Real Analysis - I
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Lecture – 11.4
Erdos S Proof on Divergence of Reciprocals of Primes

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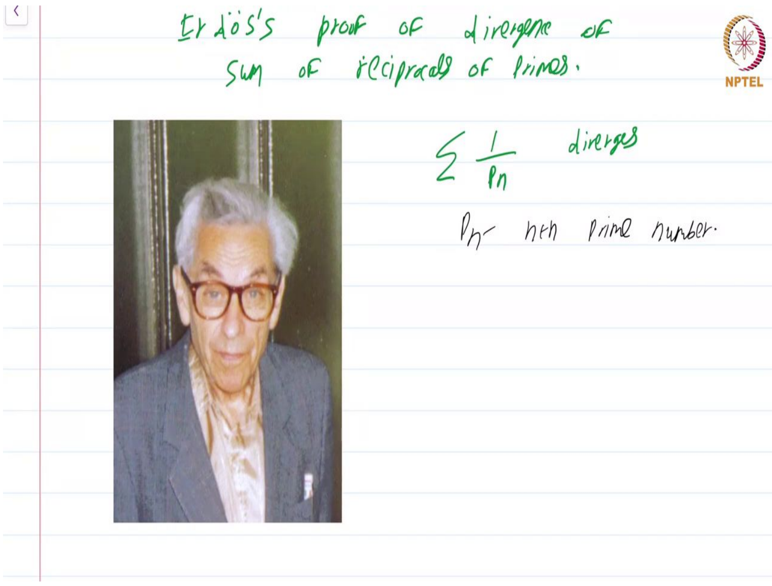

On your screen you see the portrait of the legendary Hungarian mathematician Paul Erdos. Paul Erdos was one of the most prolific problem solvers of the 20^{th} centuries. Unlike many mathematicians who are famous for grand theories, Paul Erdos is more famous for the sheer range of problems he has solved.

And, not only that many of the techniques he has used are so elementary and simple that you can probably explain such techniques to an ambitious student in 9^{th} standard or even 10^{th} standard something, a very young student.

So, I am going to present one of the most beautiful facts in mathematics that $\sum \frac{1}{p_n}$ diverges; p_n stands for the n^{th} prime number. This is actually an infinite series, there are infinitely many primes and again the proof of that is one of the most beautiful proofs in mathematics. I am sure you are familiar with that proof, so I am not going to repeat it. This says that, if you take the reciprocals of the primes, you get a series that is divergent .

In fact, I am not going to assume that there are infinitely many primes in this proof. The proof presented by Erdos will have a corollary that there are infinitely many primes. So, let us see this proof. What we are going to do is it's going to be a proof by contradiction.

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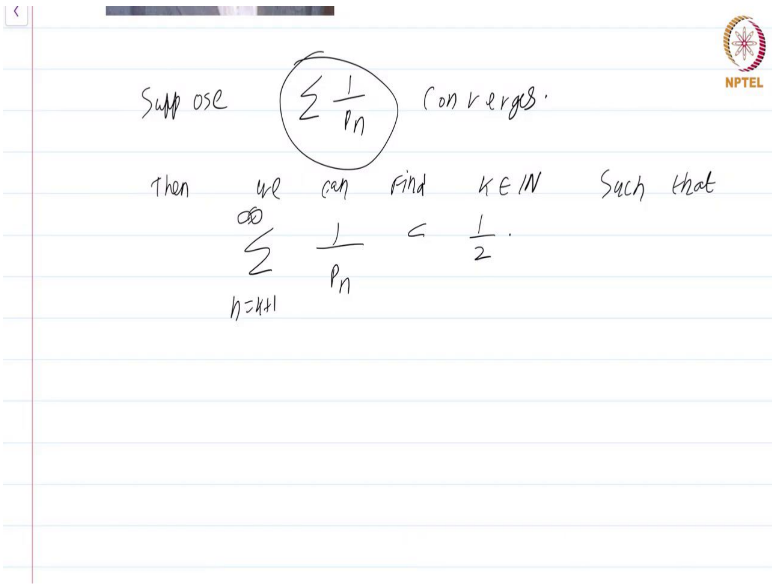

Suppose, $\sum \frac{1}{p_n}$ converges. Now, of course, here I have to make a remark, if this series turns out to be finite; I am taking it to be convergent. I am just extending the series by 0's let us say.

