Real Analysis - I

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Lecture – 12.2
Absolute Convergence Continued

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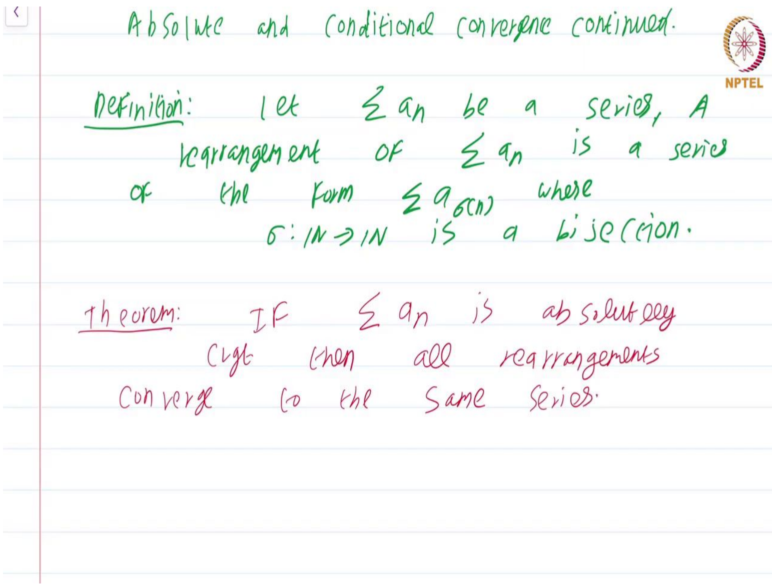

Let us continue our study of Absolute and Conditional Convergence.

Definition: Let $\sum a_n$ be a series, A rearrangement of $\sum a_n$ is a series of the form $\sum a_{\sigma(n)}$, where $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ is a bijection. So, this is capturing in a very rigorous way what it means to permute the elements of a series or rearrange the elements of a series.

In the last module, we saw that if you rearrange the terms of a conditionally convergent series all sorts of weird things can happen. So, we have not formally defined what a rearrangement is in the last module and now we have rectified that issue.

Now, what I am going to do is I am going to prove that this pathology that when you rearrange a series you get a different series cannot happen when you consider absolutely convergent series. More precisely making this precise theorem.

If $\sum a_n$ is absolutely convergent then all rearrangements converge the same limit.

So, when you rearrange a series that is absolutely convergent you cannot get a different limit.

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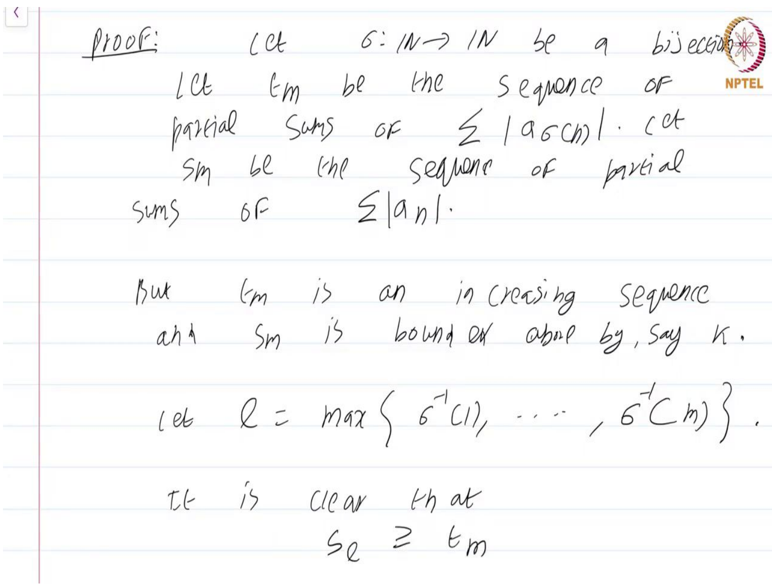

Proof: First I am going to show that a rearranged series will converge. In fact it will converge absolutely. So, let $\sigma : \mathbb{N} \longrightarrow \mathbb{N}$ be a bijection. Let t_m be the sequence of partial sums of $\sum |a_{\sigma(n)}|$.

I look at the rearranged series with absolute values of course and look at the partial sums. Let S_m be the sequence of partial sums of $\sum |a_n|$ of the original series.

We have to now show that t_m in fact is convergent, but t_m is an increasing sequence right. Simply because at each stage you are just going to add a non negative term to it and S_m is bounded above, by let us say k some quantity k. Why is S_m bounded above? Because $\sum |a_n|$ converges absolutely. convergent sequences are bounded therefore, S_m will have to be bounded

Now, what we do is the following. Let $l = \max \sigma^{-1}(1), ..., \sigma^{-1}(m)$. What I am doing is I am looking at where the first m terms of the rearranged series are coming from in the original series.

