Real Analysis - I
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Lecture - 3.2
Equivalence Relations

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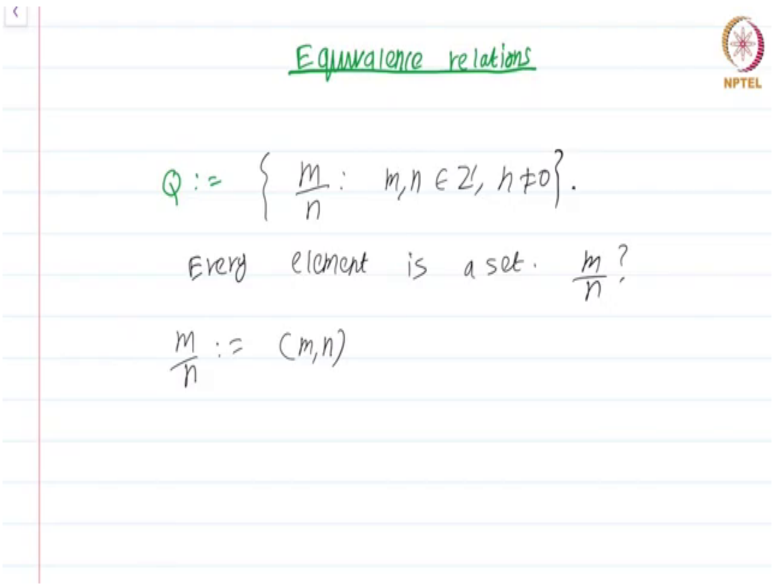

Recall that we had defined the set of rational numbers $Q := \{\frac{m}{n} : m, n \in \mathbb{Z},\ n \neq 0\}$. Now there is a problem with this definition. From our axioms of set theory we require that every element is itself a set, but it is not clear what the meaning of $\frac{m}{n}$ is ?.

What does this even mean? It is not really a set, it is just an a formal set of symbols m with a bar and then n at the bottom, it is not really clear what it is.

Now, one way to fix this is to view $\frac{m}{n} := (m, n)$; we can just introduce another notation that $\frac{m}{n}$ in actuality means just (m,n). In the module on relations, we have already seen that ordered pairs can be represented as sets; there is no problem here.

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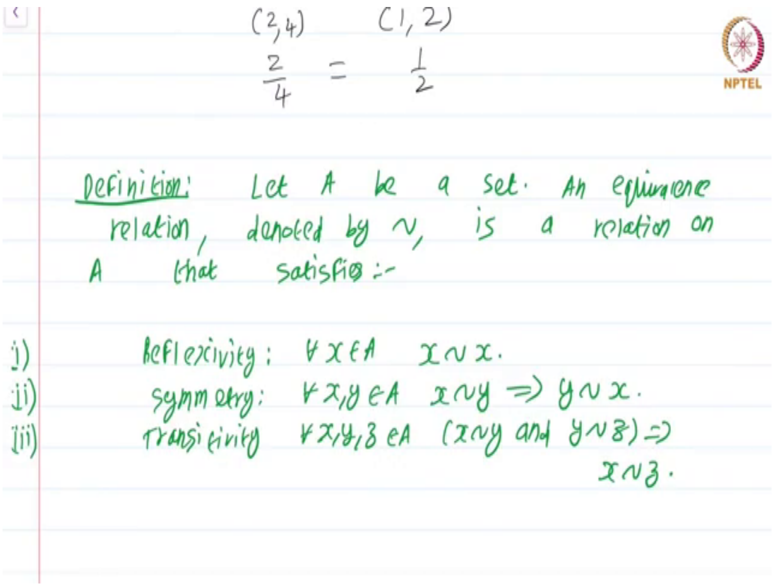

Now this new definition still has a problem. We can still view rational numbers like this; but the element which is (2,4) and the element which is (1,2) correspond to $\frac{2}{4}$ and $\frac{1}{2}$, which we all know from elementary mathematics they are both the same.

So, many elements in this collection of rational numbers are defined as ordered pairs (m,n) from integers, where $n \neq 0$, many elements themselves are equal. So, it is nice, if you could somehow redefine what equality means in a set and the notion of equivalence relation does this precisely; it is designed actually to make precise such things.

