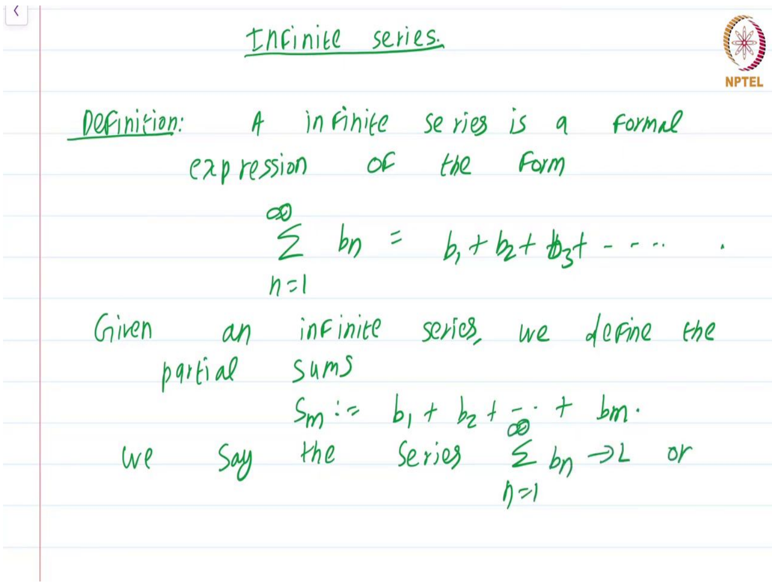
Real Analysis - I

Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 11.1
Definition and Examples of Infinite Series

(Refer Slide Time: 00:13)



We now come to the next topic, series. Infinite series are related to sequences but are not exactly coincident with the concept of sequences. Let me first make the definition before I continue with general remarks.

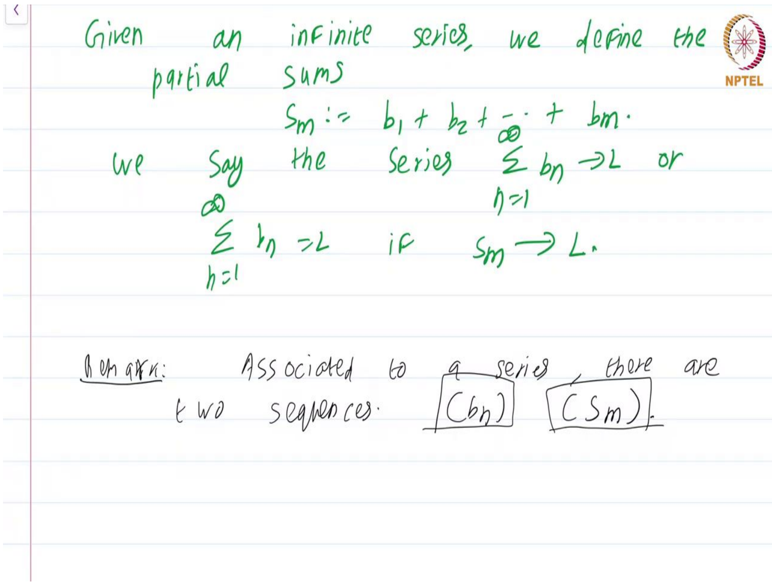
Definition: An infinite series is a formal expression of the form the form

$$\sum_{n=1}^{\infty} b_n = b_1 + b_2 + b_3 + \ldots,$$

.

So, given an infinite series we define the partial sums $S_m := b_1 + b_2 + b_3 + \ldots.$

(Refer Slide Time: 01:54)



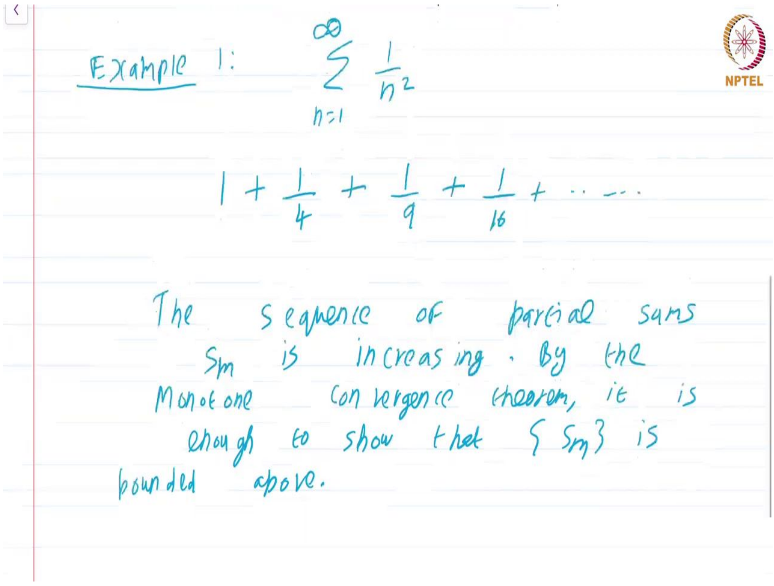
We say the series $\sum_{n=1}^{\infty} b_n \to L$ or $\sum_{n=1}^{\infty} b_n = L$ if $S_m \to L$.

