Real Analysis - I
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Lecture – 17.3
Open Covers and Compactness

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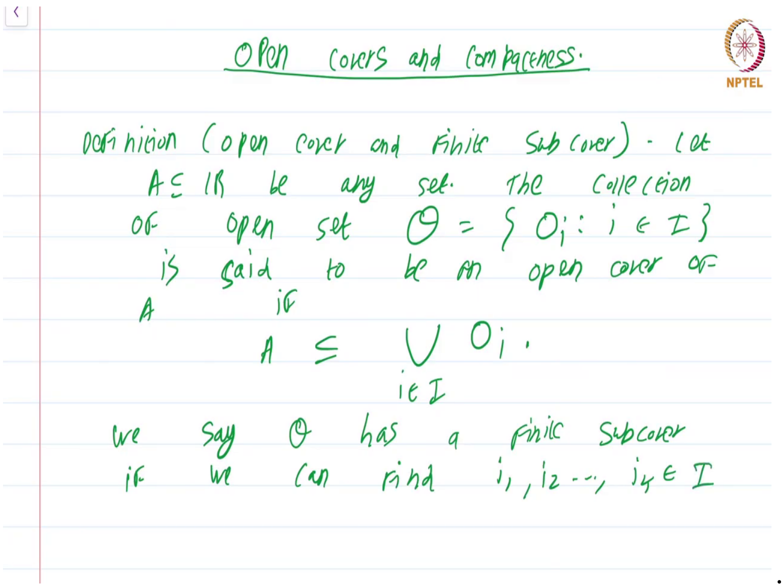

The best way to understand a complicated notion like compactness is to view it from more than one perspective. We have already characterized compact sets as precisely those with are that are closed as well as bounded. Now, I am going to give a slightly more abstract way of dealing with compact sets that is via open covers. So, we begin with the definition, this is the definition of an open cover and finite subcover.

Definition (Open cover and Finite subcover) Let $A \subset \mathbb{R}$ be any set. The collection of open sets $\mathcal{O} = \{O_i : i \in I\}$ is said to be an open cover of A, if $A \subset \cup_{i \in I} O_i$.

So, this is a fairly straightforward notion. You say a collection of open sets is an open cover, if the union covers the set. We say a finite sub collection or rather, we say O has a finite sub cover, , if we can find, $i_1, i_2, \ldots.., i_k$ in the indexing set I.

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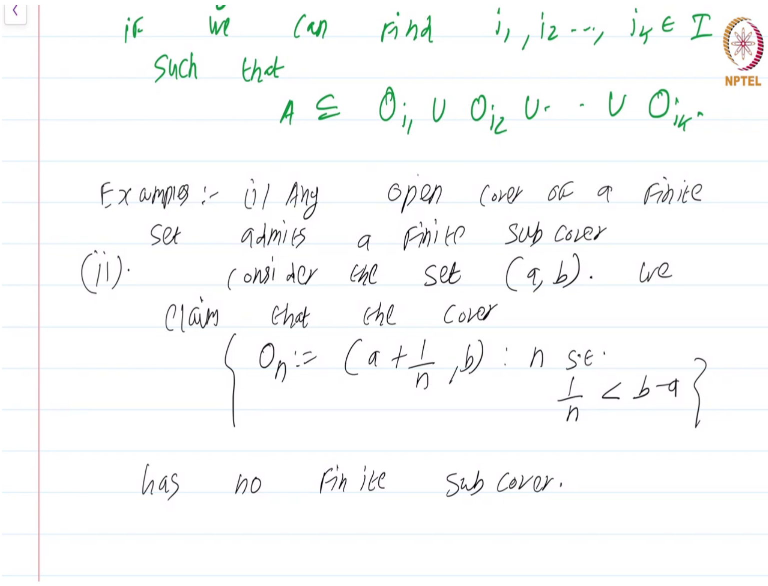

such that, A is a subset of $O_{i_1} \cup O_{i_2} \cup \quad \cup O_{i_k}$. So, if you have a finite subcollection that covers the set A, then you say that O has a finite subcover.

So, these two definitions are fairly straightforward, but they actually have a lot to do with compact sets, which is not at all clear because the definition of compact sets has had nothing to do with open covers or anything; it just said that a set is compact, if every sequence has a convergent subsequence.