So, definition; let A be a set. An equivalence relation we will usually denote equivalence relation by this tilde; denoted by $\sim$ is a relation on A that satisfies, well, it should satisfy the following three properties;

1. Reflexivity, this says that every element is related to itself $\forall x \in A \; x \sim x$.
2. Symmetry, this says that $\forall x, y \in A, \; x \sim y \implies y \sim x$.
3. Transitivity; says that, $\forall x, y \in A, \; x \sim y, \; y \sim z \implies x \sim z$

   So, this is the notion of an equivalence relation; it is a special relation on a set that is reflexive, symmetric and transitive.

If you note carefully, what we have essentially done is we have taken the characteristic property of equality and formulated it as these three properties. Remember, equality is supposed to be a logical identity in our framework; an element is obviously logically identical to itself.

If a is identical to b, then b is identical to a; so symmetry is also something that is shared by equality and equivalence relations; same thing with transitivity, we have captured the three essential properties of equality and framed it as an equivalence relation.

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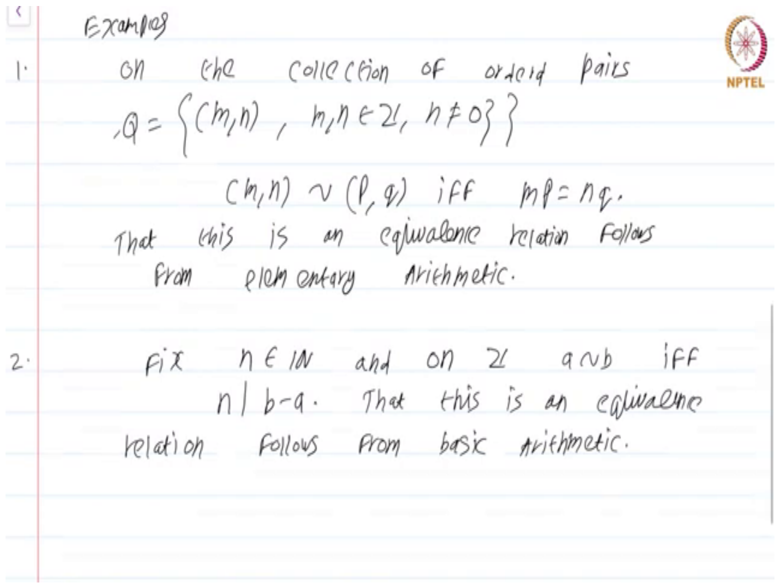

Let us see some examples. Well, the first one is the rational numbers, on the collection of ordered pairs, $\{(m,n) \in \mathbb{Z},\ n \neq 0\}$, this is what we call $\mathbb{Q}$. On this, define an equivalence relation that says that, $(m,n) \sim (p,q)\ iff\ mp = nq$.

Now, the fact that this is an equivalence relation, is just the elementary arithmetic, which I leave it to you; there is really nothing to check. This is an equivalence relation following from elementary arithmetic. So, the set $\mathbb{Q}$ of rational numbers is in fact not just the collection of ordered pairs; there is an additional equivalence relation involved, which we will make precise just in a moment when we come to partitions, ok.

Now, another example, fix $n \in \mathbb{N}$ and on the $\mathbb{Z}$, put an equivalence relation $a \sim b$ if and only if $n|b-a$. Again that this is an equivalence relation, follows from elementary arithmetic. So, this equivalence relation will give to something called integers modulo n.

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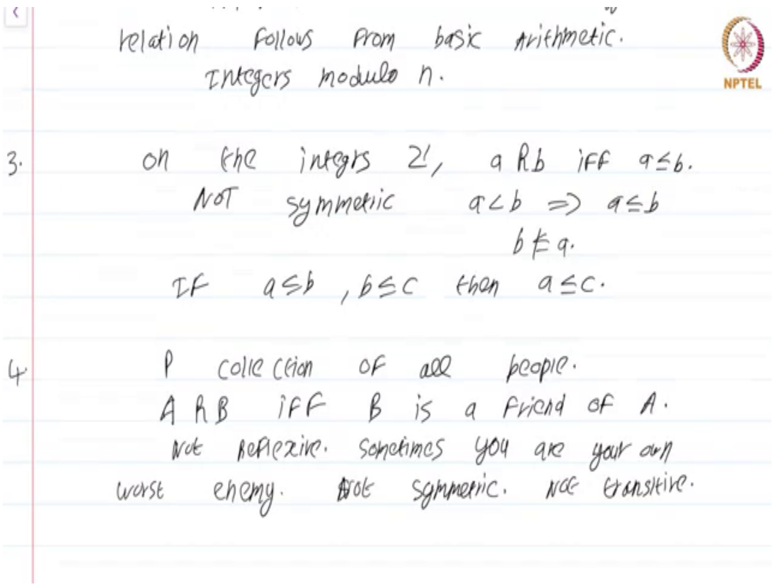