So, there are two remarks that I would like to make. The first one is

Associated to a series, there are two sequences. One is the sequence (b_n), the other is the sequence (S_m). And the convergence of the series is determined by this convergence of S_m, not by the convergence of b_n.

We will now see that if a series does converge this sequence (b_n) must also converge and it must converge to 0. That mostly acts as a negative test for series is converging. You can determine that a series does not converge by showing that b_n does not converge to 0. But the more important sum for us is this sum S_m, the sequence of partial sums. This is the one that actually plays the central role for determining convergence. So, let us see one example of a convergent series and one example of a divergent series.

(Refer Slide Time: 04:07)



So, example, let us consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$. So, it might be a good idea to write down the first few terms just to get an idea of what is happening.

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots\ldots$$

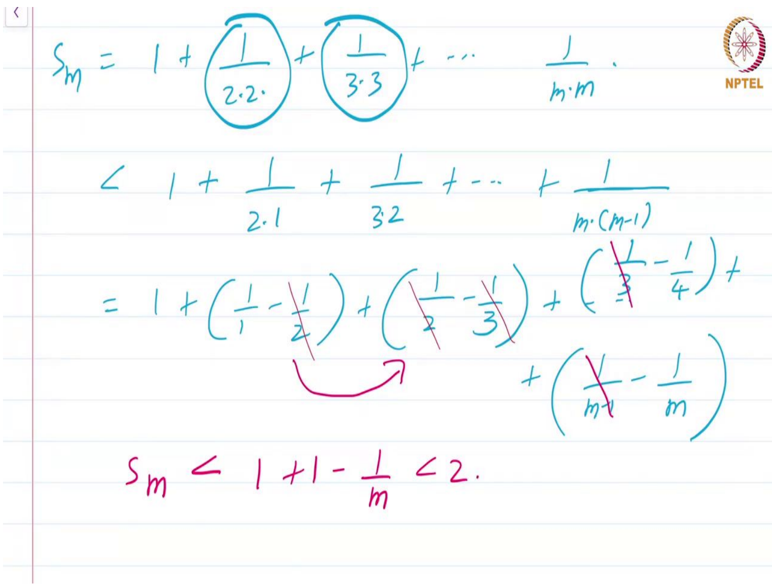
Now, what I do is, I know that the sequence of partial sums S_m is increasing simply because the series has terms that are all non-negative. So, there is nothing special about the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, any series in which you have the nth term b_n to be greater than or equal to 0 always will have this nice property. That the sequence of partial sums will be increasing.

Now, our light bulb should flicker over your head, if a sequence is increasing, then it is convergent if and only if it is bounded above ,that is the monotone convergence theorem. By the monotone convergence theorem, it is enough to show that S_m is bounded above, then we will automatically know that it is convergent.

Note the approach I am taking will give you no idea what this sum is. We will determine what this sum is at a later time in the course, it is a very interesting sum. But the approach I am taking right now I am completely choosing to ignore the limit L and directly conclude that this series converges without giving us any clue as to what L is. Actually we will get some idea what L is if you carefully follow what I am about to do.

Now, I want to show that the partial sums are bounded. So, what I want to do is, I want to bound the partial sums from the above. So, what I will do is, I will start estimating the terms of the partial sums, but at each time increasing the quantity. If I increase the quantity, then what I estimate I will get a less than or less than inequality.

