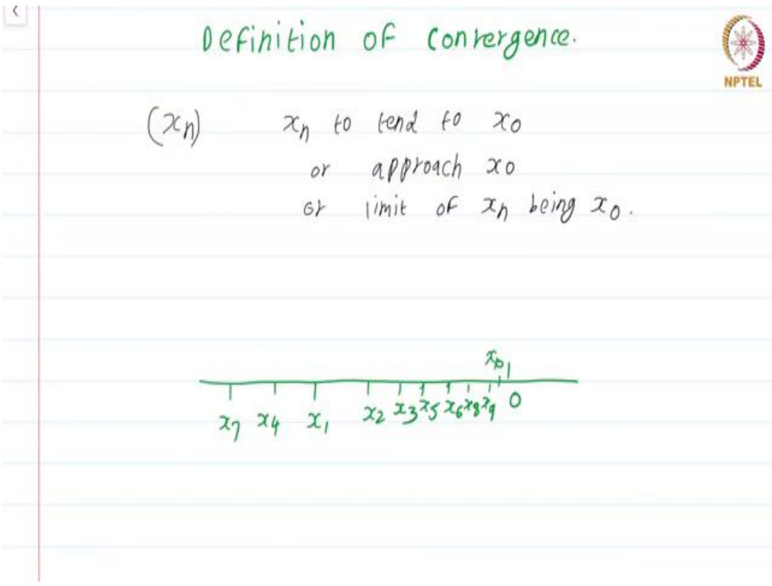
Real Analysis - I

Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture – 7.3
Definition of Convergence

(Refer Slide Time: 00:15)



In this module we shall define what Convergence means. This is therefore, the central module of this entire course. So, I suggest that you concentrate very well during this module. What we want to do is the following. We are given a sequence x_n and we want to define what it means for x_n to tend to some x_0 or approach x_0 or limit of x_n being x_0. These are the various phrases we use to describe the same thing.

So, let us draw a picture to see what we could possibly mean by convergence. So, let us draw a picture. This is the real line and we have terms of the sequence x_n, x_1 is here, let us say x_2 is here, let us say x_3 is here, let us say x_4 jumps back, then x_5 is here, x_6 is here, let us say this is 0, then x_7 is here, then x_8, x_9, x_{10}. Let us say that eventually it turns out that all the terms of the sequence keep getting closer and closer to 0, then we would like to say that x_n converges to 0, but how do we make this mathematically precise?

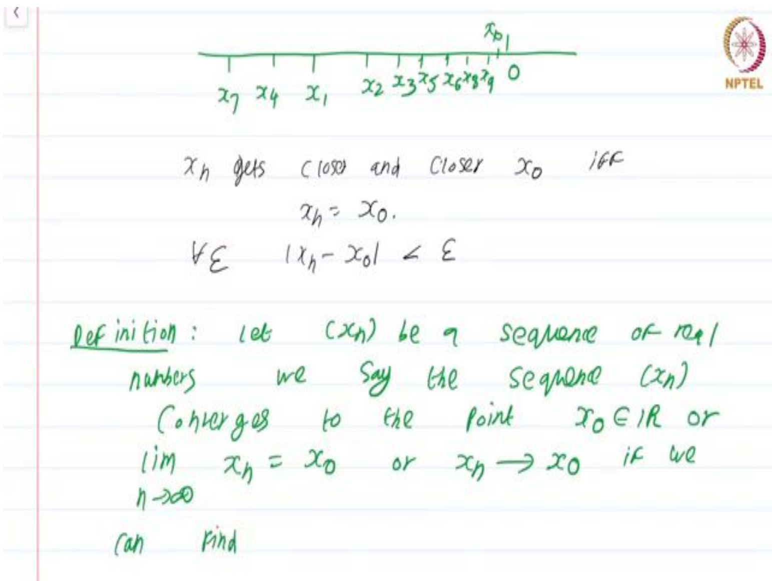
So, we have to define what gets closer and closer to 0 ok. Now, one of the great principles of mathematics is that instead of solving a really tough problem like this you try to solve a much simpler problem. What is the much simpler problem that we can solve? Well we can ask the question what does it mean for x_n to be close to x_0? We will not say x_n gets closer and closer to x_0, what does it mean that x_n gets close to x_0?