These are very important in cryptography and number theory, ok. So, let me give another example of equivalence relation. On the integers, put an equivalence relation; not an equivalence, put a relation, sorry about that, I will give a non example now. On the integers put a relation that $a\ R\ b$ if and only if $a \leq b$.

Now this is clearly reflexive, because $a \leq a$; but it is not symmetric, ok. If $a \leq b$; that means a is less than or equal to b also, then b cannot be less than or equal to a, this is not true.

So, this is not a symmetric relation, but it is transitive; if $a \leq b$ and $b \leq c$, then $a \leq c$, it is certainly transitive. So, we have given three examples, two of them are about positive examples and one is a counter example, of an example of a relation that is not having all the properties of an equivalence relation. Now let me give just one more example which is going to be a real world example.

Just for this example alone, I will violate our solemn oath; that we will only consider sets whose elements are themselves also sets, we will violate that and I will give you a real world example, ok. Now let P be the collection of all people. Define a relation that person $A\ R\ B$ if and only if$B\ is\ a\ friend\ of\ A$.

Now, we can make some interesting remarks about this real world relation; please do not take this example seriously, I just want to illustrate the mathematics with a bit of the cuff humor. This relation is actually not reflexive. Why is it not reflexive? Because we know that sometimes you are your own worst enemy, your own worst enemy. So, this is not reflexive.

Even more sad this is not symmetric, you think about it why, you can consider some other person as a really good friend, but that person may not share the same feelings, certainly this is not transitive, this is not transitive, ok. So, this is just a real world example; please do not take it seriously, it is just to illustrate the concept.

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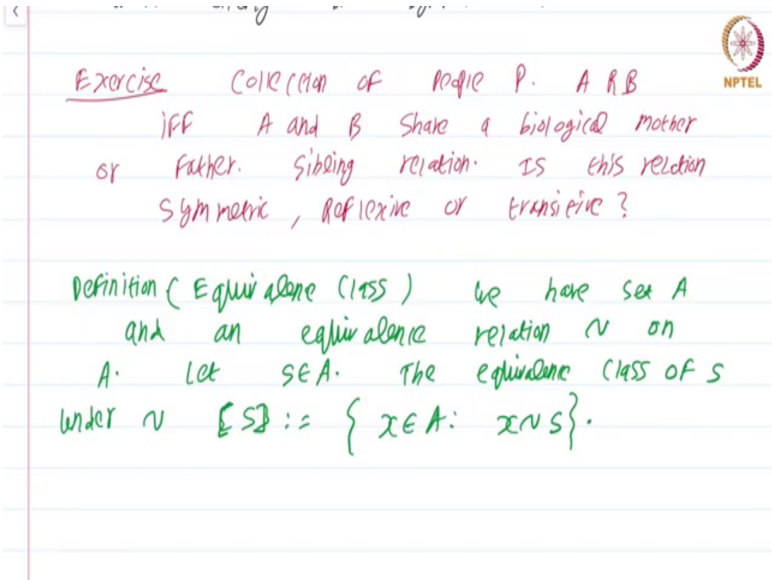

Let me give you an exercise for you to solve, exercise, Consider the collection of people P, and put a relation R where person $exercise, Considerthecollectionof people P, andputarelation Rwhereperson$ ercise, Consider the collection of people P, and put a relation R where person $ndputarelation Rwhereperson$ put a relation R where person $A\ R\ B$ if a only if A and B share a biological mother or father, ok. Now there is no necessity that you have to have the same father and mother, step siblings are also considered. So, this is the sibling relation.

Now check, which this relation is symmetric, reflexive or transitive, which properties does this relation have? Please solve this exercise, ok.

Now given an equivalence relation, I said that you can redefine equality on the set. Now I am going to make that precise by defining what is called an equivalence class.