Now, it is clear that $S_l \geq t_m$. Why is that the case? Well, look at what I have done l is the maximum of $\sigma^{-1}(1), ..., \sigma^{-1}(m)$. That means, every single term that comprises t_m will occur in this sum S_l also and possibly more terms can occur. Therefore, S_l will have to be greater than or equal to t_m.

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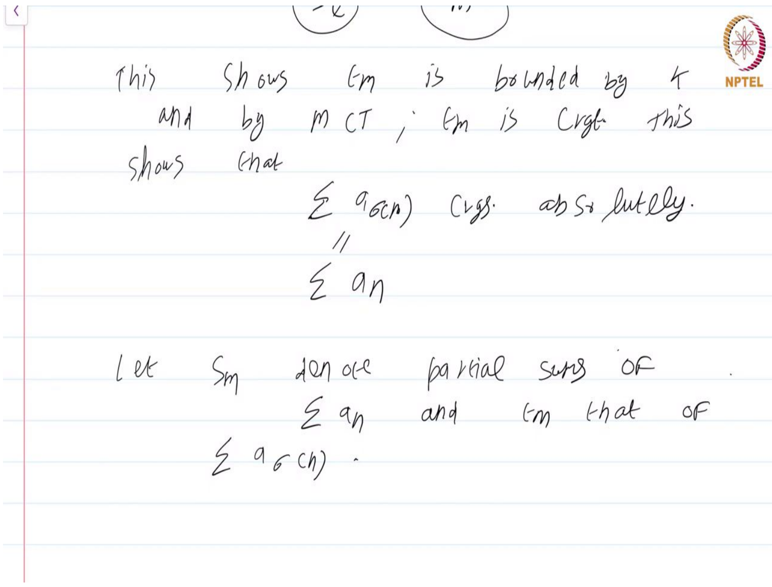

This shows t_m is bounded by k and by monotone convergence theorem t_m is convergent. This shows that $\sum a_{\sigma(n)}$ converges absolutely.

So, this does not finish the proof. We still have to show that this converges to the same thing as $\sum a_n$. Now we reuse notation and let S_m denote partial sums of $\sum a_n$ and t_m that of $\sum a_{\sigma(n)}$.

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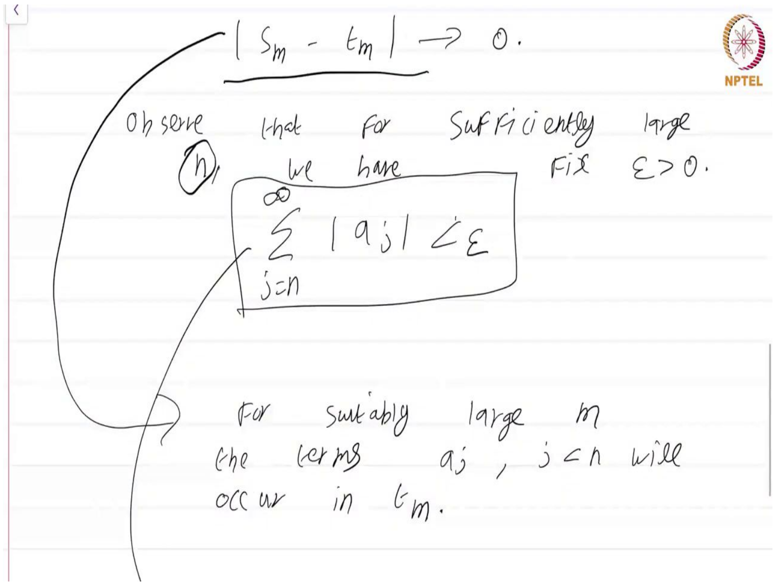

Now, we want to show that t_m converges to the same limit as S_m and we can achieve that by showing that $|S_m - t_m| \to 0$. If we can show this we are actually done right.

Now, how are we going to show that $|S_m - t_m| \to 0$. We have pulled the same trick that we pulled last time. Observe that for sufficiently large n we have $\sum_{j=n}^{\infty} |a_j| < \epsilon$. So, I have to fix $\epsilon > 0$, given any ϵ for suitably large n, we can find I mean the series $\sum_{j=n}^{\infty} |a_j|$ will be less than infinity. This is just rephrasing the fact that the series of absolute values must converge because that is there in the hypothesis.

