Real Analysis - I

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Lecture – 21.2
Classification of Discontinuities and Monotone functions

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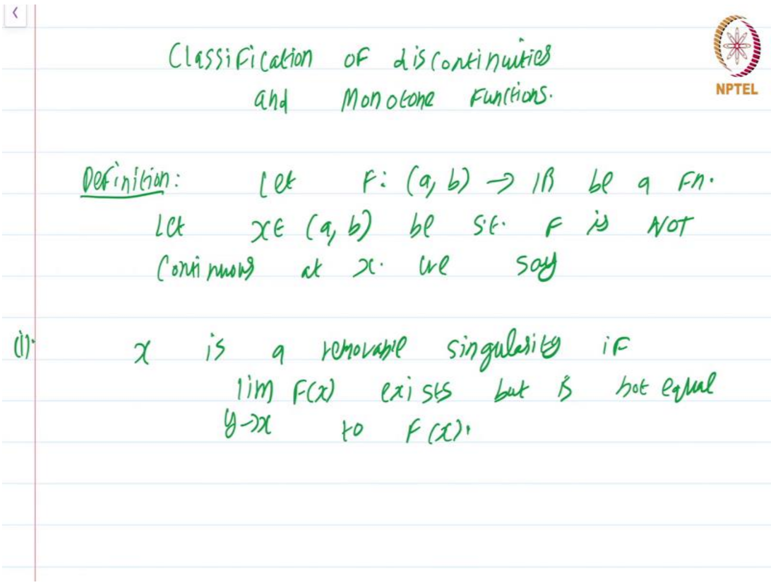

The discontinuities of a function are of 3 types. Let us begin with the definition. Definition: In this definition I am going to restrict myself to functions defined on an open set just for convenience. Let $f:(a,b) \to R$ be a function, let $x \in (a,b)$ be such that f is NOT continuous at x.

We say number 1, x is a removable singularity if $\lim_{y \to x} f(y)$ exists but is not equal to $f(x)$. That means the limit values exist, just a second, I will not call it a removable singularity, I will call it a removable discontinuity.

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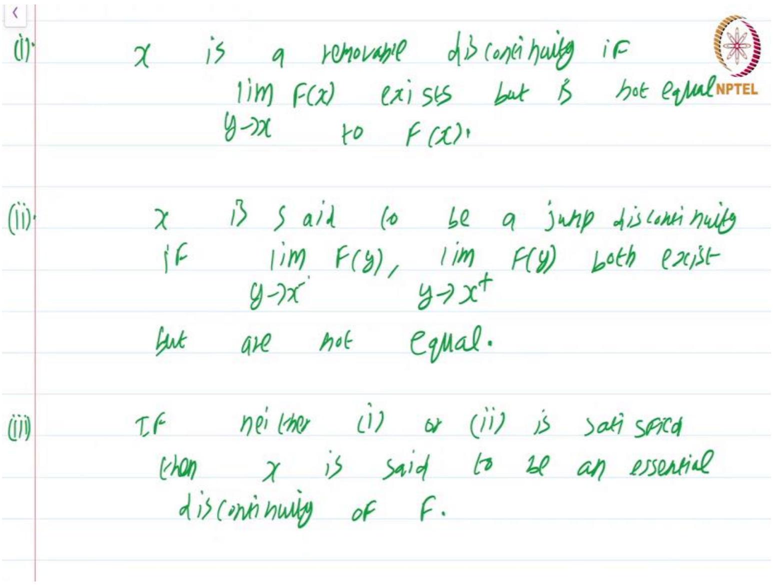

I want to reserve the term singularity for more pathological things later in the course. So, x is a removable discontinuity if $\lim_{y\to x} f(y)$ exists but is not equal to $f(x)$. Why is this called a removable discontinuity? Well, it is obvious why it is called removable discontinuity; what you do is, if we define $f(x)$ rather if we redefine $f(x)$ to be this limit value $\lim_{y\to x} f(y)$ then; obviously, you end up with a continuous function.

So, in some sense this function f has a fake singularity at, the fake discontinuity at the point x; it is not really discontinuous at the point x, the way that function has been defined at the point x is what is making it discontinuous. So, redefining that function at just that one point is going to fix the issue. So, that is why this is called a removable discontinuity.

Number 2, x is said to be a jump discontinuity if $\lim_{y\to x-} f(y)$, $\lim_{y\to x+} f(y)$ both exist, but are not equal.

