Real Analysis - I
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Lecture – 17.1
Compactness

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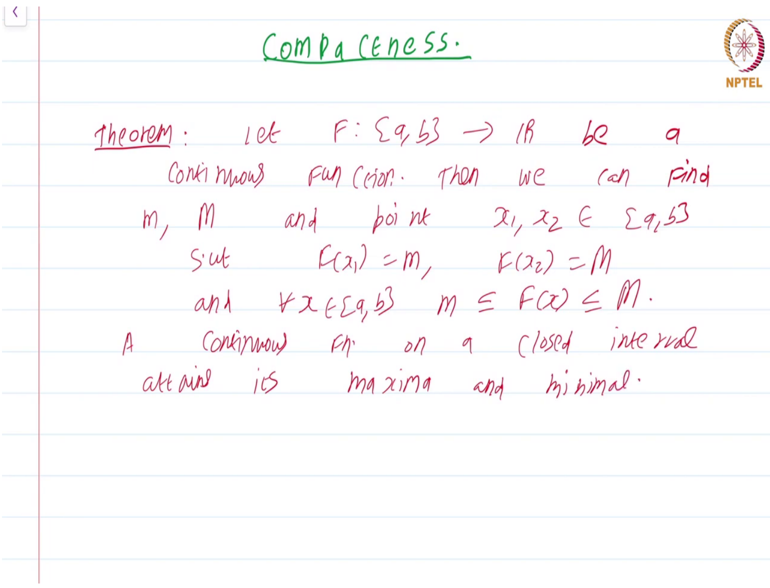

We now come to the important topological property called Compactness, before we embark and study this concept let me first motivate this by stating a very very famous theorem that you are probably seen in calculus.

Theorem: Let $F:[a,b] \longrightarrow \mathbb{R}$ be a continuous function. Then we can find m, M and points $x_1, x_2 \in [a,b]$ such that $F(x_1) = m$ $F(x_2) = M$ and for all x in [a, b], $m \leq F(x) \leq M$.

This is just a very very roundabout way of saying that a continuous function on a closed interval attains its maxima and minima. There is a maximum and a minimum value for that function and this is attained inside this closed interval. Now this result is obviously not true for continuous functions on open intervals.

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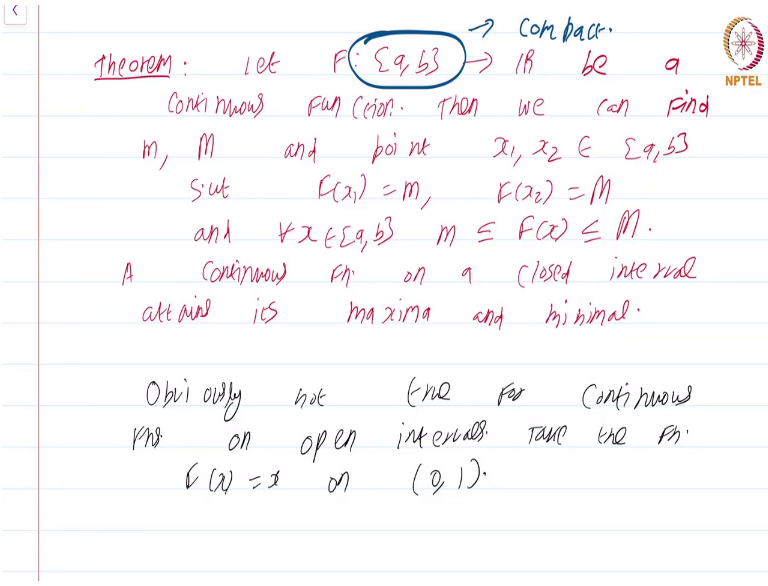

Take the function $F(x) = x$ on [0,1] for instance this function does not have a maximum or a minimum in this interval at all. So, something about the interval being closed plays a crucial role.

Now, one can prove this theorem quite easily without using anything from topology you can prove it, but the key fact is this closed set [a, b] is compact and that is what plays the crucial role in making this theorem true. The aim of analysis is not just to prove the various theorems of calculus, but it is to actually understand why these theorems are true.

Therefore, we want to get to the bottom of each theorem and to do that we need to introduce this notion of compactness which is what is forcing this theorem to be true. Another motivation as to why we must be interested in properties of the underlying subsets on which the function is defined is because the function being continuous depends on topological conditions also.

We just saw that you can characterize continuous functions using open sets, you can characterize continuous functions using sequences and topological properties like compactness and connectedness which we are going to see are also defined in terms of sequences and open sets and so on.