So, we have set A and an equivalence relation $\sim$ on A, ok. Let $s \in A$, we define the equivalence class of s under $\sim$ is the set which is denoted by square bracket [s].

$[s] := \{x \in A : x \sim s\}$.

So, given an equivalence relation on a set; take an element and collect together all those elements that are related to this given element, that is called the equivalence class under this equivalence relation.

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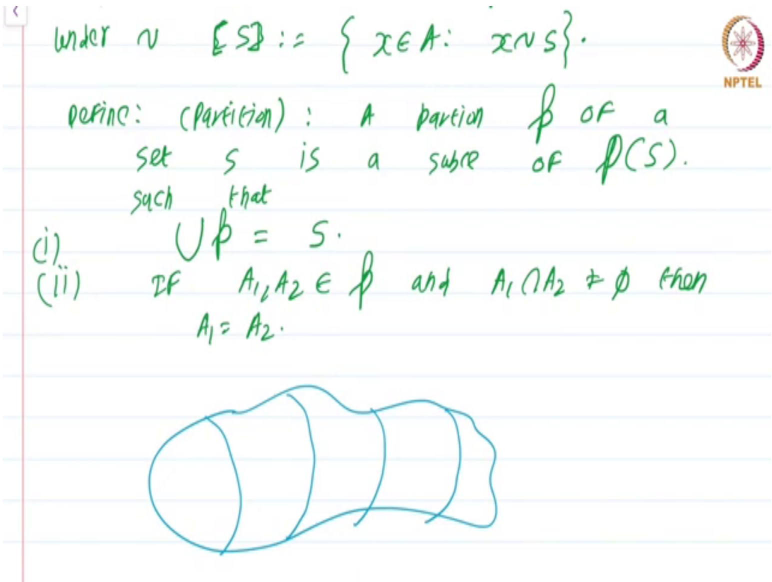

We also have a notion, an associated notion of a partition. So, let me just define the partition, a partition P of a set S is a subset of power set $P(S)$.

Recall the power set of S is the collection of all subsets of S such that, such that

(i). $\bigcup P = S$; remember each element P is itself a subset of the original set S. So, it makes sense to take the union of this collection, this is going to be the whole of S.

(ii). If A_1, A_2 is coming from P and $A_1 \cap A_2$ is non-empty, then $A_1 = A_2$.

So, a partition of a set is nothing but breaking up the set into a number of pieces such that the pieces together give you back the set and no two pieces intersect. So, it is a very intuitive thing, we can draw a picture for the partition; it suppose you have a set like this, some set, a partition is something that looks like this, each constituent part is supposed to be one of the elements of the partition.

So, now we have the notion of a partition. How is this related to the notion of an equivalence class? Well, we have the following proposition.

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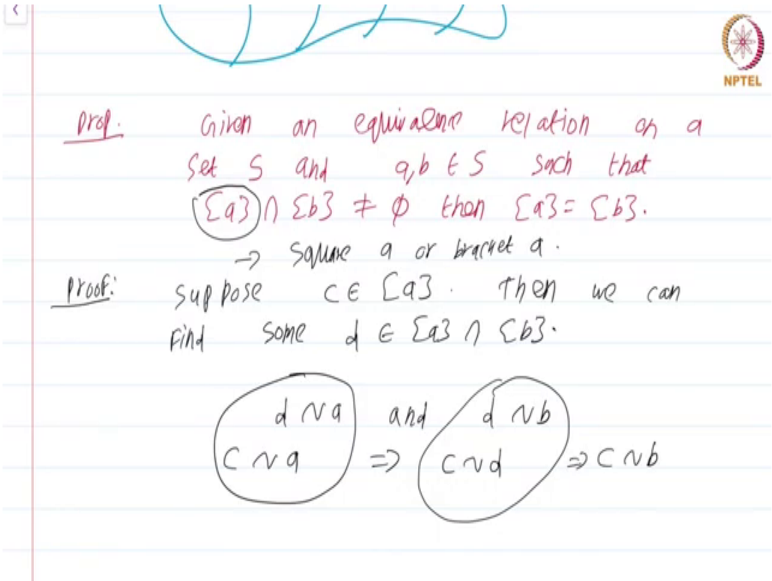

Proposition: Given an equivalence relation on a set S and $a, b \in S$ such that $[a] \cap [b] \neq \phi$, then $[a] = [b]$.