Now, what you do is you look at this $|S_m - t_m$, then for suitably large m, the terms $|a_j|,\ j < n$ will occur in t_m not $|a_j$ just a_j.

The terms a_j for j < n will occur in t_m. How do we know this? Well, we can just use the same trick that we used here the exact same trick that we used here will give us this.

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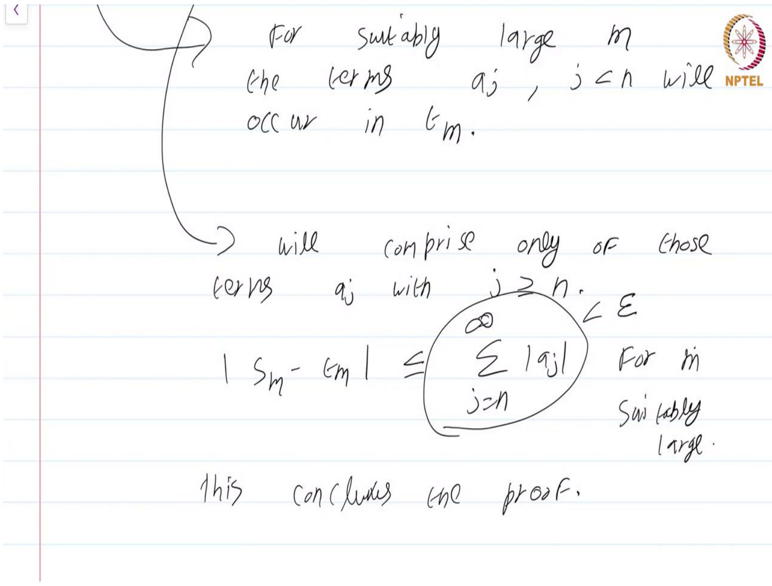

Now, this sum will comprise only of those terms a_j with $j \geq n$, because the first n terms will get cancelled. So, what happens is we get is $|S_m - t_m| \leq \sum_{j=n}^{\infty} |a_j|$ for m suitably large. And this we have already made less than ϵ.

So, here is the brief argument of what has happened. We want to show that $|S_m - t_m| \to 0$.

We choose n so large that we have this that $\sum_{j=n}^{\infty} |a_j| < \epsilon$. Then we ensure that the $|S_m - t_m||$, the first n-1 terms get cancelled, this we do by the same trick that we did for the absolute values.

So, what you will be left with is just those terms $\sum_{j=n}^{\infty} |a_j|$ that will be less than ϵ for m suitably large. So this concludes proof.

So, please go through this proof again. I am intentionally leaving a number of steps under the carpet for you to fetch.

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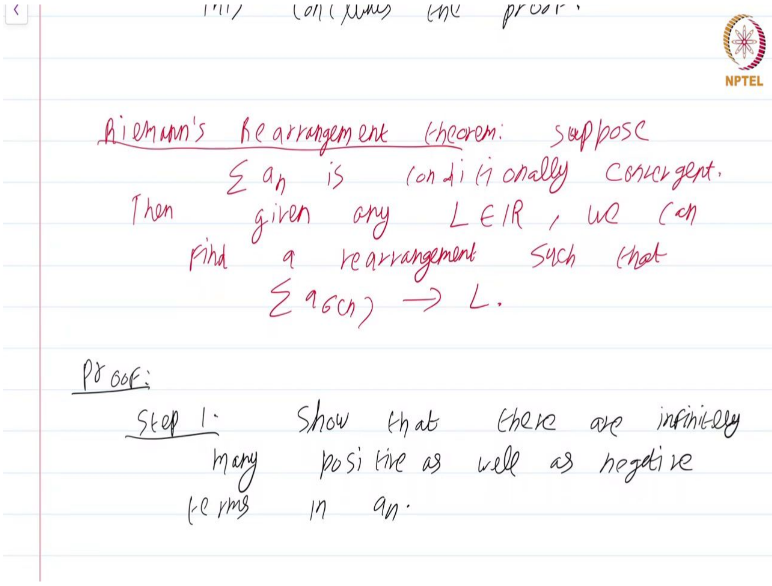

Now, we come to one of the celebrated results which I find utterly fascinating; it is called Riemann's Rearrangement Theorem.

Suppose $\sum a_n$ is conditionally convergent; that means, it converges, but not absolutely. That is you are not just given that it might or might not converge absolutely you are given that it does not converge, absolutely conditionally convergent.

Then given any $L \in \mathbb{R}$, we can find a rearrangement, such that $\sum a_{\sigma(n)} \to L$. How is that no matter what real number you choose, I can rearrange this conditionally convergent series and make it converge to L.

