Real Analysis - I
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Lecture – 13.3
Basic Properties of Adherent and Limit Points

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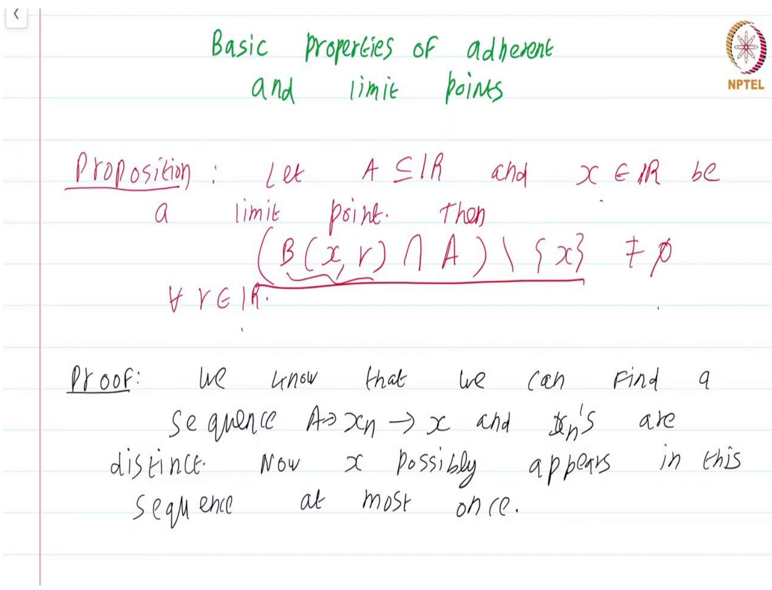

In this module, let us discuss some Basic Properties of Adherent and Limit Points. So, we immediately begin with the proposition.

Let $A \subset \mathbb{R}$ and $x \in \mathbb{R}$ be a limit point. Then $(B(x,r) \cap A) \setminus \{x\} \neq \phi \sim \forall r \in \mathbb{R}$.

What this is saying is? If you take a limit point of the set A; then whatever small open ball around that limit point that you take, there is a point in A that intersects this ball and that point is not x. That means, you can always find a point in A, that gets as close to x as you desire; but that point is not x itself. that is captured by saying that for each r, you can find a point that is there in this set.

Proof: Now, we know that, we can find a sequence $x_n \in A$ that converges to x and x_n's are distinct. Now, x possibly appears in this sequence; in this sequence at most once, right. Because this is a sequence of distinct elements; if x where to appear in the sequence, it can appear at max once.

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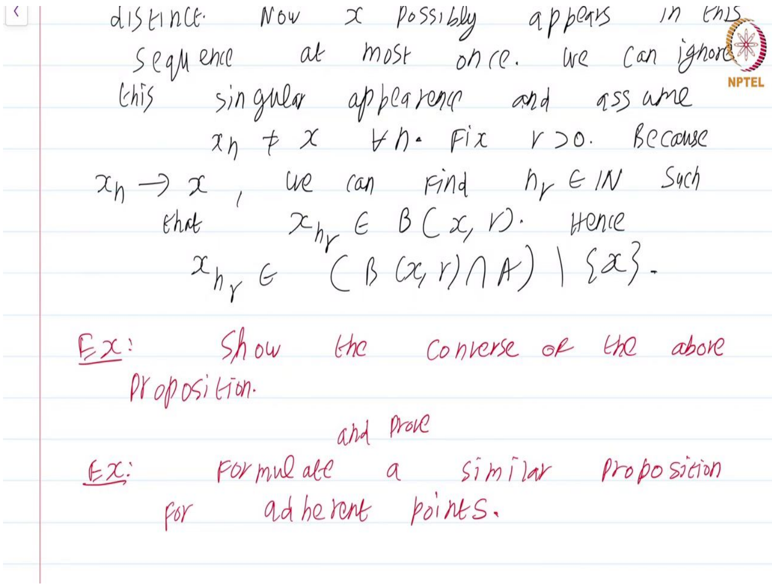

We can ignore this singular appearance and assume $x_n \neq x$ for all n. So, what I am essentially doing is I am going to remove that one appearance and construct a subsequence and re-index the subsequence and so on. I do not want to be labor the point, I will just ignore this element x and assume that the sequence x_n does not contain the element x at all, it never appears in the sequence, ok.