(Refer Slide Time: 07:05)

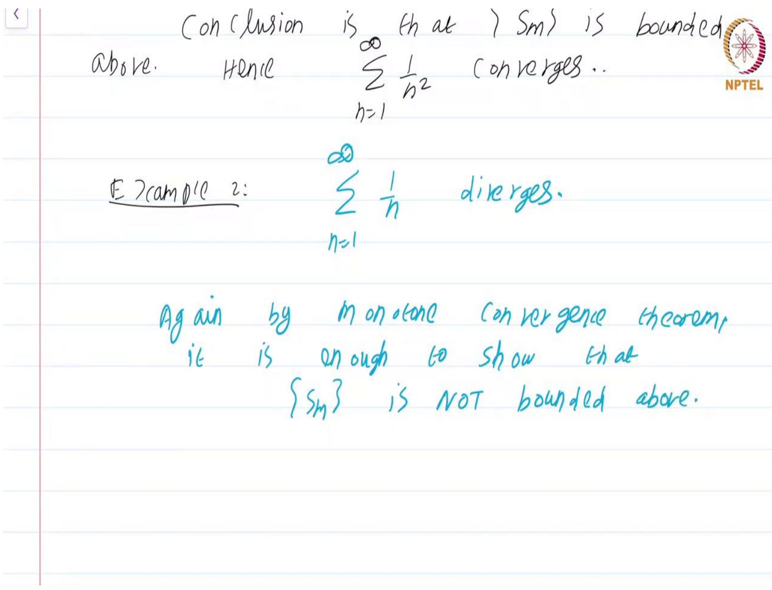


So, if that sounds vague, just wait a moment, you will understand what is going on. What I do is I write $S_m = 1 + \frac{1}{2 \cdot 2} + \frac{1}{3 \cdot 3} + ... + \frac{1}{m \cdot m}$. This is the sum of the first m terms ok. Now, as I said I want to make this less than something. So, I want to increase each one of these quantities. Now, each one of these quantities is a quotient.

To increase a quotient, the natural thing to do is to decrease the denominator. So, this is $< 1 + \frac{1}{2 \cdot 1} + \frac{1}{3 \cdot 2} + ... + \frac{1}{m(m-1)}$. Now, here is the trick: this is equal to $1 + (\frac{1}{1} - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + ... + (\frac{1}{m-1} - \frac{1}{m})$. That, this is correct is just basic arithmetic from 5th standard. How has this helped us well zoom in and you will notice something happening tunk, let me just write the next term, so that I can tunk it.

Now, I can tunk it, tunk, tunk so on you see that the second term will get cancelled with the first term of the next part right, and this will repeat. So, eventually everything up till this $\frac{1}{m-1}$ will get cancelled. So, what you will get is $S_m < 1 + 1 - \frac{1}{m}$ right which is certainly less than 2.

(Refer Slide Time: 09:27)



So, conclusion is that S_m is bounded above. Hence, hence $\sum \frac{1}{n^2}$ converges, but the limit is unknown to us currently.

So, one question, you might ask is , this is all nice and good, but there seems to be some trickery involved in finding out that $\sum \frac{1}{n^2}$ converges. And this is about the simplest series possible. Well, one of the issues in presenting mathematics is I have a limited amount of time to convey a lot of information.

Now, the first person who managed to show that this is actually a convergent series, I do not know who that person was probably did not get it in 7 minutes, which is roughly the time it has taken for us to actually show that this is convergent it probably took many, many days that person is likely to have experimented a lot with some heuristic reasoning like what I said you want to make the terms less than something.

So, you try to increase each individual term. So, you try to decrease the denominator and so on. So, he must have spent a lot of time manipulating this algebraically and finally, landed up with this proof. Now, if you were to ask me how exactly did he think of this particular line of approach, I don't really have an answer because when you experiment, you do not always experiment in a perfectly logical and linear way.

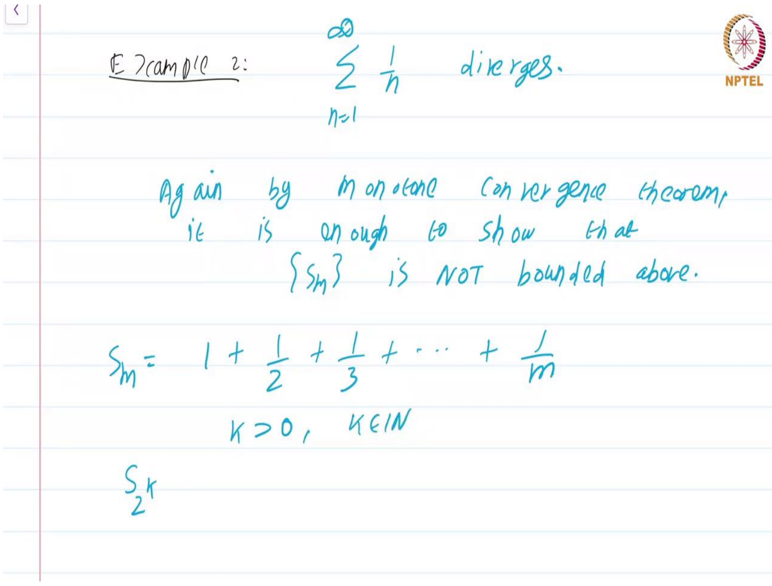
You try out various things and it clicks. So, you must always understand that there is a bit of experimentation involved in all of this guided by heuristics. You should not be randomly combining terms and hoping that life falls into place, that is not the way life works . You will have to plan systematically, but at the same time you should not be so rigid .