So, first let us see an example or let us see several examples,

(i) Any finite set that is covered or better way to phrase it is any open cover of a finite set admits a finite subcover,.

(ii) Consider the set (a,b). We claim that the cover $\mathcal{O}_n := \{(a+\frac{1}{n}, b) : n\ st. \frac{1}{n} < b-a\}$, has no finite subcover . We are considering the open set (a b), we are considering a very special cover and the claim is that this cannot have a finite subcover. Why is that the case? Well, let us see.

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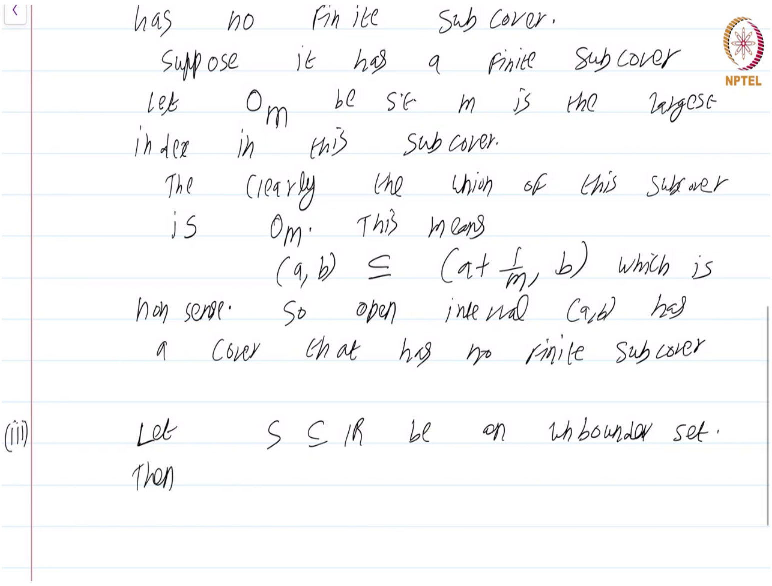

Suppose, it has a finite subcover. Let O_m be such that m is the largest index in this sub cover. Well, this is a finite subcover, it consists of elements of the form $(a + \frac{1}{n}, b)$, they must be the largest such m.

Then, clearly the union of the sub cover, is O_m right, because as n becomes large $(a + \frac{1}{n}$ contains $(a + \frac{1}{m}$, if n ¿ m right.

So, this m is the largest index. So, all other indices are less than m, therefore, $(a + \frac{1}{m}, b)$ will have to contain all such sub indices, all such sets of the form $(a + \frac{1}{n}, b)$. So, this means, we are in the peculiar situation that (a, b) is a subset of $(a + \frac{1}{m}, b)$, which is nonsense.

So, open interval (a, b) has a cover that has no finite subcover.

Now, Example (iii): let the $S \subset \mathbb{R}$ be an unbounded set.

Then, consider the cover

$\{(-n.n) : n \in \mathbb{N}\}$

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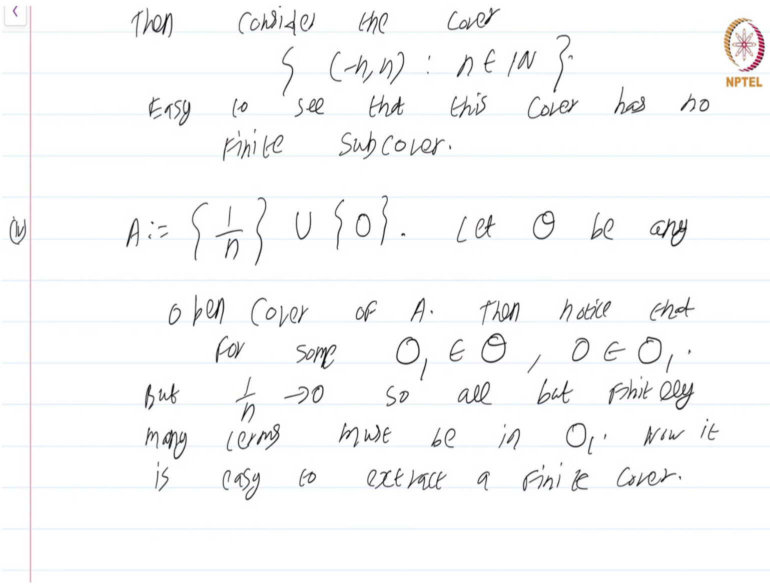

Again easy to see, easy to see, that this cover has no finite subcover.