Therefore you can logically expect an interaction between the definition of continuity which is all to do with sequences and these properties which also have to do with sequences.

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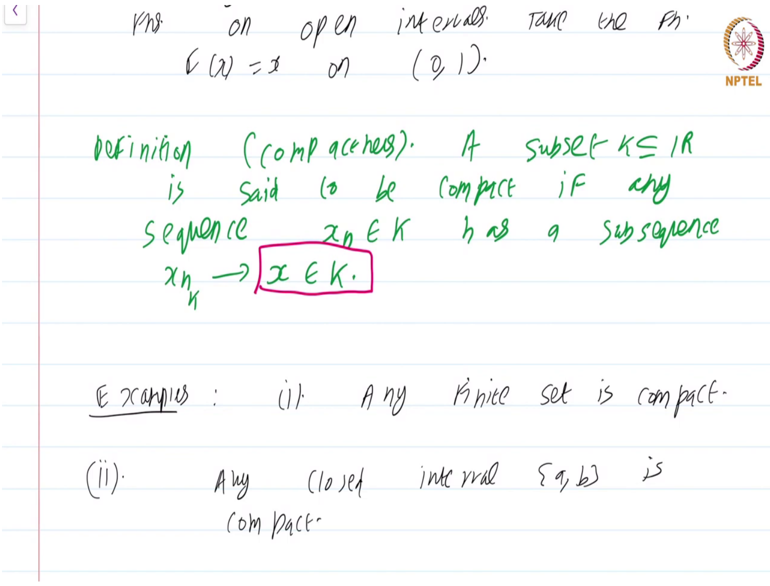

So, without further ado let me define compactness.

A subset K of R is said to be compact if any sequence $x_n \in K$ has a subsequence x_{n_k} that converges to x and this x must be in K. Let me just underline that or let me put a box around it.

Any sequence in this set must have a subsequence that converges and not only should it converge to a point that is there in the set, then we say that the set is compact, immediately we should see a few examples that any finite set is compact.

I will just verbally give the argument instead of writing it because it is a fairly straightforward argument: take a finite set, take any sequence. Then it is natural that one element of this finite set has to be repeated infinitely often there is no other way that you can construct a sequence from a finite set therefore, you have found the requisites convergence subsequence.

Example 2: Any closed interval [a, b] is compact.

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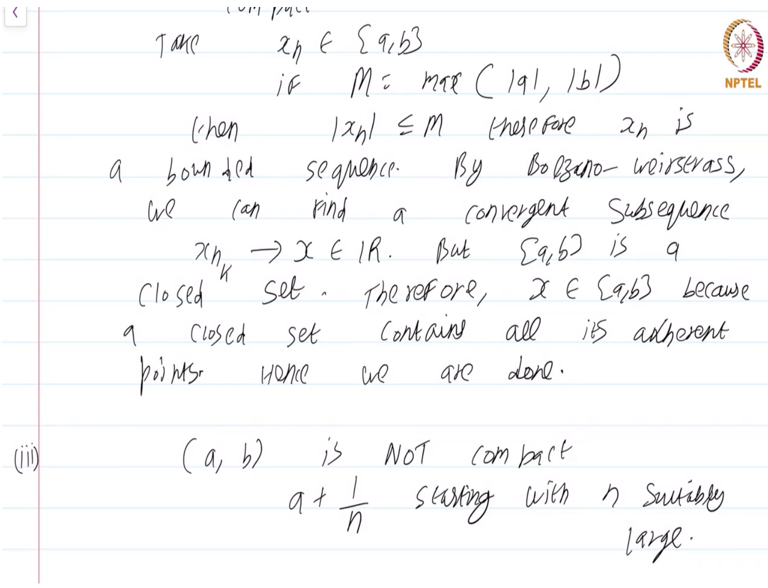

Why is this the case take $x_n \in [a, b]$? Now, [a, b] is a bounded set right I am not yet defined what a bounded set is, but because x_n's are all coming from [a, b], we can see that if $M := max\{|a|, |b|\}$ then $|x_n| \leq M$ therefore, x_n is a bounded sequence. Now, by Bolzano Weierstrass we can find a convergent subsequence x_{n_k}.

That converges to x, but this x is just going to be an element of $\mathbb{R}$, we do not really know that it is already there in the closed set [a,b], but wait a minute, but [a, b] is a closed set. Therefore, x must belong to [a, b] because a closed set contains all its adherent points, that is the definition of a closed set, hence we are done.