So, suppose $\sum \frac{1}{p_n}$ converges then we can find $k \in \mathbb{N}$, such that $\sum_{n=k+1}^{\infty} \frac{1}{p_n} < \frac{1}{2}$. The tail of a convergent series can be made as small as you desire, that is all I am using here.

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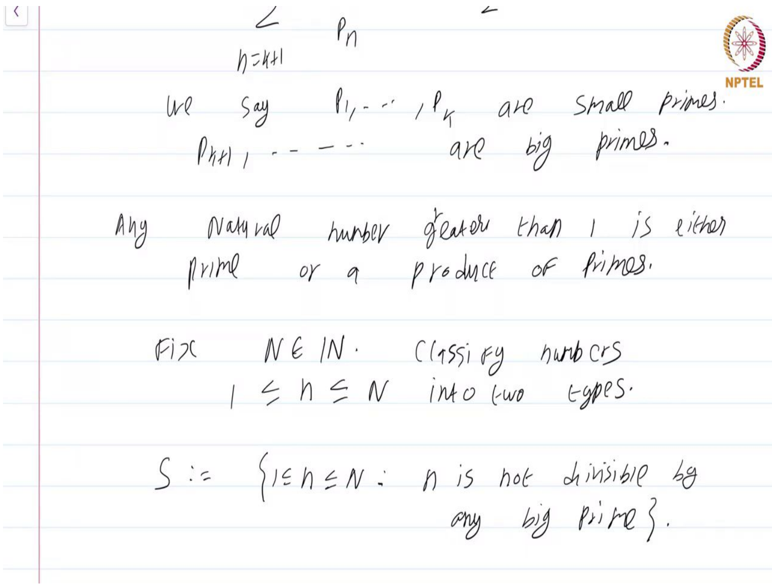

Now, let us divide the collection of primes into two sets. We say we say $p_1, \cdots, p_k$ are small primes. $P_{k+1}, \cdots$ are large primes. Let us call them big primes, that sounds better are big primes. So, I have divided the collection of primes into two parts. The first k primes are called small, beyond that they are called big primes .

Now, I need to use one fact from very basic number theory which you have no doubt, learnt in school. Any natural number higher than 1 or greater than 1 is either prime either prime or a product of primes. This is called the fundamental theorem of arithmetic. They are one of the most foundational facts of number theory.

So, fix $N \in \mathbb{N},$ we are going to classify numbers $1 \leq n \leq N$ into two types. I am going to call these two sets.

$$S := \{1 \leq n \leq N : \ n \ is \ not \ divisible \ by \ any \ big \ prime\}$$

That means, from the fundamental theorem of arithmetic that I have stated above, this n is going to be a product of primes; none of those primes can be a big prime that is what this means.

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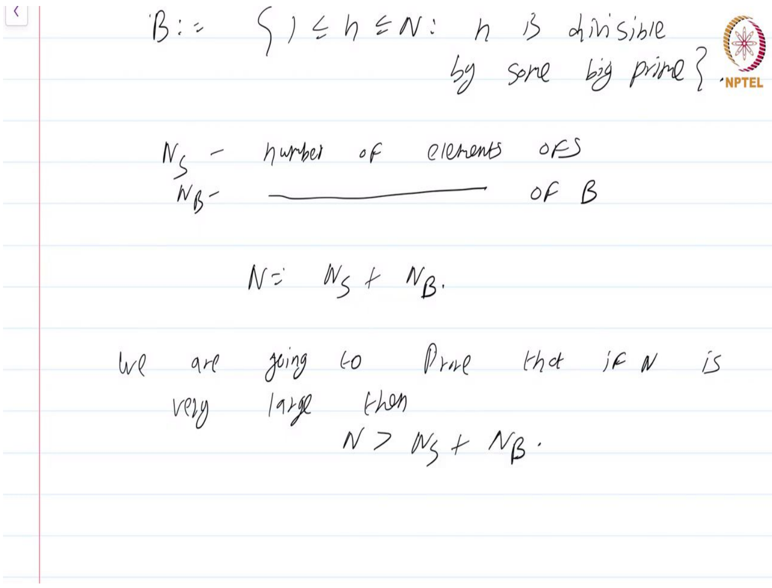

Then, I am going to call the set B,

$$B := \{1 \le n \le N : \ n \ is \ not \ divisible \ by \ some \ big \ prime\}$$

So, we have classified all the numbers $1 \le n \le N$ into these two sets S and B. Just pause the video for a moment and think about where the number 1 falls into this classification.