Example (iv): Consider the set $A = \{\frac{1}{n}\} \cup \{0\}$. Let $\mathcal{O}$ be any open cover, let me call this set A be any open cover of A .

Then, notice that, for some $O_1 \in \mathcal{O},\ 0 \in O_1$. Because this is an open cover of A some element in this cover has to contain the element 0 right, but $\frac{1}{n}$ converges to 0. So, all, but finitely many terms must be in O_1.

Now, it is easy to extract a finite subcover, I just choose various elements from this cover $\mathcal{O}$, that contain the first few elements of the sequence, beyond these first few terms all the terms of the sequence have to belong to this set O_1, just because $\frac{1}{n}$ converges to 0. Now, we have seen several examples. These examples should sort of suggest to you that this is an equivalent characterization of compactness.

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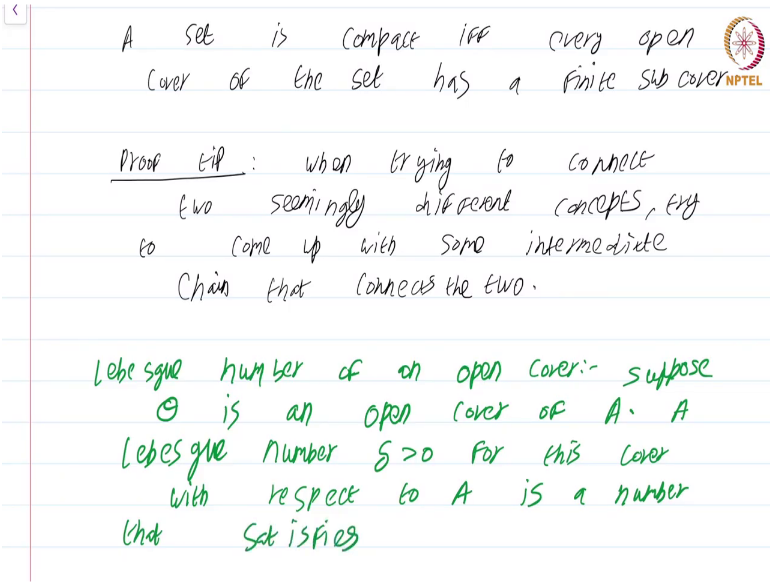

The characterization is as follows;

A set is compact if and only if every open cover of the set has a finite subcover. Now, this notion of open covers seems a fair bit more complicated than that of just compactness defined via sequences. I am going to show this theorem that a set is compact if and only if every open cover has a finite subcover.

Here is a proof tip: when trying to connect two seemingly different concepts, try to come up with some intermediate chain that connects the two.

Now, here I want to connect sequential compactness, that is, A set is compact if any sequence has a convergence subsequence that converges to an element of the set, with this notion of open covers and finite sub covers.

So, what I will do is, I will state an intermediate concept that ties both Lebesgue number, Lebesgue number of an open cover of an open cover.

Suppose, $\mathcal{O}$ is an open cover of A. A Lebesgue number note these choice of word it is a Lebesgue number $\delta > 0$ for this cover with respect to A is a number that satisfies $B(x, \delta)$ is contained in some element of $\mathcal{O}$. Let us say, I will just give it a name for concreteness, a subset of O_x, which is an element of $\mathcal{O}$.