Well now it is subjective. If you were to consider a regular scale the rulings of a scale are 1 millimetre apart or in more fancy language, the least count of a scale is 1 millimetre. If x_n and x_0 are less than 1 millimetre away, then at least from the perspective of this scale, they are very very close, we cannot distinguish really between x_n and x_0 if x_n is at max 1 millimetre away.

So, from the perspective of this scale x_n is very very close to x_0. On the other hand suppose you have some finer scale and you have attached to it some sort of magnifying glass so, that you are still able to see let us say the least count is now one thousandth of a millimetre, then x_n and x_0 being 1 millimetre apart might not be close to each other, under the magnifying glass of this new ruler they might look really far away.

Now, let us say you have an electron microscope that is able to resolve all the way up to nanometres, then x_n must be really close to x_0 from the perspective of this electron microscope. So, the point is that x_n must get closer and closer to x_0 means irrespective of the device that I use, x_n must be very very close to x_0.

(Refer Slide Time: 04:25)



If you think about it for a moment, if we crudely analyze this, we will be forced to conclude that x_n is gets closer and closer; closer and closer to x_0 if and only if $x_n = x_0$ if you do a crude analysis of what we are doing here ok. Because what we are essentially saying is that $|x_n - x_0| < \varepsilon$ for all ε and we have solved, we have rather proved a proposition that says that in this scenario x_n is actually equal to x_0.

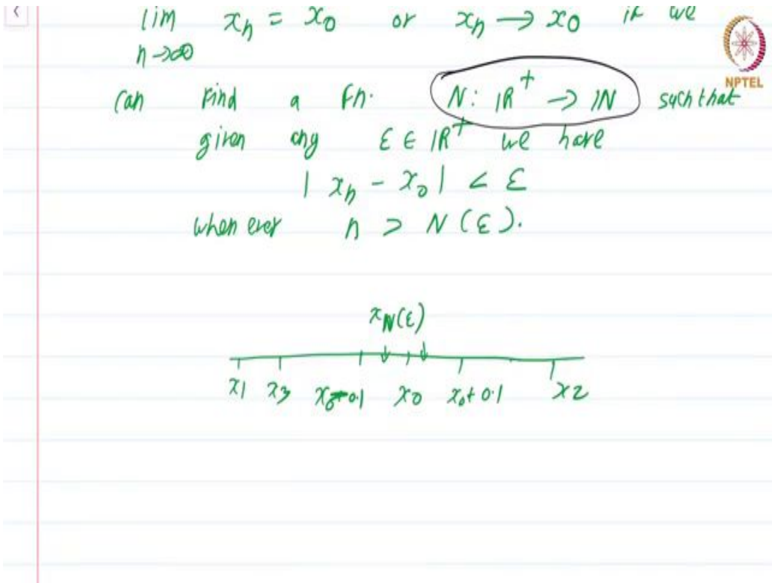
But wait a minute, x_n is not an individual term, its not just one object. its actually a sequence. So, x_n gets closer and closer to x_0 does not mean that every term in the sequence x_n is closer and closer to x_0, we want x_n to get closer to x_0 as we increase n. That means, suppose I give you a ruler whose least count is 1 millimetre and you are able to say well x_1 may not be 1 millimetre away from x_0, x_2 may not be, but I guarantee that from x_{100} onwards $|x_{100} - x_0|$ is less than 1 millimetre.

So, what this is essentially saying is that you provide the ruler or the least count or the closeness or the level of approximation that you desire, then I am able to provide for you a value of n such that beyond that n all the terms in the sequence satisfy that level of closeness or that level of approximation.

So, what we have to essentially do is that to show that x_n converges to x_0, we must show that no matter what ε is provided; however, small it be I must be able to provide an n such that whenever the sequence you have beyond that point n in that sequence, then $|x_n - x_0| < \varepsilon$ and this is the motivation for the central definition of this course the definition of convergence.

Definition: Let x_n be a sequence of real numbers we say the sequence x_n converges to the point $x_0 \in \mathbb{R}$, or $\lim_{n\to\infty} x_n = x_0$ or $x_n \longrightarrow x_0$, we will use both notations $\lim_{n\to\infty} x_n = x_0$ and $x_n \longrightarrow x_0$, x_n converges to x_0 is just denoted by an x_n right arrow $x_0 . \in \mathbb{R}$, if we can find a function; if we can find a function $N : \mathbb{R}^+ \longrightarrow \mathbb{N}$ such that, $\mathbb{R}^+$ here denotes the set of positive real numbers, such that given any $\varepsilon \in \mathbb{R}^+$ we have $|x_n - x_0| < \varepsilon$ whenever $n > N(\varepsilon)$.