Well, again the terminology is very explanatory. When you approach from the left and when you approach from the right both limits exist, but they are not equal. So, there is a jump happening at the point x ok. So again the terminology is very nice.

3. So, two cases of mathematicians having vivid and nice terminology, means that the third terminology is going to be ridiculously bad; if neither 1 or 2 is satisfied then, x is said to be an essential discontinuity of f.

So, I mean I think the idea behind it is there is the situation is bad; there is nothing really you can do about the discontinuity if it is neither the removable discontinuity or it is a jump discontinuity; therefore, you just say it is an essential singularity, there is nothing you can do about it. So, I do not know what really motivated this terminology here. So these are the 3 types of discontinuities that can happen.

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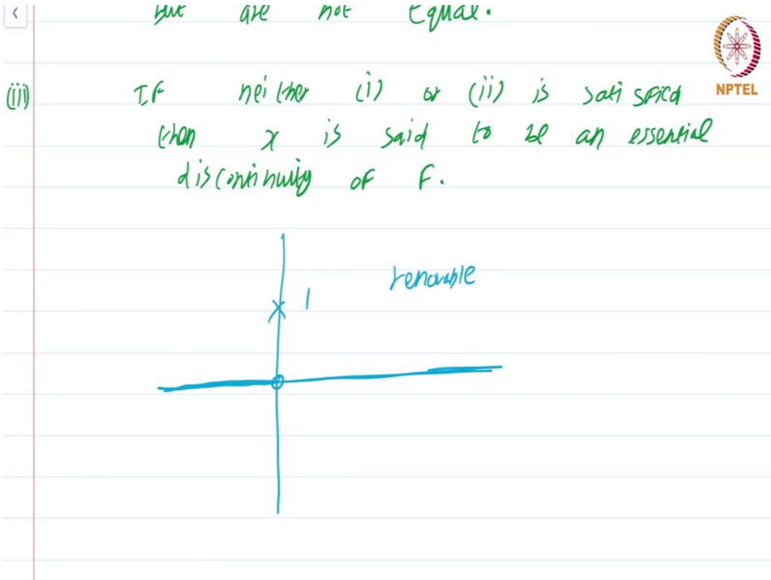

Let us draw pictures to illustrate these rather than give actual real examples because they are rather easy to actually give. So, suppose you had a function that is 0 on the negative real line, 0 on the positive real line, but for some reason it is 1, it is 1 at the point 0 then clearly this is a removable singularity.

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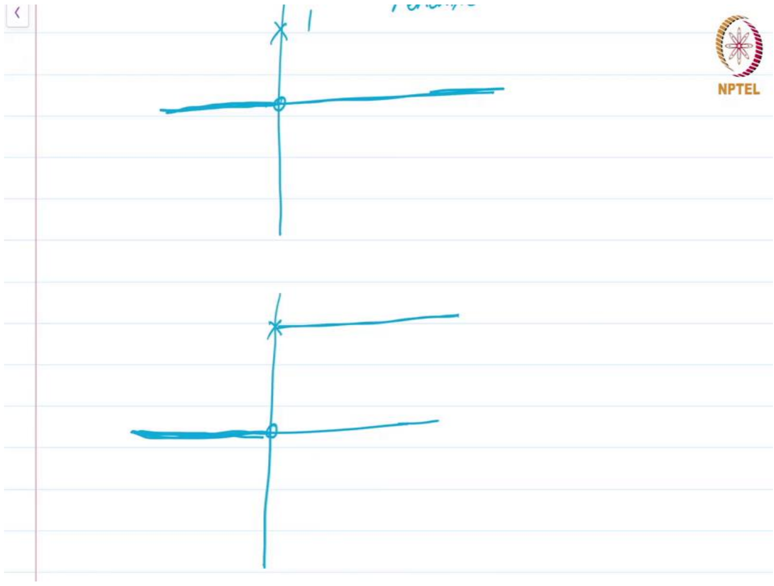

You just have to redefine $f(0) = 0$ to make that discontinuity go away. The second is a jump discontinuity, well that is the famous heavy side function you can just take I will just take it to be 0 on the negative real axis and 1 on the positive real axis this point I can define it whatever I want it is still going to be a jump discontinuity. What is essentially happening is that the left hand limit is 0 whereas the right hand limit is 1.

