)$ for various functions $f(x)$.

So, one recommendation is $f(x) = x^n$. We have already seen x and x^2, higher powers what happens. And can you temper $\sin\left(\frac{1}{x}\right)$ in different ways? How can you adjust the behavior at the origin by multiplying by various other functions $f(x)$? And see what happens ok.

So, we have now seen some examples of differentiation, and we have also seen that the derivative does not always behave nicely. Even when you temper it appropriately the $\sin\left(\frac{1}{x}\right)$ function, there seems to be some problem with the derivative. Now, in a later module, I will show Darboux theorem which says that the derivative will nevertheless satisfy the intermediate value property.

Now, why is this important? If the derivative satisfies the intermediate value property, then it cannot have jump discontinuities. The derivative can only have such wild oscillatory discontinuities, cannot have jump discontinuities and that is for a later day. This is a course on real analysis, and you have just watched the module on examples of differentiation.