(Refer Slide Time: 07:55)



So, let us see what this definition is trying to say, we have a sequence x_n of real numbers, we want to say that the sequence x_n converges to x_0. The notation for this is $\lim_{n\to\infty} x_n = x_0$ or $x_n \longrightarrow x_0$.

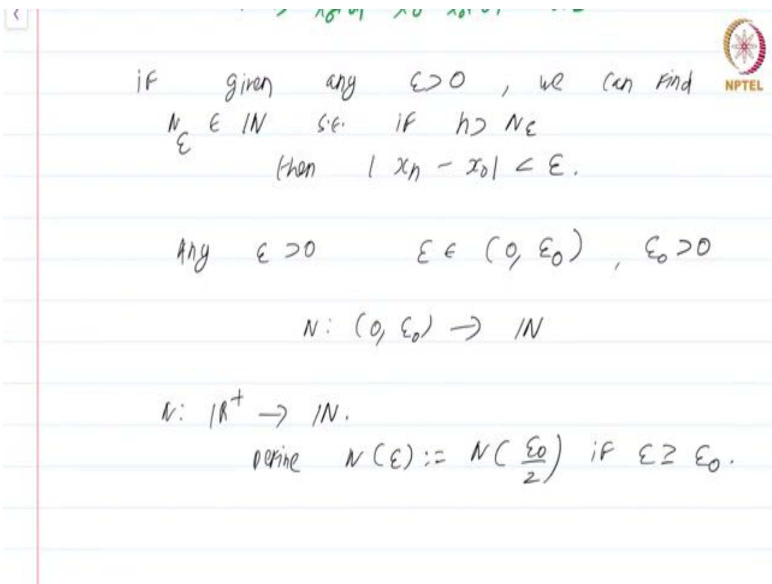
To demonstrate that x_n converges to x_0, we must produce a function n from R plus to the natural numbers. If somebody gives you an ε, this ε is supposed to be the level of desired approximation or you can think of it as the least count of the scale, this $N(\varepsilon)$ is supposed to be the point at the sequence beyond which the terms of the sequence satisfy this desired level of approximation.

That means, given any ε, if I choose $n > N(\varepsilon)$, $|x_n - x_0|$ must be less than ε. Note this must not be true just for $n = N(\varepsilon)$, it should be true for all natural numbers greater than $N(\varepsilon)$. That is essentially saying pictorially that if somebody gives 0.1 as their level of desired approximation $x - 0.1$ here.

Somebody gives you 0.1 as the desired level of approximation what this says is that x_1, x_2, x_3 so on might fluctuate outside this desired level of approximation, but the moment you cross $N(\varepsilon)$, then all the terms of the sequence are forced to be within this interval $[x_0 - 0.1, x_0 + 0.1]$.

So, the definition captures our intuition that as you go further and further along the sequence, the terms of the sequence are all clustered nearer and nearer to the desired point x_0, ok. Now, let me just make a remark. I have said that we should be able to find a function $N : \mathbb{R}^+ \longrightarrow \mathbb{N}$.

(Refer Slide Time: 11:23)



The textbook definition states this slightly differently, it states, the preliminaries are the same, you have a sequence x_n we say that x_n converges to x_0 if given any $\varepsilon > 0$, we can find $N_\varepsilon \in \mathbb{N}$ such that if $n > N_\varepsilon$ then $|x_n - x_0| < \varepsilon$.

The standard textbook definition just states that given any ε we can find an N_ε such that if $n > N_\varepsilon$ then $|x_n - x_0| < \varepsilon$. The standard textbook definition of convergence and the definition we have given in terms of this function $N : \mathbb{R}^+ \longrightarrow \mathbb{N}$ is exactly the same, only difference is I have chosen to treat N_ε in the definition of convergence given in the textbook as a function of ε. The reason is that it clarifies that there is a relationship between ε and N_ε, in fact; you can construct a function that does the job for you.