Note that this is the second property of a partition; if two subsets of P do not intersect, I mean intersect, then they have to be identical. So, let us see a proof of this, proof.

So, what we do is, we have to show every element of [a] is in [b], and every element of [b] is in [a].

So, suppose $c \in [a]$, then we can find, first of all we can find some $d \in [a] \cap [b]$. Now let us write down what is the meaning of$d \in [a] \cap [b]$.; it just means that $d \sim a$ and $d \sim b$, because the equivalence class is the collection of all elements that are related to the particular element b or a.

Now, c is in a just means that $c \sim a$. Now putting these two together with transitivity gives $c \sim d$. Now $d \sim b$ and $c \sim d$; means that $c \sim b$, again by transitivity.

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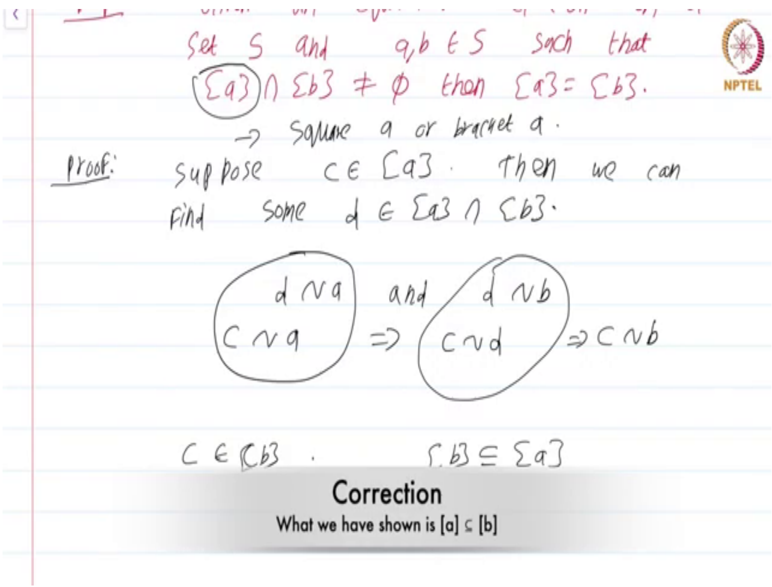

So, what does this show? This shows that $c \in [b]$. Now, the thing is we must still show. So, what is this show? Ultimately what we have shown is that, $[b] \subset [a]$. But we have used no property of the [a] and [b] other than the fact that they are disjoint.

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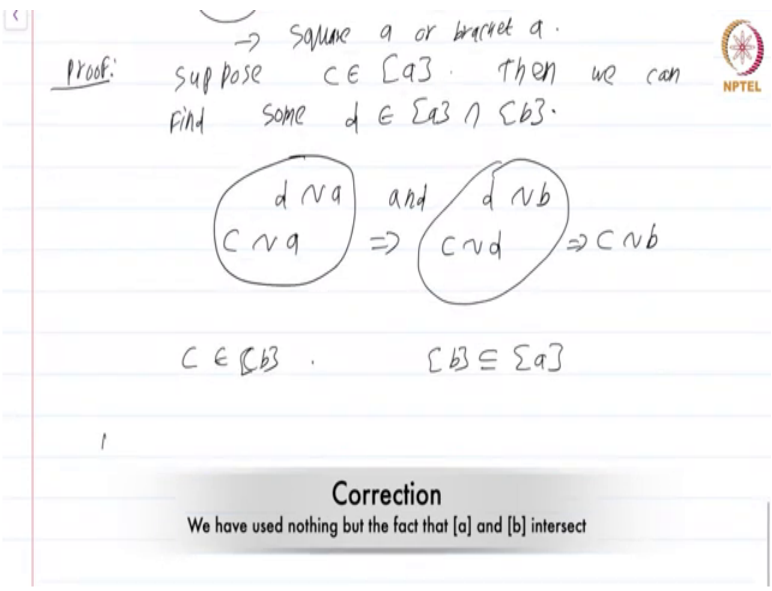