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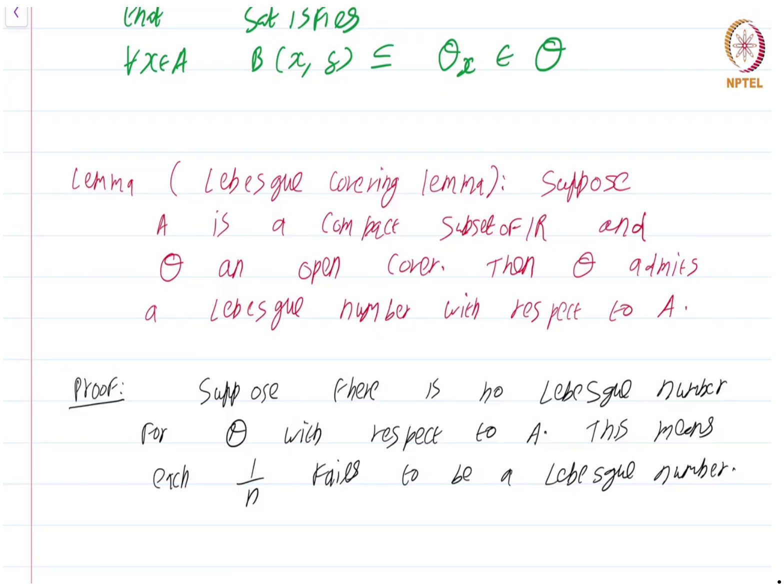

So, $\forall x \in A, \quad B(x,\delta) \subset O_x \in \mathcal{O}$. Note the order of quantifiers, what this is saying is you pick an element x of A and consider the ball of radius delta centered at x, there will be a corresponding O_x in this cover $\mathcal{O}$ such that $B(x,\delta)$ is fully contained in O_x.

Note you will understand why I said a Lebesgue. If, δ works certainly $\frac{\delta}{2}$ will also work as a Lebesgue number. So, will $\frac{\delta}{3}$. So, will $\frac{\delta}{10,000}$ right. So, a Lebesgue number for an open cover with respect to a set A is some sort of uniform estimate, on how the collection $\mathcal{O}$ covers this set A. Now, here is the key theorem or rather lemma, this is called the Lebesgue covering lemma.

Suppose A is a compact subset of $\mathbb{R}$, $\mathcal{O}$ an open cover. Then, $\mathcal{O}$ admits all a Lebesgue number with respect to A. If you have a compact set and some open cover, then that open cover has a Lebesgue number with respect to this compact set.

So, this Lebesgue covering lemma is going to be the connection between open covers and compactness. So, let us do this proof in great detail.

Suppose, there is no Lebesgue number for O with respect to A. This means, each $\frac{1}{n}$ fails to be a Lebesgue number right. There are no Lebesgue numbers therefore, $\frac{1}{1}, \frac{1}{n}$ cannot be a Lebesgue number. Now, how do we exploit this?

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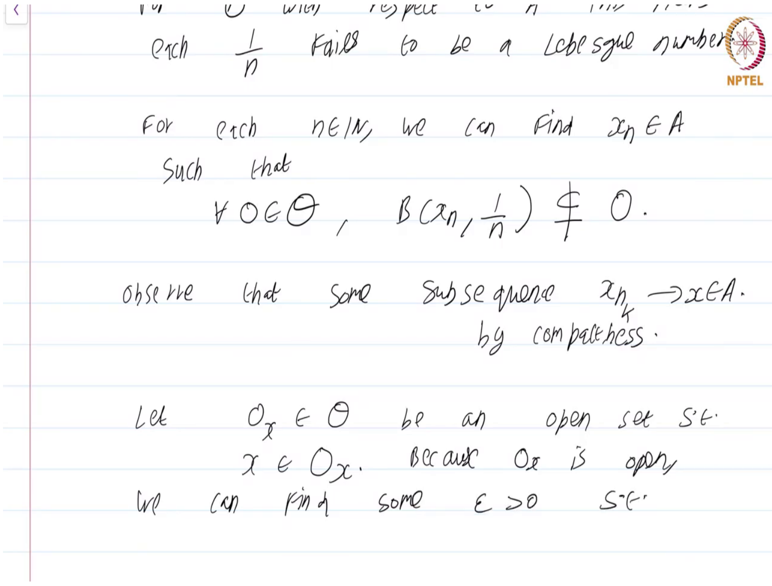

Because, each $\frac{1}{n}$ fails for each $n \in \mathbb{N}$, we can find $x_n \in A$ such that, for all O in this $\mathcal{O}$, $B(x_n, \frac{1}{n}) \not\subseteq O$. What is the condition that a particular number is a Lebesgue number? It is a number such that for each element x in the set A, $B(x, \delta) \subset O_x$ for some element O_x coming from the cover $\mathcal{O}$.