Now, we will unravel the definition, this textbook definition in great detail in a future module. But for the time being convince yourself that the standard textbook definition and the functional notation we have given are exactly the same. The reason why I have given a functional notation is because in the standard definition the order of quantifiers is the key and its difficult to appreciate how the order of quantifiers can greatly influence the meaning of a statement.

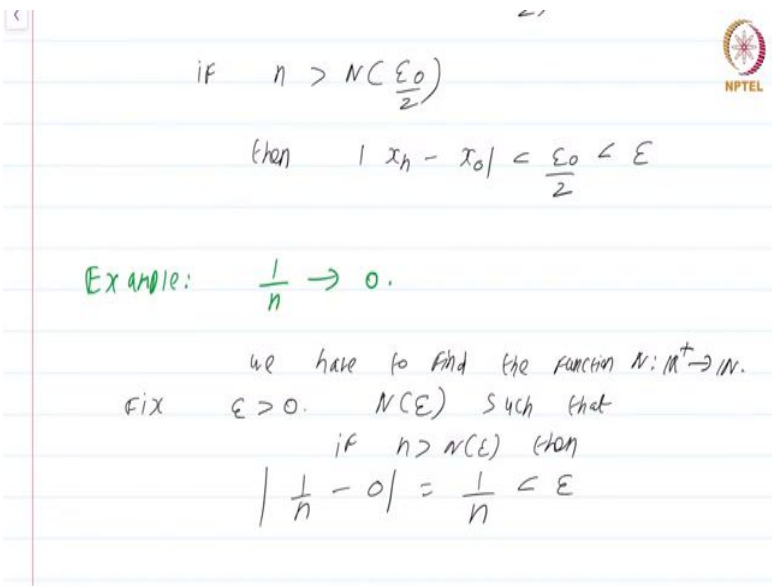
So, we will do a deep dive of this definition in a future module, till then familiarize yourself with both the definition in terms of a function $N:\mathbb{R}^+ \longrightarrow \mathbb{N}$ as well as a standard textbook definition. One more remark, in both definitions you need not do this for any $\varepsilon > 0$; you need not do this for any $\varepsilon > 0$.

If you could do it for $\varepsilon \in (0, \varepsilon_0)$ where $\varepsilon_0 > 0$, if you can satisfy, if you can find a function, in other words, if you can find a function from $N:(0,\varepsilon_0) \longrightarrow \mathbb{N}$ such that the definition of convergence is satisfied for all $\varepsilon \in (0, \varepsilon_0)$, open interval, then you are actually done.

You can make this function capital N actually from $N:\mathbb{R}^+ \longrightarrow \mathbb{N}$. I urge you to pause the video for a few moments and think about why it is sufficient to just treat ε in a given interval $(0, \varepsilon_0)$, you need not consider all possible values of ε. Just pause the video for a moment and think about it. I hope you got how to proceed from $N:(0,\varepsilon_0) \longrightarrow \mathbb{N}$ all the way from $N:\mathbb{R}^+ \longrightarrow \mathbb{N}$, well all you do is define $N(\varepsilon) = N(\frac{\varepsilon_0}{2})$ if $\varepsilon \geq \varepsilon_0$.

Just define this function, suppose you have satisfied the definition of convergence for all $\varepsilon \in (0, \varepsilon_0)$, then define the new function N, I am choosing to use the same name for convenience, $N:\mathbb{R}^+ \longrightarrow \mathbb{N}$ by $N(\varepsilon) = N(\frac{\varepsilon_0}{2})$ whenever $\varepsilon \geq \varepsilon_0$.

(Refer Slide Time: 15:23)



Why does this work? Well if $n > N(\frac{\varepsilon_0}{2})$, then $|x_n - x_0| < \frac{\varepsilon}{2}$, but if $\varepsilon \geq \varepsilon_0$ this is certainly less than ε as well ok. So, the definition of convergence is satisfied for this new function $N : \mathbb{R}^+ \longrightarrow \mathbb{N}$ ok.

So, we will do a deep dive soon. I just want to give one example in this particular lecture, we will study this example in some great detail in the next module. Let me just state it as a proposition because or rather example is better, let me just state it as an example. Example: $\frac{1}{n}$ converges to 0 ok.

