Real Analysis - I
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Lecture – 6.3
Uncountability of the Real Numbers

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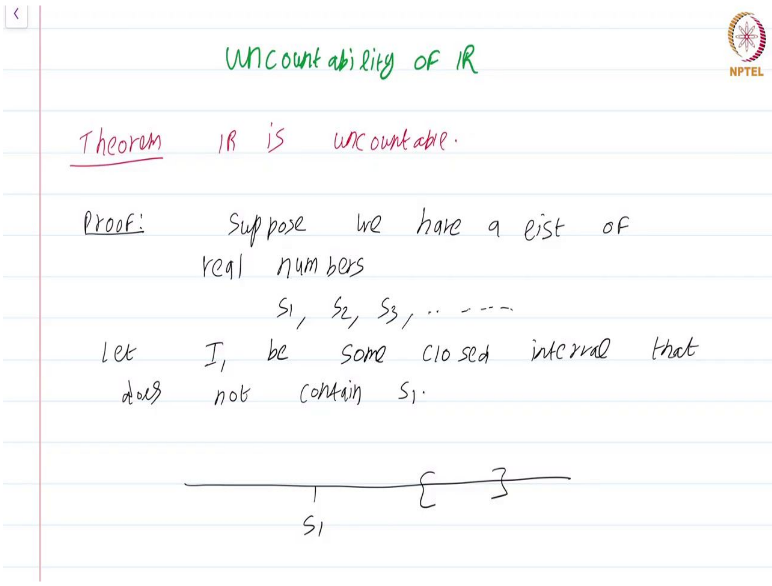

In this module, let us prove that the set $\mathbb{R}$ is uncountable. We will use the Nested intervals property again to prove it; the common proof given in most textbooks is via decimal expansion of Real Numbers. Since I have not made precise the decimal expansion of real numbers and doing so is quite tedious, I prefer to use this approach.

Theorem: $\mathbb{R}$ is uncountable, the statement is simple enough. Let us prove this. Proof; suppose we have a list of real numbers $s_1, s_2, s_3, ..$ so on, ok. Let I_1 be some closed interval that does not contain s_1. Well why does such an interval exist? Well that is obvious, say s_1 is here, just choose some interval like this, fine.

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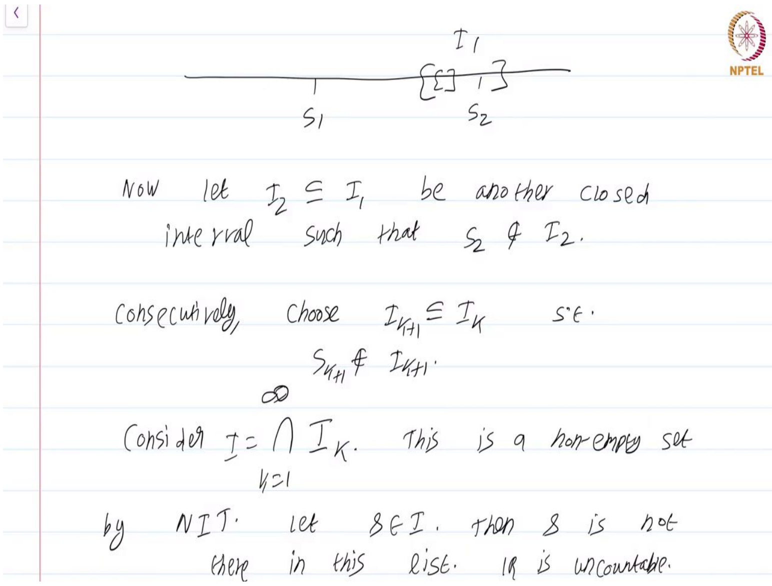

Now, this is I_1, let $I_2 \subseteq I_1$ be another closed interval; another closed interval such that $s_2 \notin I_2$. Why does such an interval exist? Well again just choose some say s_2 is here, just choose this interval, ok.

Now, consecutively choose I_k subset or rather $I_{k+1} \subseteq I_k$ such that $s_{k+1} \notin I_{k+1}$, this can always be done. Consider $I = \cap_{k=1}^{\infty} I_k$, k running from 1 to ∞; this is a non empty set, nonempty set by Nested Intervals Theorem, ok.

Let s be in this intersection which I will just call I; let $s \in I$, then s is not there in this list, ok. So, what we have done is, we have started out with a list, $s_1, s_2, s_3, ..$ of real numbers and I have shown that there is some element that is not there in this list, but is a real number.

So, this proves that there is no way to list out the real numbers and if you recall our discussions on cardinality; listing out the elements is another equivalent condition for being countable, so that means $\mathbb{R}$ is uncountable.

So, the set of real numbers is an explicit example of a set that is uncountable and this also shows that the number of real numbers that are there exceeds the number of rational numbers in some sense; because the rational numbers is a countable set. This is a course on real analysis and you just watched a module on the uncountability of $\mathbb{R}$.