No $\frac{1}{n}$ can be a Lebesgue number, that is our assumption; that means, no matter what n you choose, $\frac{1}{n}$ fails to be a Lebesgue number. So, somehow the definition of Lebesgue number is not satisfied for $\frac{1}{n}$, that can only happen if you can find a point x_n such that $B(x_n, \frac{1}{n})$ is not an element of $\mathcal{O}$ and this must be true for all the sets O coming from the collection $\mathcal{O}$ right. Now, you can guess what is going to happen? Observe that some subsequence, x_{n_k} must converge to x in A, why, by compactness. Compactness ensures that there will be a subsequence that converges to an element in A. Now, what do I do with this? Well, if I will state it this way.

Let, $O_x \in \mathcal{O}$ be an open set such that $x \in O_x$. Why is there such an open set in the collection now simply because this is an open cover of A. So, every element of A has to be present in at least 1 element of the cover $\mathcal{O}$. So, I am just picking an open set O_x such that $x \in O_x$. Because O_x is open, we can find some $\epsilon > 0$, such that $B(x, \epsilon) \subset O_x$.

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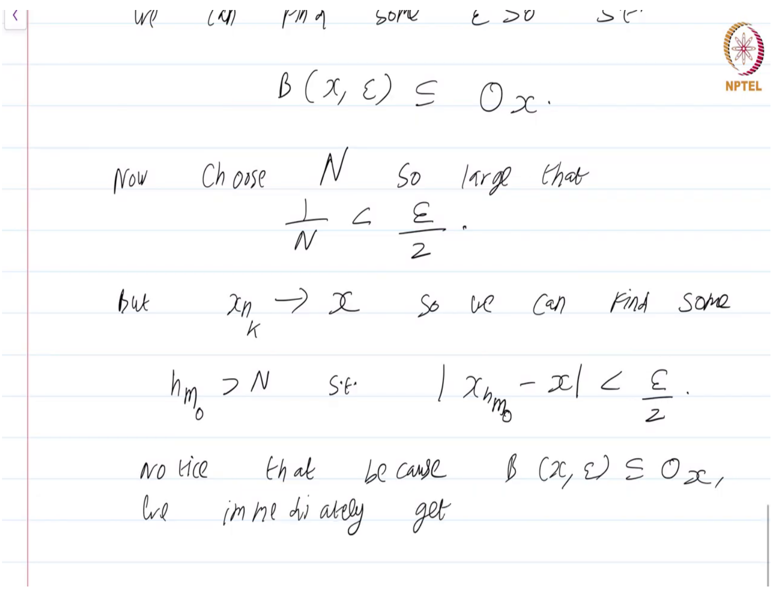

Now, we are getting somewhere we have found a ball $B(x, \epsilon)$ which is fully contained in O_x. Now, choose n_k so large that $\frac{1}{n_k} < \frac{\epsilon}{2}$. But x_{n_k} converges to x right. So, let me not use the word n_k here let me just use N.

So, there is less confusion because of notation repetition, but x_{n_k} converges to x. So, we can find some let us say some $n_m > n$, such that $|x_{n_m} - x| < \frac{\epsilon}{2}$.

Because, x_{n_k} converges to x we can find $n_m > N$ such that $|x_{n_m} - x| < \frac{\epsilon}{2}$. Now, let me change the notation slightly. I do not want to give the indication that x_{n_m} is a different subsequence all I mean is m_0.

Notice that, because $B(x, \epsilon) \subset O_x$. We immediately get, $x_{n_{m_0}} \in B(x, \epsilon)$.