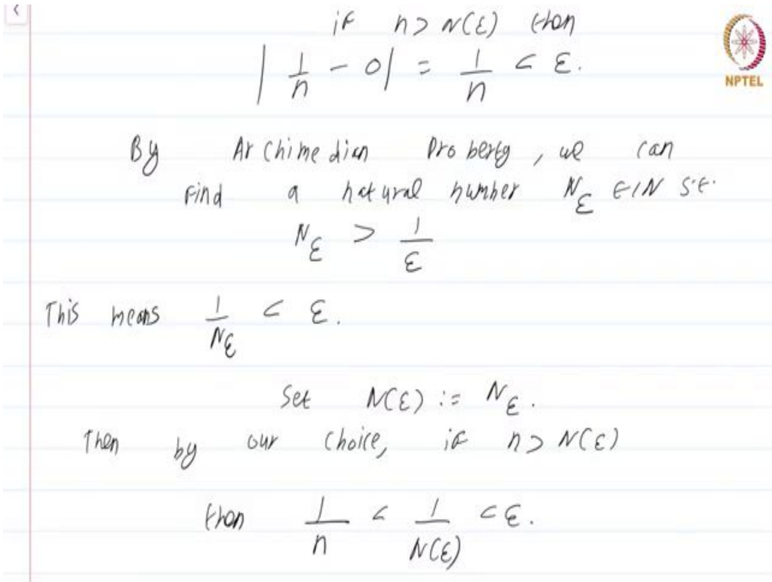
Now, this might seem obvious to you, but we have a formal definition and we want to satisfy that what is what should we do to show that $\frac{1}{n}$ converges to 0, let us display the definition again you are given the sequence x_n, now the sequence is $\frac{1}{n}$. We have to find x_0, but x_0 is also given to us, 0. We have to show that given any $\varepsilon \in \mathbb{R}^+$, we have $|x_n - x_0| < \varepsilon$ whenever $n > N(\varepsilon)$.

As you would have realized by now the heart of the matter is finding out what this N function is. So, this $N : \mathbb{R}^+ \longrightarrow \mathbb{N}$ is the key. How do we find this function? So, what do we have to do? We have to find. So, first we have to find the function $N : \mathbb{R}^+ \longrightarrow \mathbb{N}$, ok.

Now, fix $\varepsilon > 0$. Now, we have to find an appropriate $N(\varepsilon)$, note that for each choice of ε, it suffices if I find out one value for which the definition of convergence works, I can just take all these values together and construct the function N. So, what I am essentially doing is I am fixing an arbitrary $\varepsilon > 0$ and I am going to find an appropriate choice of the value $N > \varepsilon$, if I can do this for each ε, then I have a well-defined function.

So, fix $\varepsilon > 0$, we have to find an $N(\varepsilon)$ such that if $n > N(\varepsilon)$ then $\frac{1}{n} - 0 = \frac{1}{n} < \varepsilon$ ok. I have to find some value $N(\varepsilon)$ such that this is satisfied.

(Refer Slide Time: 18:52)



How do I do that? Well by Archimedean property we can find a natural number N_ε such that $\frac{1}{N_\varepsilon}$ or rather $N_\varepsilon > \frac{1}{\varepsilon}$, right. In other words, $\frac{1}{N_\varepsilon} < \varepsilon$ ok. Now, just set $N(\varepsilon) = N_\varepsilon$. Then by our choice by our choice, if small $n > N(\varepsilon)$, then $\frac{1}{n} < \frac{1}{N(\varepsilon)} < \varepsilon$, because $n > N(\varepsilon$.

(Refer Slide Time: 20:38)

So, then this choice of $N(\varepsilon)$ works in the definition of convergence, right. So, the net upshot is we have found a function $N(\varepsilon)$ that does the job for us and we have used the Archimedean property to do it. Observe that the way this proof is written you could have used either this definition using functions that I have given or the standard textbook definition. All I have done is, I have found for a fixed ε, I have found this $N(\varepsilon)$, all I am doing is I am just setting $N(\varepsilon)$ to be just this N_ε.

Whichever way you prefer, you can phrase the proofs, it really does not matter both definitions are identical and if you understand this particular proof, it should be clear to you why both definitions, the standard textbook definition and the definition in terms of the function $N : \mathbb{R}^+ \longrightarrow \mathbb{N}$ are exactly the same, they are just the same thing in a different disguise, ok.

So, this is a very simple example. After a deep dive of the definitions, of both definitions, we will proceed and give many more examples. This is a course on real analysis and you have just watched the module on definition of convergence.