So, all these remarks are just an excuse for me to give example 2 which will have similar manipulations example 2 , $\sum \frac{1}{n}$ diverges. The series $\sum \frac{1}{n^2}$ converges, but the series $\sum \frac{1}{n}$ diverges.

Well, this might be a bit shocking to you, but you can do this following experiment on a calculator or a computer and try to see what exactly happens by summing up 1 by n for the first 10,000 terms the first 1 lakh terms, the first million terms and so on, and you will see that it slowly increases it is it diverges, but it diverges very very slowly.

So, again by monotone convergence theorem, by the way this should be implicit in my by the way I am talking. If a series does not converge, we say it diverges and let me just make that clear because I do not think I wrote it down explicitly in the definition. Again by monotone convergence theorem it is enough to show that S_m is not bounded above, then I would be done.

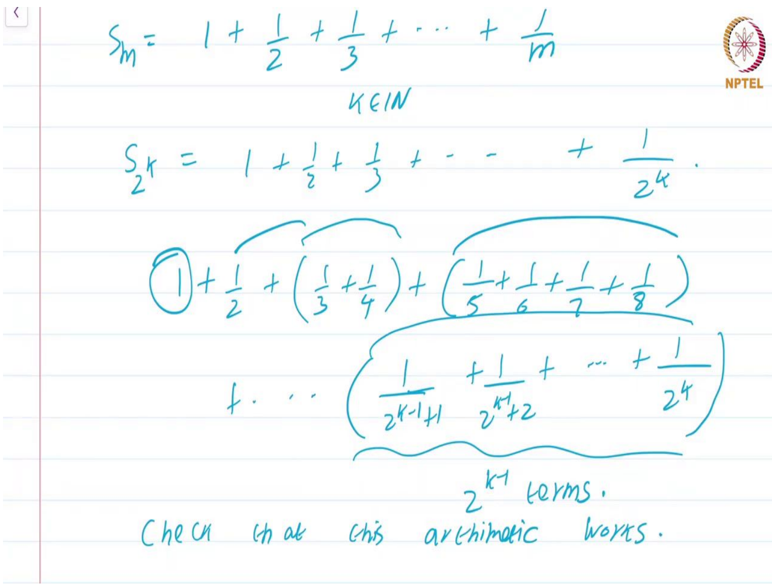
(Refer Slide Time: 13:18)



So, let us write down S_m it will be $1+\dfrac{1}{2}+\dfrac{1}{3}+....+\dfrac{1}{m}$. Now, what I am going to do is, I am not going to consider S_m, I am going to consider S_{2^k}, k > 0.

If I can show that this S_{2^k}, so of course, I must write $k \in \mathbb{N}$. That is better than writing k > 0, which is ambiguous. If I can show that this S_{2^k} itself is unbounded or not bounded above, then certainly S_m is unbounded above.

(Refer Slide Time: 14:03)



So, what will this S_{2^k} = $1+\frac{1}{2}+\frac{1}{3}+....+\frac{1}{2^k}$, this is a bit silly. It really does not seem like I have done anything other than substitute $m=2^k$, but this is going to make a world of difference in the next step.

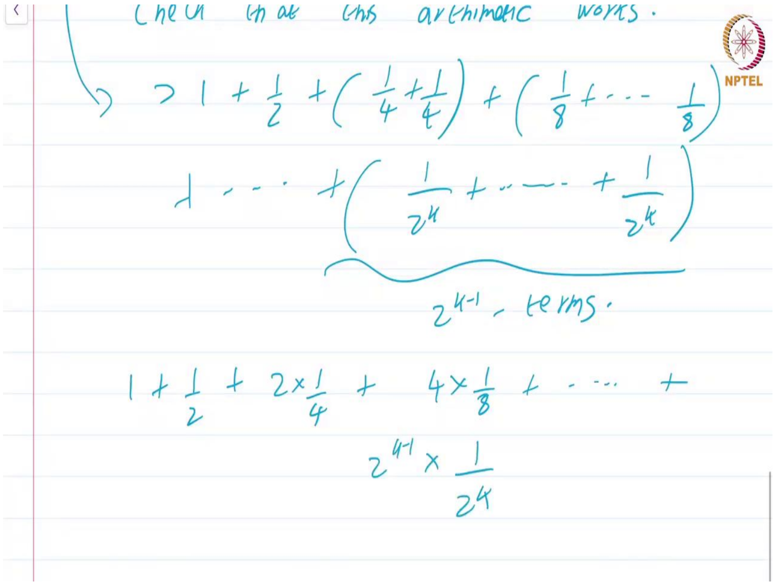
I write this as $1+\frac{1}{2}+(\frac{1}{3}+\frac{1}{4})+(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8})....+(\frac{1}{2^{k-1}+1}+\frac{1}{2^{k-1}+2}+.+\frac{1}{2^k}$. And here there are 2^{k-1} terms. You see what I have done, I combined two terms first, then I combined four terms, then obviously, I am going to combine eight terms so on, and finally I am combining 2^{k-1} terms.