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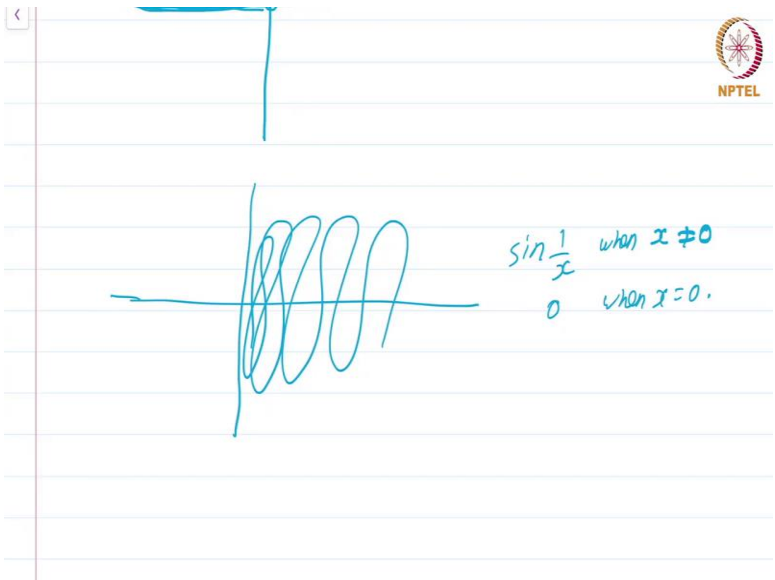

And for the third example of an essential discontinuity, we have the very famous example which is my greatest enemy to showcase my poor drawing skills it is this function; take $\sin\left(\frac{1}{x}\right)$, ok and of course, I have to draw it for the negative I'm not going to embarrass myself twice. So, I'm not even drawing it on the negative x axis. $\sin\left(\frac{1}{x}\right)$ when $x \neq 0$ and let us just say 0 when $x = 0$.

This function certainly has a discontinuity at the point 0. Neither the left hand limit nor the right hand limit actually exist at the point 0; therefore, this is an essential singularity ok. Now, I'm going to prove a very interesting theorem about the singularities of a monotone function; recall a monotone function is one that is either increasing or decreasing.

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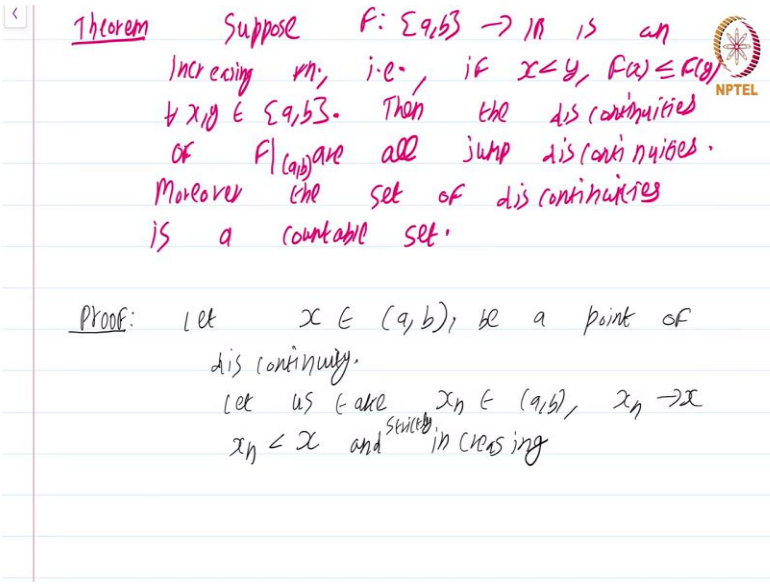

I will just state it for the increasing case, it is similar for the decreasing case, exact same proof will work with slight modifications. Suppose $f: [a, b] \to R$ is an increasing function; that is, if you do not recall the definition, if $x \leq y$, $f(x) \leq f(y)$ for all $x, y \in [a, b]$, ok.

Suppose you have an increasing function, then the discontinuities of f are all jump discontinuities, they are all jump discontinuities. Moreover, the set of discontinuities, that is, you put all the points of discontinuities into a set; the set of discontinuities is a countable set.

So, monotone functions are going to be continuous at almost all the points of closed interval $[a, b]$, whenever there is a discontinuity it is going to be a jump discontinuity ok.

So, proof: Now let $x \in [a, b]$ be a point of discontinuity. Now since I have defined discontinuities only for open intervals what I am going to do is for ultra precision I am going to restrict f to the open interval (a, b), ok.