I hope you got that 1 is actually an element of S, simply because it is not divisible by any big prime. In fact, it is not divisible by any prime at all. Now, it is clear that if I call N_S, the number of elements of S and N_B the number of elements of B.

Then a moment's thought should convince you that $N = N_S + N_B$, after all right. Now, what I am going to do is prove that if N is very large; how large that is? Well, we will see in a moment, if N is very large then $N > N_S + N_B$, which is nonsense. How are we going to show this? Well, we deal with each case separately.

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N_B

$n \in B$

p_i, $i \geq k+1$

$n = q p_i$, $q \leq \frac{N}{p_i}$

Each element in the set B is

$q p_i$, $i \geq k+1$, $q \leq \frac{N}{p_i}$

For a fixed i, there are exactly

$int(\frac{N}{p_i})$ numbers in B that

are divisible by p_i.

Let us look at N_B; let us try to see whether we can count N_B. So, take an element $n \in B$. Then if you think for a moment, it is going to look something like this; some prime is going to divide it. Let us say p_i. So, $i \geq k+1$ is going to divide it. Recall that k was the number $p_1, \dots, p_k$ are the small primes.

So, some p_i, $i \geq k+1$, is going to divide it. So, this number n is going to look like some qp_i and obviously, this $q \leq \frac{N}{p_i}$ right. So, because the elements of the set B are all less than or equal to N. It is obvious that this quotient q has to be less than or equal to $\frac{N}{p_i}$; otherwise $p_i q > N$, which is not possible.

So, what we are going to do is in other words. In other words each element in the set B is some qp_i, $i \geq k+1, q \leq \frac{N}{p_i}$. For a fixed i, there are exactly integer part $int(\frac{N}{p_i})$ numbers in B that are divisible by p_i.

What I am trying to say here is this $int(\frac{N}{p_i})$ just strips away the decimal portion. This N need not be divisible by p_i. So, if I get let us say 4.7, I am just going to treat it as 4, that is what this integer part $int(\frac{N}{p_i})$ means. So, what I am trying to say is look at all those numbers in the set B that are divisible by p_i. What are they going to be?

They are going to be $p_i, 2p_i, 3p_i, \cdots$ all the way until you reach a number qp_i such that $(q+1)p_i > N$. And how do I get what this qp_i is? I precisely take the integer part of $\frac{N}{p_i}$ right, that will precisely capture what this final qp_i is going to be which is just less than or equal to N. So, this will count all those numbers that are divisible by p_i.

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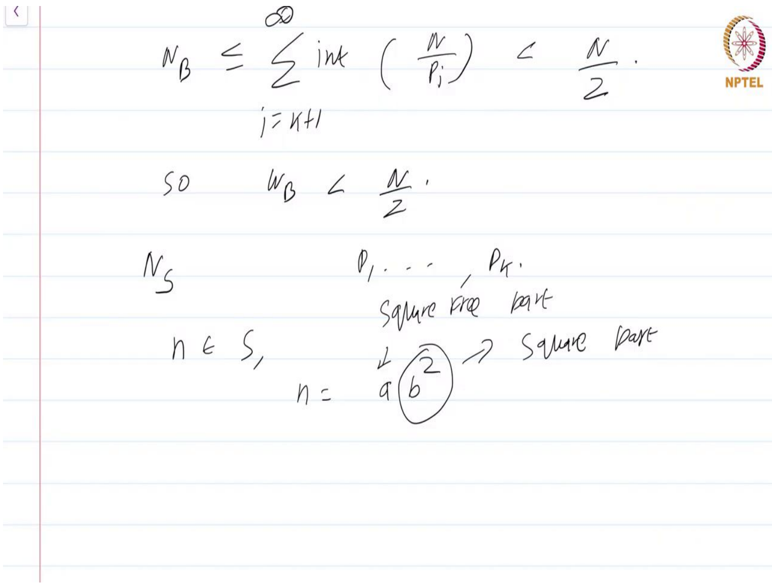

Now, do this for every prime and you will understand that, this $N_B < \sum_{i=k+1}^{\infty} int(\frac{N}{p_i})$. Simply because, each such term will count the number of numbers that are divisible by that particular p_i. I just sum it across all the big primes.