Now, check that this arithmetic works. I cannot just say I am combining terms if there are not enough terms for me to combine. Arithmetic works that are just summing up a very finite set of terms of a geometric series.

So, $2+4+....+2^{k-1}$, you will immediately see that that is $2^{k-1}-1$ or something like that or very close to that, or rather $2+4+....+2^{k-1}$ will be exactly 2^{k-1}, and there is a 1 here. So, put together there are 2^k terms. So, I am accounting for every single term and it is possible to split up this series in this way, this finite series in this way.

Now, what is the moral of the story? I want to make this term greater than something and show that what it is greater than sort of diverges to infinity right. So, what I want to do is, I want to get a greater than bound. So, what I will do is I will have to make each one of these quantities larger right. Sorry I want to make each one of these terms smaller not larger, that means, I want to increase the denominator.

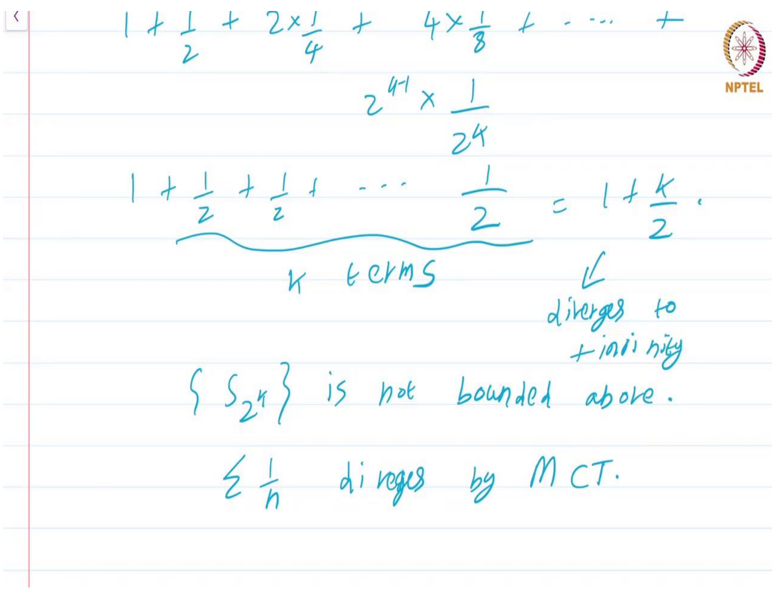
(Refer Slide Time: 16:58)



And it should be fairly easy to guess the next step, it is $1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + ... + \frac{1}{8}) + ... + (\frac{1}{2^k} + . + \frac{1}{2^k})$

So, what will this give us? This will give us $1 + \frac{1}{2} + 2 \times \frac{1}{4} + 4 \times \frac{1}{8} + ... + 2^{k-1} \times \frac{1}{2^k}$ right. Well that is just going to be $1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ... + \frac{1}{2}$ and so forth.

(Refer Slide Time: 17:50)



How many such terms they will be k halves, k terms like this . So, this will immediately give $1+\frac{k}{2}$. This will give $1+\frac{k}{2}$, which diverges to infinity; which means the set S_{2^k} is not bounded above. So, consequently by the monotone convergence theorem $\sum \frac{1}{n}$ diverges.

So, here we see that the behaviour of $\sum \frac{1}{n}$ is quite different from behaviour of $\sum \frac{1}{n^2}$. And we are able to see the difference in the way we combine the terms and do this algebraic manipulation . Now, what I will do is, in the next module I will state and prove the Cauchy condensation tests that sort of takes into account these manipulations that we are doing and makes them into a general theorem.

This is a course on Real Analysis and you have just watched the module on Definitions and Examples of Infinite Series.