So that, because I have not really defined what the meaning of discontinuities at these endpoints are; so, for precision I am going to just restrict the function f to this open interval (a, b). So, let us take a point $x \in (a, b)$ be a point of discontinuity, we have to somehow show that it is a jump discontinuity.

Well, how do we do that, what we are going to do is let us take $x_n \in (a, b)$, x_n converging to x; $x_n < x$ and increasing. In fact, I am going to demand that it is strictly increasing really does not make a difference, but I can demand of x and whatever I want.

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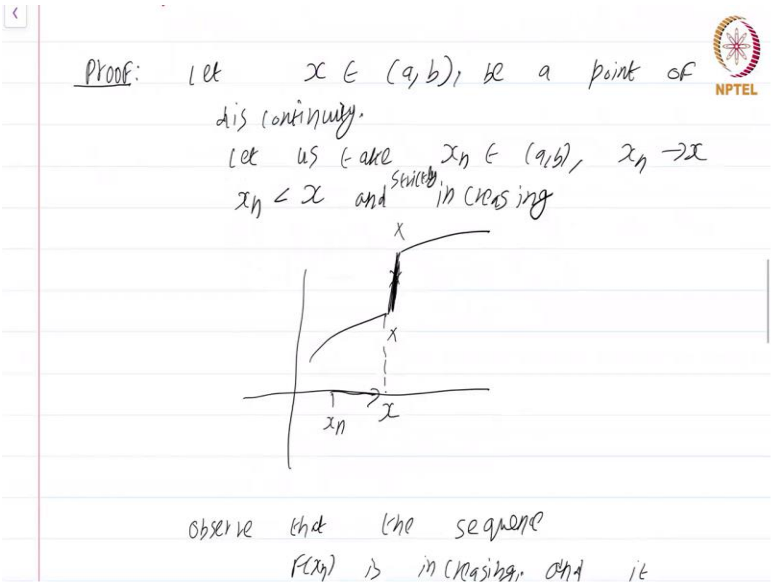

So, what I am going to do is. So, I have this function. I am going to draw the discontinuity as a jump discontinuity because that is what it is going to be, I have this increasing function. I am focusing on this point x, what I am doing is I am taking a sequence x_n that is converging to this point x ok. So, this sin curve type thing I have drawn is just a doodle it really has no place here. So, I have this sequence x_n which is converging to x.

And x_n is strictly increasing and x_n's are all less than x ok. How does this help? Well, observe that the sequence $f(x_n)$ is increasing, we cannot say that it is strictly increasing because all I have assumed is that the function f is increasing. I not assumed that the function f is strictly increasing. I have just assumed that if $x < y$, $f(x) \leq f(y)$.

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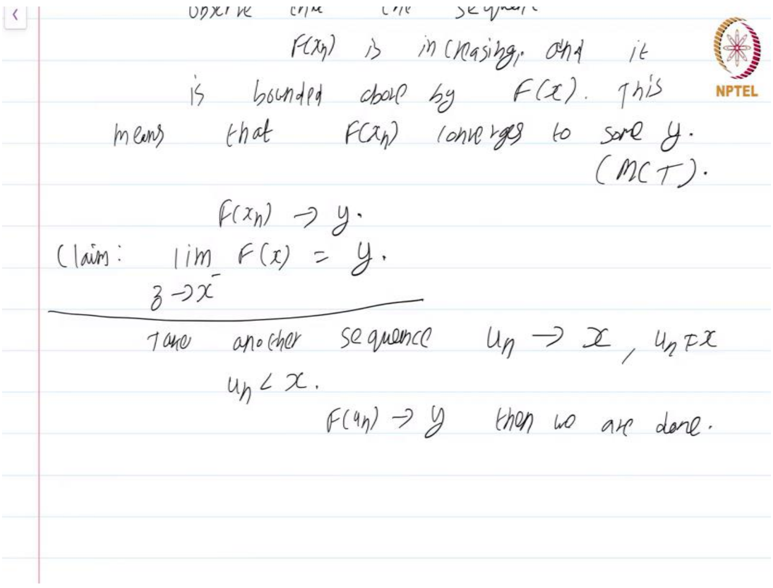

So, observe that the sequence $f(x_n)$ is increasing and it is bounded above by $f(x)$ right, because the, see the function f is an increasing function. So, the sequence $f(x_n)$ is bounded above by $f(x)$, this means that $f(x_n)$ converges to some y, why? This is the monotone convergence theorem. If you have an increasing sequence that is bounded above, then it is convergent.