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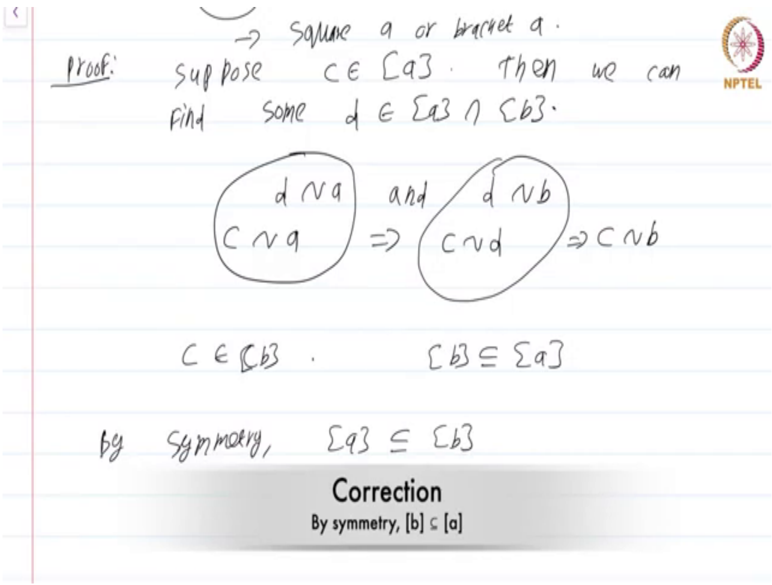

So, by symmetry $[a] \subset [b]$. Note when I say by symmetry here, I do not mean symmetry of the equivalence relation. What I mean is, the symmetry of the nature of our argument; there is nothing about a or b special when we showed that [b] is a subset of [a], the same argument works for [a] is a subset of [b]. [a] = [b].

So, this is promising; what has happened is, if you take two equivalence classes, either they are going to be completely disjoint or they are going to be the same.

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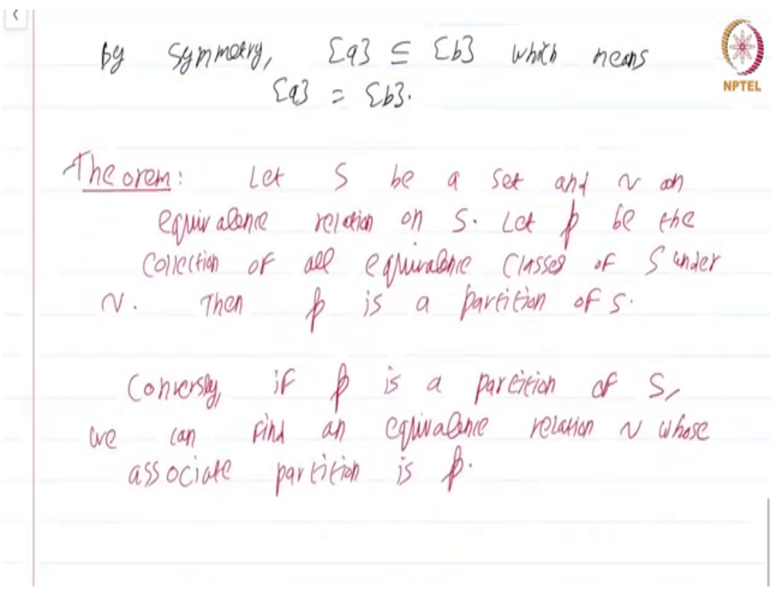

So, you should have anticipated the theorem that is about to come up, Theorem: Let S be a set and $\sim$ an equivalence relation on S. Let P be the collection of all equivalence classes of S under the relation $\sim$. Then P is a partition of S.

Conversely, if P is a partition of S, we can find an equivalence relation $\sim$ whose associated partition is P. So, what does this theorem say? It says that, if you start with a set that is equipped with an equivalence relation; you consider the collection of all equivalence classes that gives you a partition of the set.

Conversely you start with a partition of the set; we can define an equivalence relation in such a manner that, if you consider the partition generated by that equivalence collection, equivalence relation. In other words the collection of all equivalence classes under this equivalence relation, then you get back the original partition, ok.