Example 3 is sort of an anti example: open set (a, b) is NOT compact, why? because we can just take the sequence $a + \dfrac{1}{n}$, I mean starting at n suitably large, because the width of this interval need not be greater than 1 starting with n suitably large. This sequence converges to the point a therefore, any subsequence will have to converge to the point a and the definition of compactness cannot be satisfied.

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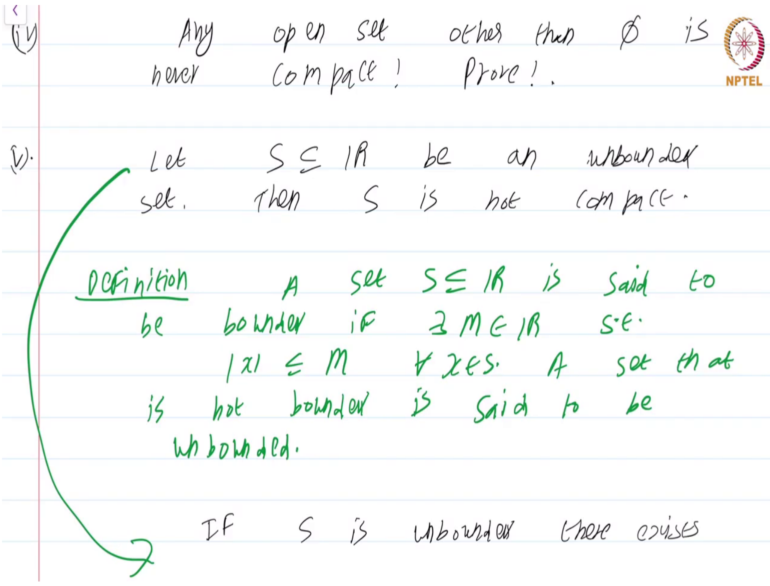

In fact, any open set other than the empty set is never compact.The proof is not hard just think about why this is true finally,

One other example: let $S \subset \mathbb{R}$ be an unbounded set.

I will just define what unbounded is in a moment, but I want to collect the examples, then S is not compact. So, let me just define for you bounded and unbounded sets and we will come back to this example.

Definition: A set $S \subset \mathbb{R}$ is said to be bounded if $\exists M \in \mathbb{R}$ such that $|x| \leq M \ \forall x \in S$. A set that is not bounded is said to be unbounded.

Now, why is an unbounded set not compact. Well if S is unbounded there exists $x_1 \in S$ such that $|x_1| \geq 1$, because 1 is not a bound there must be some element in that set whose modulus is greater than 1. Similarly, we can find $x_n \in S$ such that $|x_n| \geq n$.

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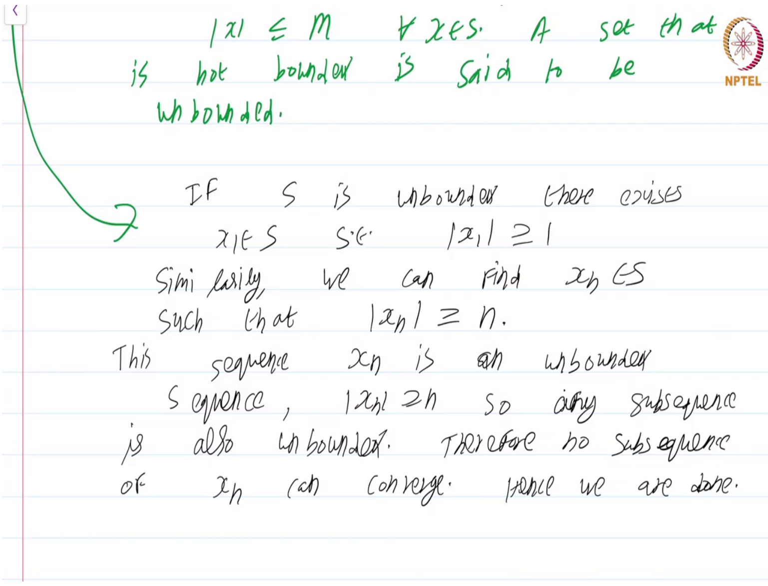

So, this sequence (x_n) is an unbounded sequence not only is it an unbounded sequence we have $|x_n| \geq n$. So, any subsequence is also unbounded. Therefore, no subsequence of x_n can converge. Hence proved, hence we are done.

In the next module we will state and prove the very famous Heine Borel theorem which completely characterizes compact sets in $\mathbb{R}$. They are precisely those sets which are closed and bounded.

This is a course on Real Analysis and you have just watched the module on Compactness.