Real Analysis - I
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Lecture – 15.4
Relationship between Limits and Continuity

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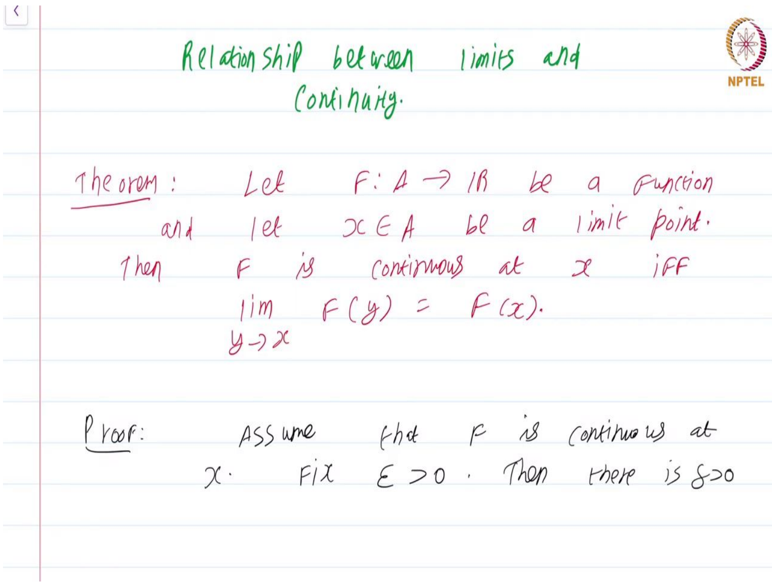

In this module, we are going to see what the Relationship between Limits and Continuity are. Since the definition of a limit was more or less almost exactly the same as the definition of continuity; there is no shock that both are intimately related to each other. The theorem is as follows.

Theorem: Let $F : A \longrightarrow \mathbb{R}$ be a function and let $x \in A$ be a limit point.

Note: I am taking $x \in A$, because I am going to talk about continuity at the point x and you can talk about continuity at a point only if that point is there in the domain. Whereas, to talk about limits, you need not require the point x to be actually in the domain of F, that is not essential;

Then F is continuous at x if and only if $\lim_{y \to x} F(y) = F(x)$.

A function is continuous at a point if and only if the limit as you approach that point is equal to the functional value. If you recall a slightly more elaborate version of this is what was used as the definition of continuity in school; the left hand limit is equal to the right hand limit is equal to the functional value, right. Let us prove this and it is not hard.

Assume that, F is continuous at x; fix $\epsilon > 0$.

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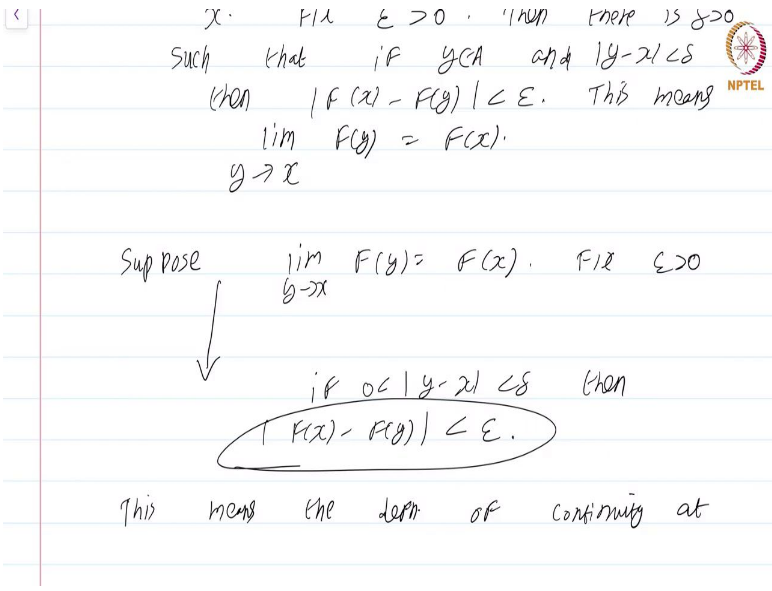

Then there is $\delta > 0$, such that if $y \in A$ and $|y - x| < \delta$; then $|F(x) - F(y)| < \epsilon$. This is just the definition of continuity, which I am writing down for the seventeenth time, if my count is correct.

So, what does this mean? Well, this means $\lim_{y \to x} F(y) = F(x)$; it is word for word the same thing as the definition of limit, except that $0 < |y - x| < \delta$ is missing, but who cares and this statement says more than what is required for me, right.

So, immediately from the fact that f is continuous at x, it immediately follows that $\lim_{y \to x} F(y) = F(x)$. What about the other direction?

Suppose, $\lim_{y \to x} F(y) = F(x)$ and I have to write something down for the eighteenth time now. Fix $\epsilon > 0$, then I am practically getting bored of this, something, if $0 < |y - x| < \delta$, then $|F(x) - F(y)| < \epsilon$.

But this is exactly the same as the definition of continuity, except the point y = x is allowed; but at y = x, this inequality is easily satisfied, because $|F(x) - F(x)| = 0$. So, this means the definition of continuity at x is satisfied; because $|F(x) - F(x)| = 0$, that is why, right.

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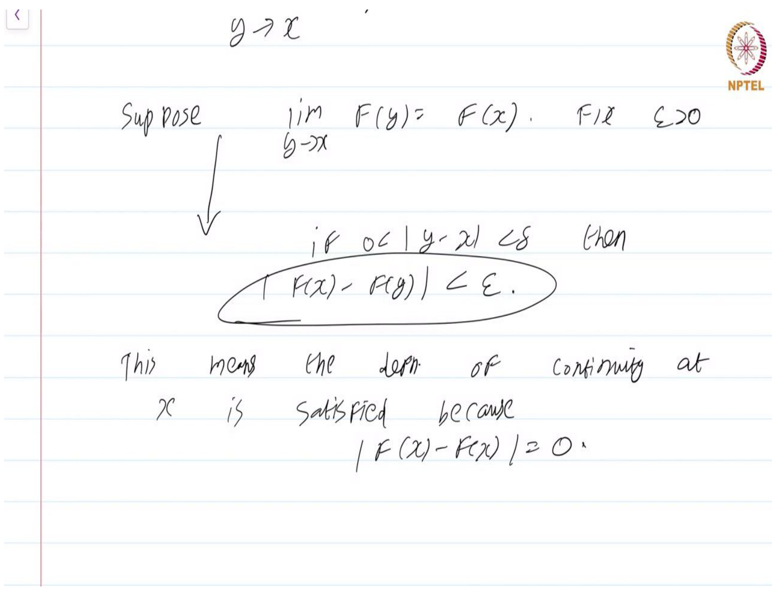

So, because the definitions were so similar, it is sort of trivial to see that continuity and limits are intimately tied to each other.

This is a course on Real Analysis and you have just watched the module on the Relationship between Limits and Continuity.