Now this y need not be equal to $f(x)$; there is no reason on earth why this y should be equal to $f(x)$. I am not going to claim that at all. All I know is that $f(x_n)$ converges to y ok. Now I claim that this will force $\lim\limits_{z \to x-} f(x) = y$, this is the claim, ok.

Now, I am going to leave it to you to go back to the definition of one sided limits and check that the argument I am about to give is 100 percent proper. So, what we do is the following, take another sequence u_n converging to x, $u_n \neq x$, $u_n < x$. Actually $u_n < x$ will cover the case $u_n \neq x$, I am just emphasizing $u_n's$ are never equal to x. So, I m just going to take any sequence u_n converging to x from the left, if I can show that $f(u_n)$ converges to just y then we are done.

Essentially, what I am doing is no matter what sequence you choose that converges to x, but from the left then the image sequence must converge to y then the $\lim\limits_{z \to x-} f(z) = y$ is true, ok.

So, this is what I want you to check from the definition of one sided limit it is just of 2 or 3 lines of verification ok, why is this the case? Well, think about what is happening.

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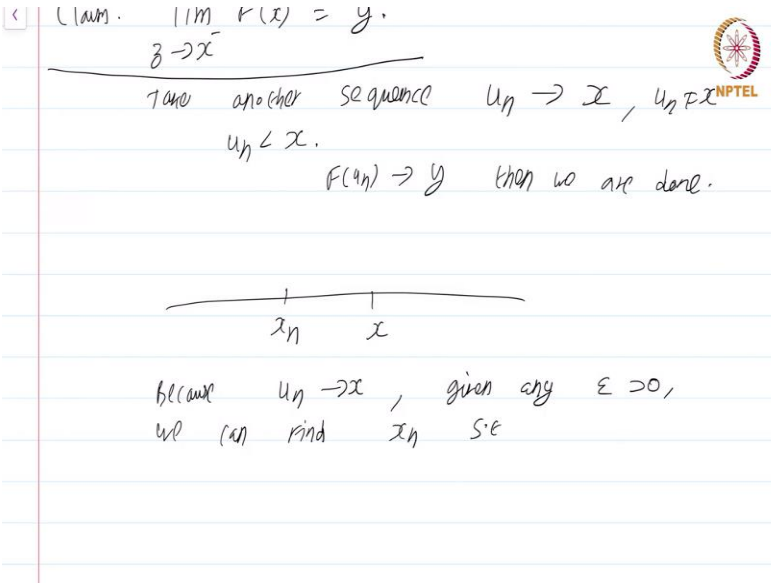

We have this point x and we have the sequence x_n ok, now because u_n converges x.

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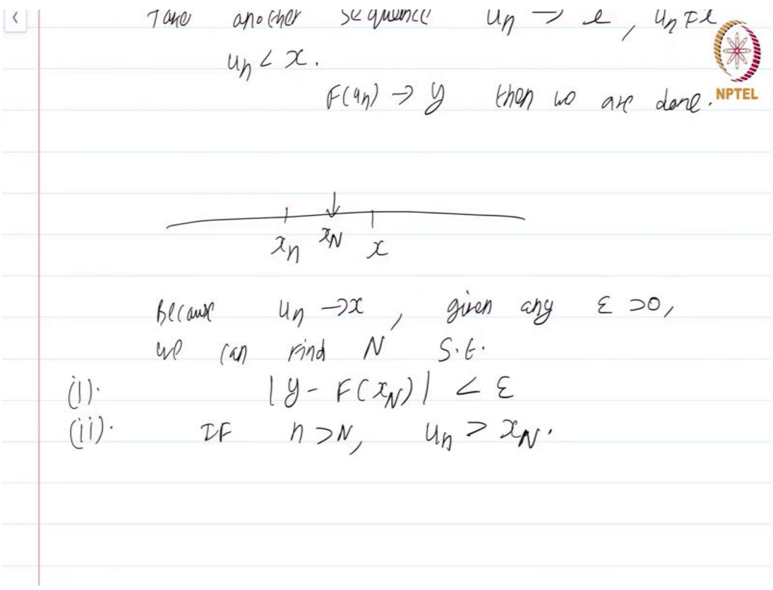

Given any $\varepsilon > 0$, we can find capital N such that two things happen, 1, first of all $|y - f(x_N)| < \varepsilon$. Well, we can certainly do this simply because $f(x_n)$ converges to y ok.