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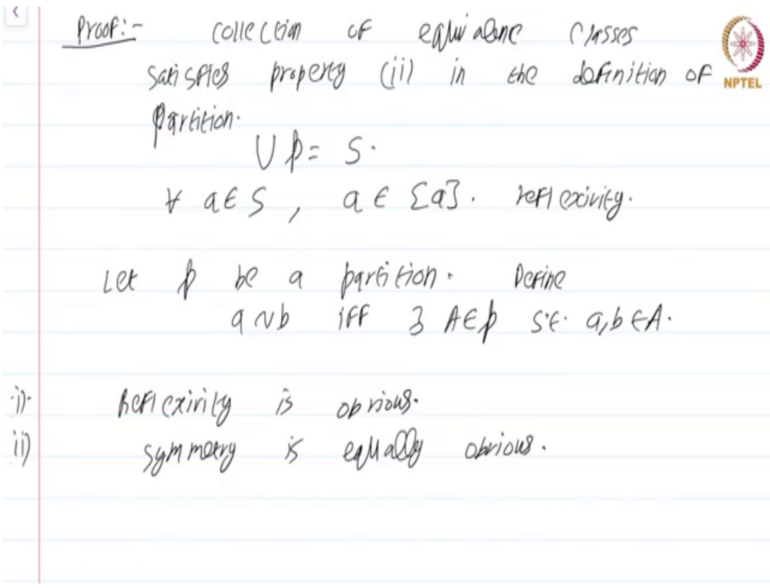

So, proof. So, we know that collection of equivalence classes satisfies property (ii) in the definition of equivalence relation of partition. That is we just showed that in the previous proposition that if two equivalence classes intersect then they must be identical, which is exactly what condition (ii) in the definition of partition states.

So, all we have to do is check condition (i); what do we have to check? We have to check that $\bigcup P = S$; but this is obvious, because $\forall a \in S$, $a \in [a]$, this just comes from reflexivity, this is just reflexivity, ok. So, both properties of partition are satisfied by the collection of equivalence classes.

Now, let us start with the partition P. We have to show the converse and construct an equivalence relation such that the partition of equivalence classes of that equivalence relation gives you back the original partition P. Let P be a partition. Define $a \sim b$ if and only if $\exists A \in P \ st. \ a, b \in A$.

So, what I do is, I look at this partition, it will consist of several sets which are subsets of the given set S; I put the relation that a is related to b if and only if there is some element in the partition, in other words some subset which is present in the partition such that both elements a and b belong to A.

Now, first, reflexivity is obvious. Why? Because $a \in A$. two, symmetry is equally obvious, symmetry is equally obvious; if $a, b \in A$, then $b, a \in A$.

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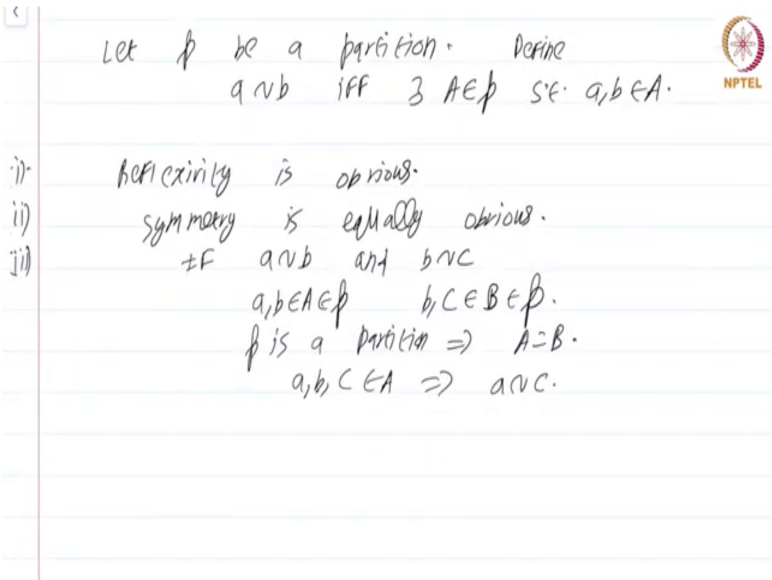

Third, transitivity is also equally easy, if $a \sim b$ and $b \sim c$; that means, $a, b \in A$, and $b, c \in B$ ok, both should be elements of P.

But A and B are not disjoint, they have the element b in common. So, because P is a partition, $\implies A = B$; that means, $a, b, c \in A$, which means that $a \sim c$.. So, the equivalence relation that you get from the partition is straightforward; you declared two elements to be related if and only if they belong to the same subset.

Now, for the last part I have to show now that the partition that is obtained from the equivalence classes under this relation is the same as the original partition; there is really nothing to prove. So, I just want you to pause the video and just think about why this is true. So, this concludes the module on equivalence relations.

Thank you.