Fix r > 0; because $x_n \to x$, we can find $n_r \in \mathbb{N}$, such that $x_{n_r} \in B(x, r)$. This is just one of the equivalent ways of formulating sequences, convergence of sequences, the neighborhood or the open topological formulation that we dealt with. For sufficiently large n, the terms of the sequence must in fact all be in this ball. I will not be able to find just one n_r, after which all the elements of the sequence must be in B(x,r).

Hence $x_{n_r} \in (B(x,r) \cap A) \setminus \{x\}$ because this x_{n_r} cannot be the element x. Simply because we have assumed that the sequence, x_n does not contain the point x as any of its terms. Well, not only is this proposition true; the converse of this proposition is also true, which I am going to leave as a nice exercise in sequences.

Exercise: Show the converse of the above proposition. The converse says that if for each $r \in \mathbb{R}$, we can find a point in $(B(x,r) \cap A) \setminus \{x\}$, then x must be a limit point of the set A.

Now, in some sense, limit points seem to be points for which terms in the set A get arbitrarily close to A; but not just terms in the set A, these terms in the set A x excluding this point. Now, another exercise for you; formulate a similar proposition, formulate and prove a similar proposition for adherent points. Well, let me just give a hint; the only thing that will happen is you do not have to do set minus x.

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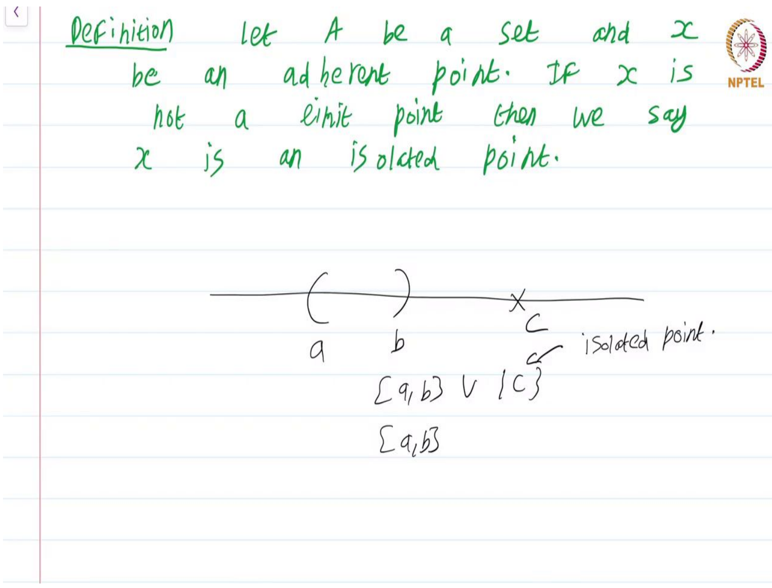

Now, all this motivates the following definition;

Let A be a set and x be an adherent point. If x is not a limit point, then we say x is an isolated point.

So, an isolated point is an adherent point that is not a limit point. Well, how can a point be an isolated point?

Well, let us look at an example; look at again the real line and look at this set, I am just taking an open set (a,b), then I am taking another point c somewhere far away. Now, we already know that the set of adherent points is going to be the closed set [a, b] union this single point c.

And we already know that the set of limit points is going to be this closed set [a,b]. So, this point c is going to be an isolated point. So, from the formulation of this previous proposition that we have that, a point is a limit point then $(B(x,r) \cap A) \setminus \{x\} \neq \phi$.

And the converse which I have left for you as an exercise combining this; we have the following proposition which just follows from understanding the meaning of the definitions and the previous proposition.