Second is, if small $n > N$, $u_n > x_N$. This requires maybe 30 seconds of thought, well think about what is happening x_n is somewhere over here, what I am claiming is if you choose N. So this I really cannot do, we can find N, M we can find N, M, I got a little bit too ambitious.

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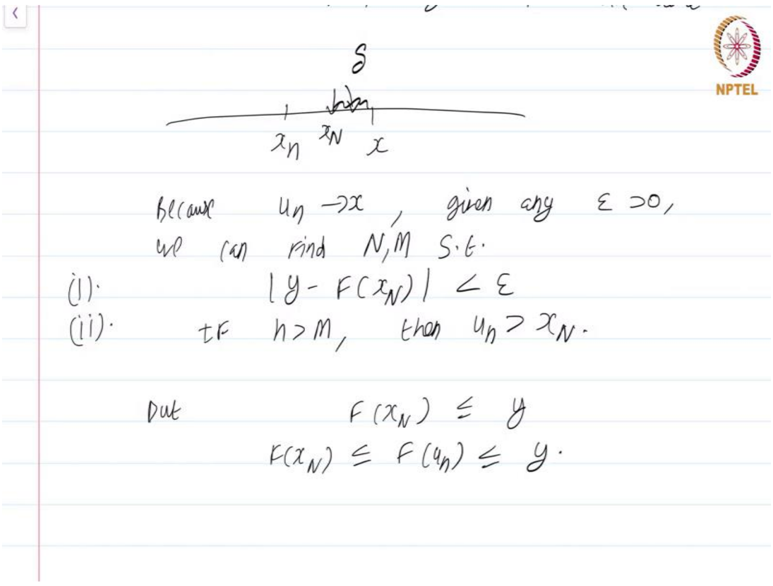

So, if small $n > M$ then $u_n > x_N$, this is what I want. Well, why can I guarantee this because $u_n \to x$. There is a suitably large capital M such that if $n > M$, then u_n must lie in this interval right.

So, there is some finite distance between x_n and x, call that distance δ; all I am saying is for this delta there is a corresponding n which I am just calling capital M ok. So, this is just requires a little bit of thought, all I m doing is, since u_n converges to x, I can force it to be greater than x_N for suitably large n that is what I have written down ok.

Now, how does this help? But, we actually have more data than what I have written down. We in fact have, I have written $|y - f(x_N)|$, I could have forgotten about the modulus we actually have $f(x_n) \leq y$, right and in fact, we have $f(x_n) \leq f(u_n) \leq y$, ok for $n > M$.

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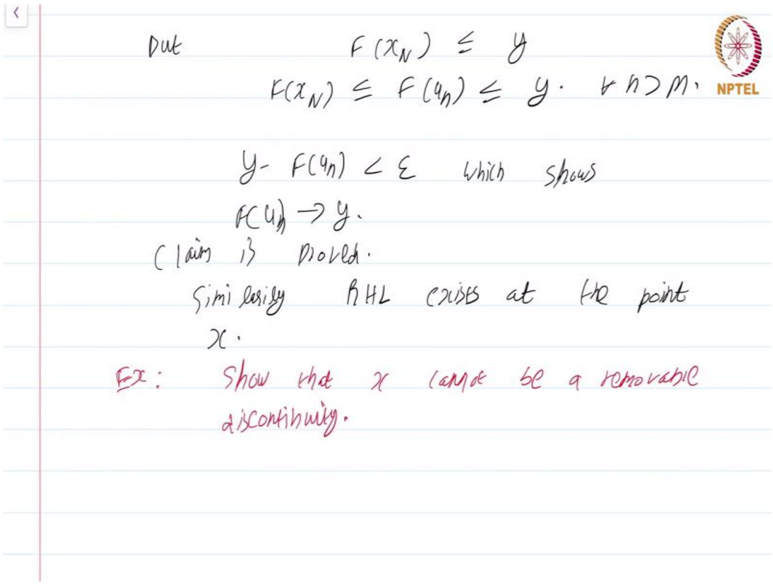

In other words in other words $y - f(u_n) < \varepsilon$; there is no need to put the modulus because we know that $f(u_n) \leq y$, ok, which just shows which shows $f(u_n) \to y$ ok. So, the claim is proved. So, the left hand limit of the function f exists at the point x ok.