Now, immediately from our assumption that these big primes were chosen, so that $\sum \frac{1}{p_n} < \frac{1}{2}$. What we get is, this is less than or equal to or strictly less than $\frac{N}{2}$.

So, $N_B < \frac{N}{2}$, that is very nice. Now, let us try to estimate N_S. The number of of numbers that are divisible by exclusively by small primes ; recall there are k of them $p_1, \ldots, p_k$.

Now, take a number $n \in S$ again, you can write it as a product of primes. Now, what I am going to do is to group together all those primes that are occurring exactly once in the prime factorization. That means, the non-squares. So, what I am going to do is I am going to write $n = ab^2$. So, these are the square part and these are the square free part.

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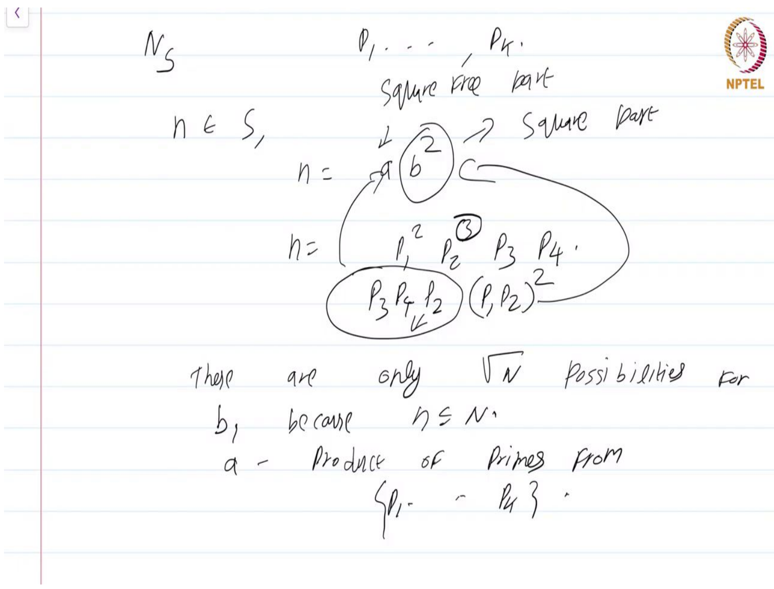

So, for a concrete illustration, if $n = p_1^2 p_2^3 p_3 p_4$ then this will be written as 4 . So, this is the square free part. So, I am going to club together all those primes that occur at least twice together, not at least twice; I am going to club all the squares together; there might be some leftover powers.

Like for instance here the number p_2 even though it occurred as a cube this p_2 is left over. So, I am going to gather together all the squares and call it b^2 and the rest of the parts which each prime will occur only once will be called 'a' . Now, what I am going to do is I am going to count the possibilities for a and b.

Now, the first observation is that there are only square roots of N possibilities for b right. Because, if b were to exceed $\sqrt{N}$ then b^2 will exceed N which is simply not possible because, this n is because $n \leq N$, that is the reason right.

So, the number of possibilities for this number b, there are $\sqrt{N}$ possibilities. What about a? How many possibilities are there? Well, they are just going to take some subset. So, what is a? a is going to be a product of primes from $p_1, \cdots, p_k$ right. But, you should take each prime only once.

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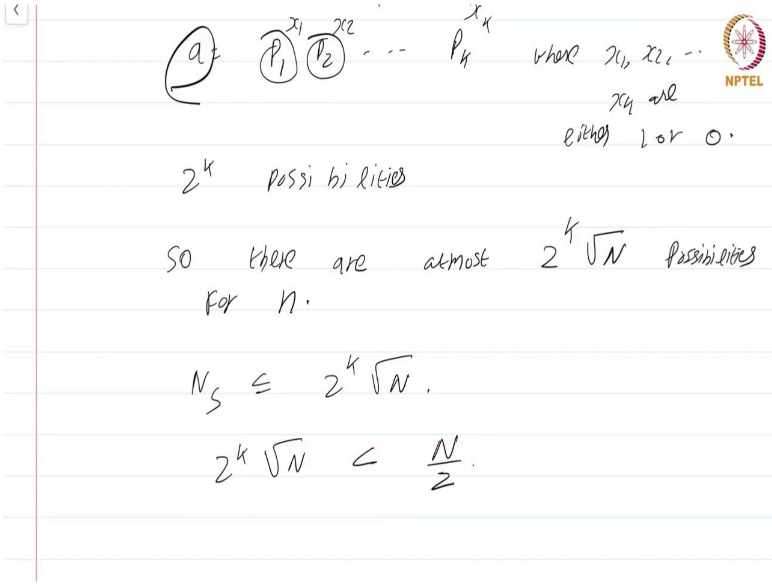