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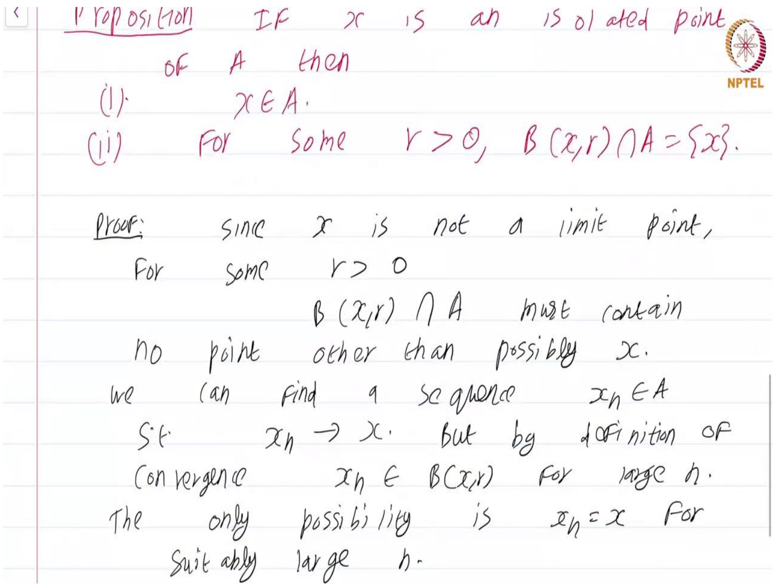

Proposition: (i)If x is an isolated point of A, then first of all x must be in A.

You cannot have an isolated point of A that is not an element of A; this is simply not the case for limit points, you could have limit points that are not, that are not elements of the set A.

(ii) for some r > 0; $B(x,r) \cap A = \{x\}$. The converse of the previous proposition said that, if a point is going to be a limit point; that means that every intersection B(x,r) intersect A must contain some element apart from x.

The converse means, that whenever it is the case that you have a point, such that every such intersection is non-empty; non-empty as it does is contains an element apart from x, then that point must be a limit point. But here B(x,r) intersect A is just the $\{x\}$ for some r > 0, we must have this.

Proof: Since x is not a limit point for some r > 0. B(x,r) intersect A must contain no point other than possibly x, right.

We can’t directly say that x is there, because I am not assuming; I mean that is the first part, then possibly x. Since x is not a limit point for some r > 0; B(x,r) intersect A must contain no point other than possibly x. Now, we can find a sequence, $x_n \in A$, such that $x_n \to x$.

But, by definition of convergence, $x_n \in B(x,r)$ for large n. The only way this is possible is, the only possibility is $x_n = x$ for suitably large n. This immediately shows both parts of the proposition.

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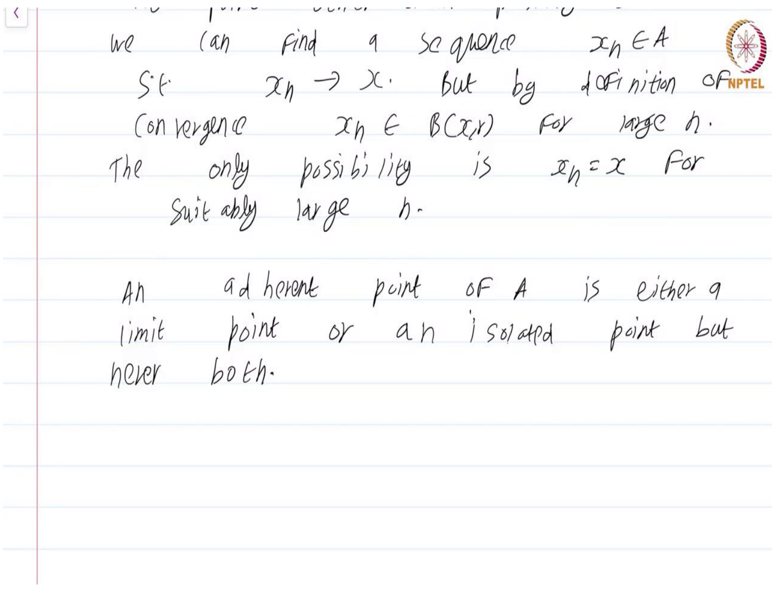

So, we have got an interesting thing: an adherent point of A is either a limit point or an isolated point but never both. So, here are some basic properties of adherent and limit points. In the next module, we will see basic properties of open and closed sets.

This is a course on real analysis and you have just watched the module on basic properties of limit and adherent points.