Now, this result is a bit technical and this is going to be one of the first really long exercises in this course. I am going to leave you with a number of steps on how to prove this and you are going to fill up the details ok.

So, let us see how this goes.

Step 1: Show that there are infinitely many positive as well as negative terms in a_n. It cannot happen that a_n has only finitely many positive terms it cannot happen that a_n has only finitely many negative terms.

This is very easy, this is just a warm up this will actually Step 2 will be something even stronger and that is what will be required. But I think it is a good idea to start with a warm up to get a sense of what is going on.

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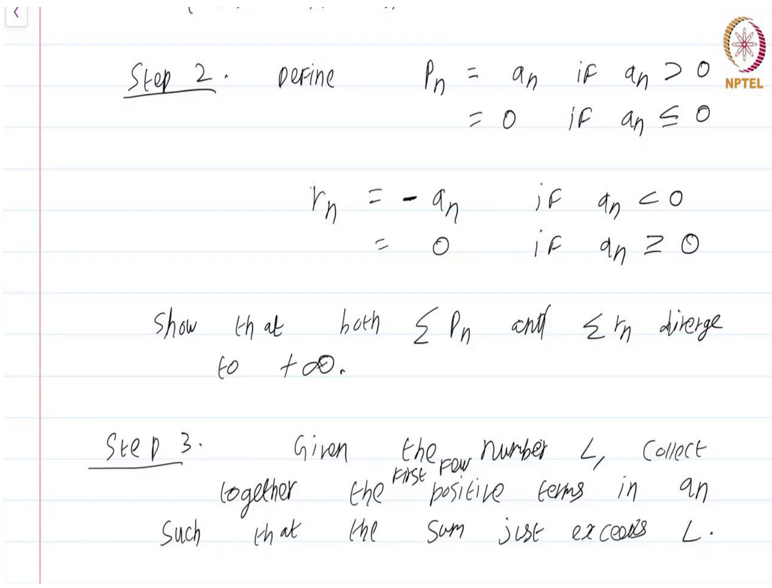

Note for step 1 you have to use the fact that the series does not converge absolutely. That means, you will be showing that if there are only finitely many positive terms then the series actually converges absolutely.

Step 2 Define $P_n = a_n \; if \; a_n \geq 0, = 0 \; if \; a_n \leq 0$. I am defining a new sequence P_n, P stands for positive here whose value will be a_n, if $a_n > 0$ and whose value will be 0 if $a_n \leq 0$.

So, next I define a negative I do not want to use n, so maybe I will just use r_n, $r_n = -a_n \; if \; a_n < 0, = 0 \; if \; a_n \geq 0$.

So, r_n is collecting together all the negative terms of the series. So, P_n collects together all the positive terms, r_n collects together all the negative terms.

Step 2 :If you do not mind I will put a minus sign here, because it will make the statement I am about to write simpler to show that both P_n or rather $\sum P_n$ and $\sum r_n$ diverge. In fact, they diverge to $+\infty$.

So, what this says is that if you sum up all the positive terms and if you sum up the negative of all the negative terms both diverge to plus infinity , this is step 2. Again you have to use the fact that the series $\sum a_n$ does not converge absolutely to show this.

Step 3: Given the number L what you do is collect together the positive terms in a_n, such that the sum just exceeds L.

So, this step might require a bit of elaboration since I am expecting you to fill up the details this is the following.

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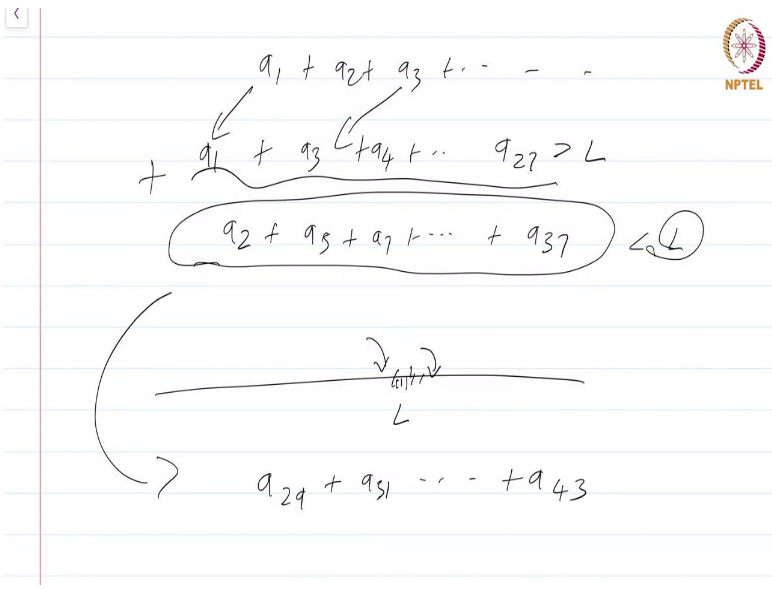

You have $a_1 + a_2 + a_3 + \ldots\ldots$, some of these terms are positive, some of these terms are negative. What I am doing is I am collecting together all the positive terms say a_1 is positive then I put it here then I am putting a_3. Let us assume that a_2 is negative, let us assume that a_4 is positive.