So, what you can do is $a = p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$, where $x_1, x_2, , , , , , x_k$ are either 1 or 0 right. So, these are all the possible a's. They are constituted by primes coming from $p_1, \cdots, p_k$, but the powers cannot be higher than 2. Simply because, these are the square free part, if there was a power higher than 2 then that would not even occur here.

So, this is going to be all those $p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$, where $x_1, x_2, , , , , , x_k$ are all 1 or 0. And, there are precisely 2^k possibilities. That means there are 2^k possibilities; for each p_1, p_2, I have two choices 1 or 0. So, multiplying all together there are 2^k possibilities. So, there are at most $2^k \sqrt{N}$ possibilities for n. There are 2^k possibilities for the number a and there are $\sqrt{N}$ possibilities for b.

Therefore, put together there can be only $2^k \sqrt{N}$ possibilities for n. In other words, this argument shows that $N_S \leq 2^k \sqrt{N}$. Now, how does this help?

Well, we have full flexibility of the choice of N, k is fixed because we have fixed k to be the first I mean the k to be the number such that the sum of the primes greater than p_k greater than I mean sum of the reciprocals of primes greater than p_k is going to be less than half. That is how we fix k. k is fixed, but N is completely up to S. All we want is to find N such that $2^k \sqrt{N} < \frac{N}{2}$, then we will get a contradiction.

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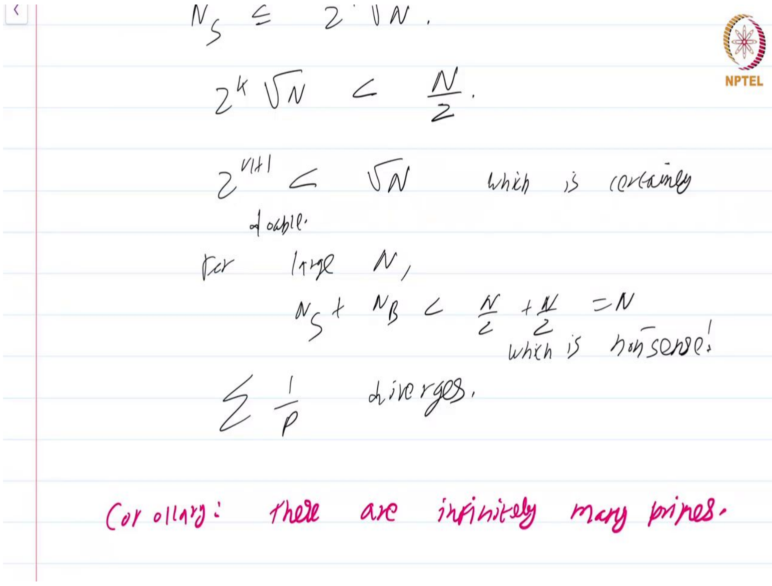

In other words we just want $2^{k+1} < \sqrt{N}$ right, which is certainly doable for large enough N. It will be doable just certainly doable; which means for large N, $N_S + N_B < \frac{N}{2} + \frac{N}{2} = N$, which is nonsense.

So, this completes the proof of the fact that $\sum \frac{1}{p}$ diverges. If you carefully look through the proof we have not used the assumption that there are infinitely many primes. So, as a corollary, we get corollary there are infinitely many primes.

So, this is the hardest example in this course actually. This example is somewhat different from all the others that we have seen in the chapter on series. I just wanted to show this proof because it illustrates some of some deep thinking that can go into proving something as simple as this.

This is a course on Real Analysis and you have just watched the module on the sum of the reciprocals of primes is divergent.