Now, similarly, right hand limit exists at the point at the point x ok. Now I am going to end this proof by giving you a very very easy exercise. Show that x cannot be a removable singularity, cannot be a removable a discontinuity.

I keep using the word singularity that is because I am a complex analyst. So, you if you learn complex analysis, you will understand why I keep saying removable singularity. I apologize in advance. So, if I say singularity, I am interesting you with the task of interpreting it as discontinuity ok, show that x cannot be a removable discontinuity that will complete the proof.

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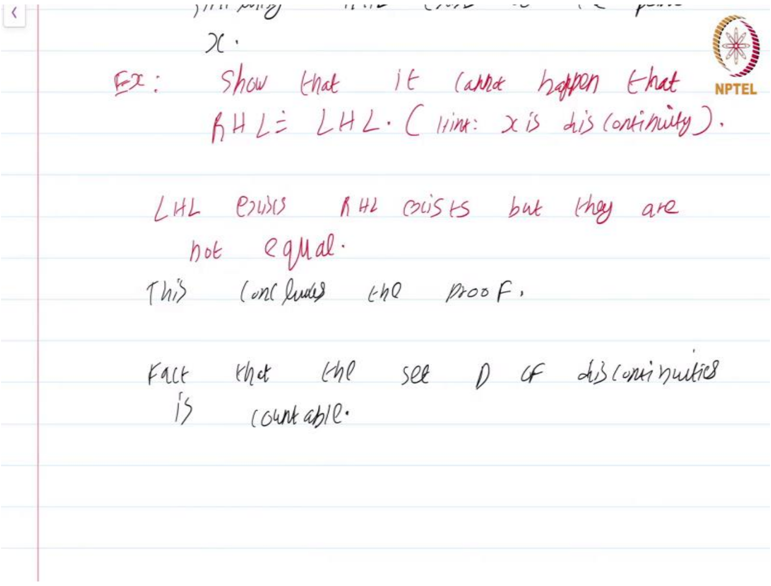

That means left hand limit exists, right hand limit exists, but they are not equal, ok. So, since the right hand limit and the left hand limit exist. In fact, let me just rephrase this exercise in a slightly more palatable form.

Show that it cannot happen, it cannot happen that right hand limit is equal to left hand limit ok. This is a better way of phrasing this exercise. Show that it cannot happen that the right hand limit is equal to the left hand limit. Hint: x is a discontinuity; remember that x is a discontinuity. So, I am just phrasing the exercise in a slightly different form which will which is I think more clear.

So, left hand limit exists, right hand limit exists and I am leaving you to show that left hand limit cannot be equal to the right hand limit ok. Now this will this concludes the proof; in fact, this concludes the proof. At least the proof of the first part that all the singularities are jump discontinuities, all the discontinuities are jump discontinuities.

So, now, the second part, the second part is this fact that the set D of discontinuities is countable. Now this proof requires this machinery of oscillation right, what you do is the following what you do is the following.

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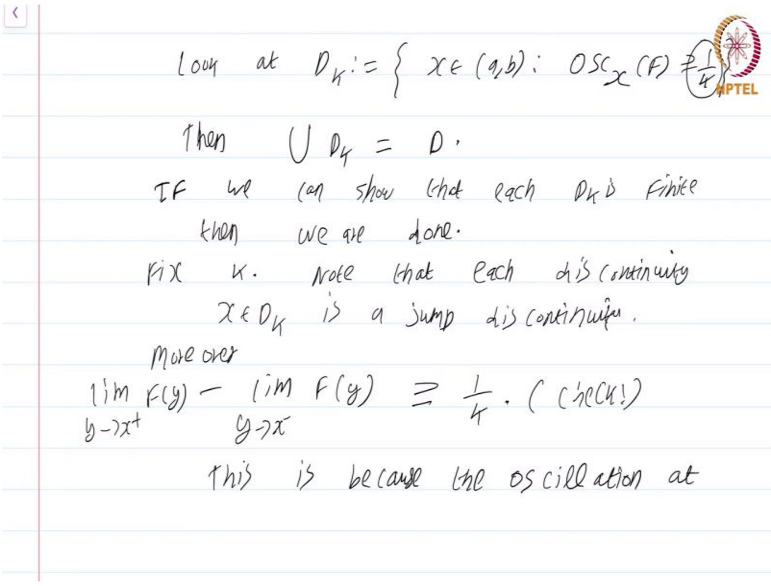

Look at $D_k \coloneqq \{x \in (a,b): osc_x(f) \geq \frac{1}{k}\}$. At a point of discontinuity by an exercise that I have given, I must add an easy exercise, that I have given you, can show that the oscillation will have to be non – zero, ok and in fact, it has to be a positive quantity just by the way the oscillation is defined.