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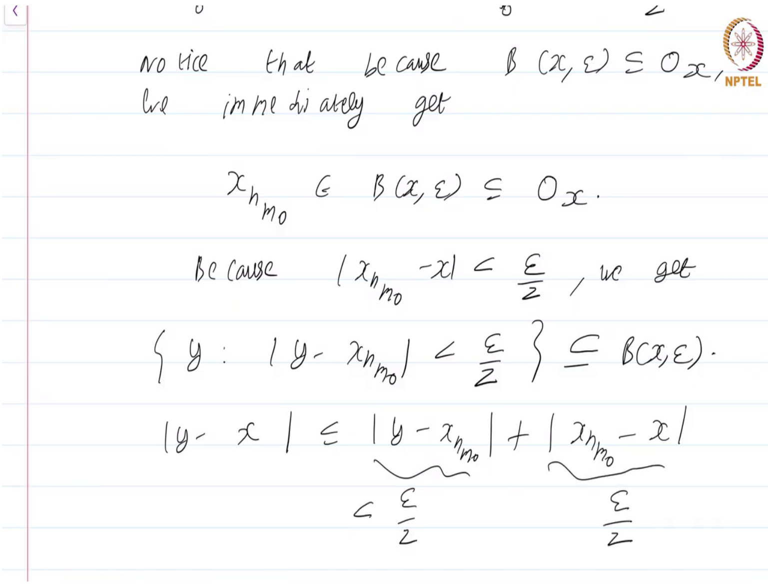

And, not only that because $|x_{n_{m_0}} - x| < \frac{\epsilon}{2}$, we get the set $\{y : |y - x_{n_{m_0}}| < \frac{\epsilon}{2}\}$ this set is going to be a subset of $B(x, \epsilon)$. Why is that the case?

Well, $|y - x| \leq |y - x_{n_{m_0}}| + |x_{n_{m_0}} - x|$. But, this is less than $\frac{\epsilon}{2}$ and this quantity is also less than $\frac{\epsilon}{2}$, because $n_{m_0} > n$ and we have this condition.

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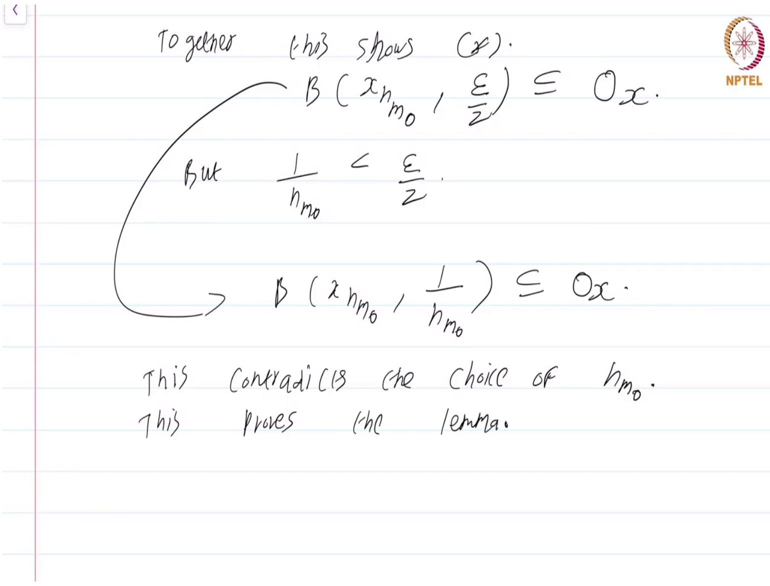

Together this claim together shows *. Well, what is the net consequence of all this? We have got $B(x_{n_{m_0}}, \frac{\epsilon}{2})$ is fully contained in this element O_x. But, wait a second $\frac{1}{n_{m_0}} < \frac{\epsilon}{2}$ right. Why is that? Because $\frac{1}{n}$ itself is less than $\frac{\epsilon}{2}$; therefore, $\frac{1}{n_{m_0}} < \frac{\epsilon}{2}$ wait a second; that means, $B(x_{n_{m_0}}, \frac{1}{n_{m_0}})$ is contained in O_x.

But this contradicts the definition or rather the choice, the choice of n_{m_0}. We had intentionally chosen this point $x_{n_{m_0}}$ as a point such that no element in this open cover can contain $B(x_{n_{m_0}}, \frac{1}{n_{m_0}})$, but we have shown that $B(x_{n_{m_0}}, \frac{1}{n_{m_0}})$ is contained in O_x, this proves the theorem.

So, this is a somewhat involved proof, but this acts as the intermediary that allows us to move from compactness defined via sequences and subsequences converging to this more abstract notion of open covers. In the next module, we will show that open covers having finite sub covers is just a different formulation of compactness.

This is a course on Real Analysis and you have just watched the module on Open Covers and Compactness.