I am collecting together the first few positive terms because $\sum P_n$ diverges to infinity there will come a point that let us say a_{27} such that a_{27} just exceeds L. That means, if I sum up the first 26 terms I should not exceed L, but the moment I sum up the first 27 positive terms. Sorry, I made a mistake that there may not be 27 of them because some of them will be negative.

So, let me rephrase the last statement: what I want to say is if I sum up the first positive terms coming from a_1 to a_{26} it should be less than r. But if I sum up the positive terms coming from a_1 to a_{27} that should exceed L.

So, you are going to keep adding the positive terms till you just cross L do this. So, this is going to be the first few terms of our new modified series, such that it first exceeds L. Now what you do is the following you start clubbing together the negative terms.

So, in particular a_2 is a negative term because it is not appearing in the previous thing you start summing up the negative terms, let us say $a_5, a_7, \ldots$. Let us say till a_{37}, I am plucking numbers out of the air. Do not first too much on why I am choosing these particular numbers.

So, a_1, a_2, a_5, a_7. I am adding all the negative numbers, note that this series $\sum r_n$ also diverges to $+\infty$

So, what will happen is when you add enough of these terms to the previous set of terms there will come a point where it will become less than l again right. Where that when you sum up the first few positive terms and the first few negative terms it will become negative again, go all the way till it just crosses l, but from the other side.

So, intuitively the picture is L is here you first landed here then you are slowly decreasing by adding terms and then just when you cross L again you stop.

So, now what you do is the following you keep adding positive terms again. Let us say $a_{29}, a_{31}, \ldots, a_{43}$, again I am plucking numbers out of the air. And what should happen now well it should start increasing because I am adding positive numbers till it just crosses over l again.

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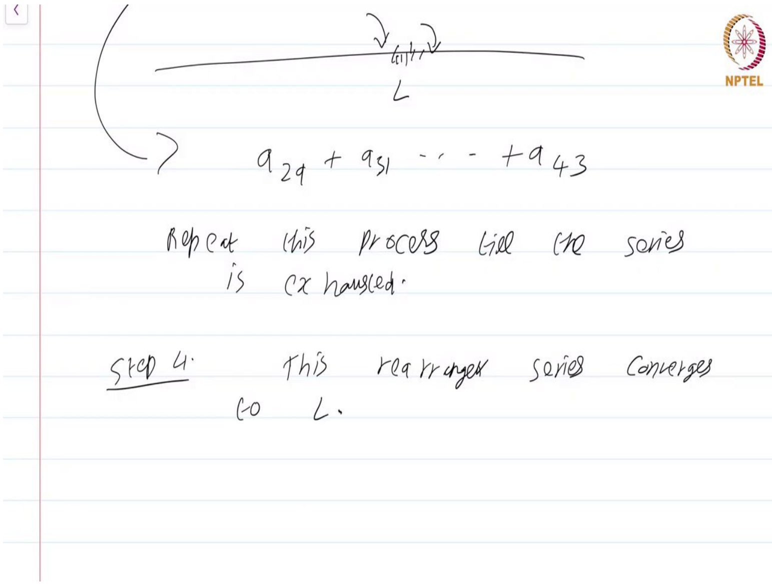

Now, repeat this process, keep adding positive terms till you just cross L then keep adding negative terms till you just cross L again from the other side. Then you are just ping ponging between the left side of L and the right side of L, repeat this process till the series is exhausted. That means, technically you cannot do it because it is an infinite series, but this can be made precise by repeating this process till the series is exhausted. So, you have found a rearrangement.

So, Step 4 is to show that this rearranged series converges to L fine. So, this will not be hard because at each stage you notice that it jumps left and right between L and the magnitude of the jump will become smaller and smaller.

Because the original series converges therefore, the n^{th} term of the original series must converge to 0. I am essentially handing you the proof on a silver platter for the final step, but they are still all the details for you to fill. So, I urge you please fill up the details and finish the proof entirely on your own.

So, this concludes this module you are watching a course on Real Analysis.