So, look at those points where the oscillation is at least $\frac{1}{k}$, then $\cup D_k = D$, ok. Now, observe if we can show that each D_k is finite then we are done; because a countable union of finite sets is certainly countable, then we are done.

So, fix k, we are going to focus our attention on D_k, ok. Now note that each discontinuity continuity $x \in D_k$ is a jump discontinuity fine. Moreover, $\lim_{y \to x+} f(y) - \lim_{y \to x-} f(y) \geq \frac{1}{k}$, right. This is because, check this check; I will just write this is because the oscillation at the point x is at least $\frac{1}{k}$.

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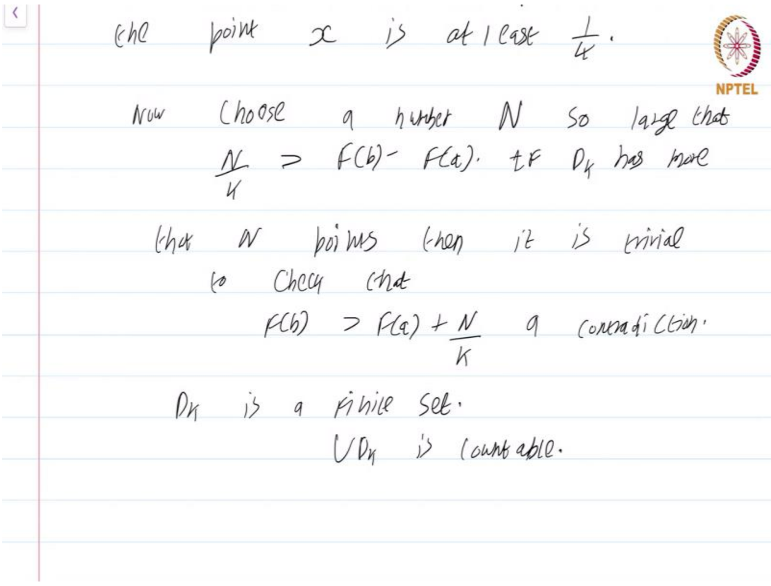

Because the function is having an oscillation at least $\frac{1}{k}$ at the point x it has to be the case that the right hand limit minus the left hand limit must also be at least $\frac{1}{k}$, one of the crucial things to understand why this must be the case is to also observe that, let us see, where is the graph? $f(x)$ has to be somewhere in between these two, in between this jump.

The value of $f(x)$ has to be in between this jump because the function is an increasing function; it cannot be here. So, essentially $f(x)$ is going to be sandwiched between the jump and this jump has to be at least $\frac{1}{k}$ because the oscillation at that point is $\frac{1}{k}$ and $\frac{1}{k}$ is not going to contribute that much oscillation ok.

Now choose a number let us say capital N. So, large that capital $\frac{N}{k} > f(b) - f(a)$, this is always possible because $f(b) - f(a)$ is going to be a finite quantity.

If D_k has more than capital N points; that means, there are at least N places where the jump is at least $\frac{1}{k}$; then it is trivial to check that $f(b) \geq f(a) + \frac{N}{k}$ in fact, not greater than or equal to in fact strictly greater than it is going to be strictly greater than $f(a) + \frac{N}{k}$.

So, what is essentially happening is you start at $f(a)$ each time you encounter a jump you are going to increase the value by at least $\frac{1}{k}$ and this is since this is a monotone function it is an increasing function you are going to preserve at that level at least and you are going to jump again at $\frac{1}{k}$ when you encounter the next discontinuity and this is going to happen at least capital N times.

So, $f(b) > f(a) + \frac{N}{k}$, a contradiction, right. So, because at each jump there is a jump of $\frac{1}{k}$ happening there cannot be too many jumps. So, D_k is a finite set and the net upshot is union of D_k is countable.

So this concludes the proof. There is a very nice idea in this proof where we use the oscillations to capture how much the functions jump and limit the number of such jumps because it cannot jump to exceed $f(b)$.

This is a course on Real Analysis, and you have just watched the module on Classification of Discontinuities and Monotone functions.