Real Analysis - I
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Lecture - 3.1
Axioms of Set Theory

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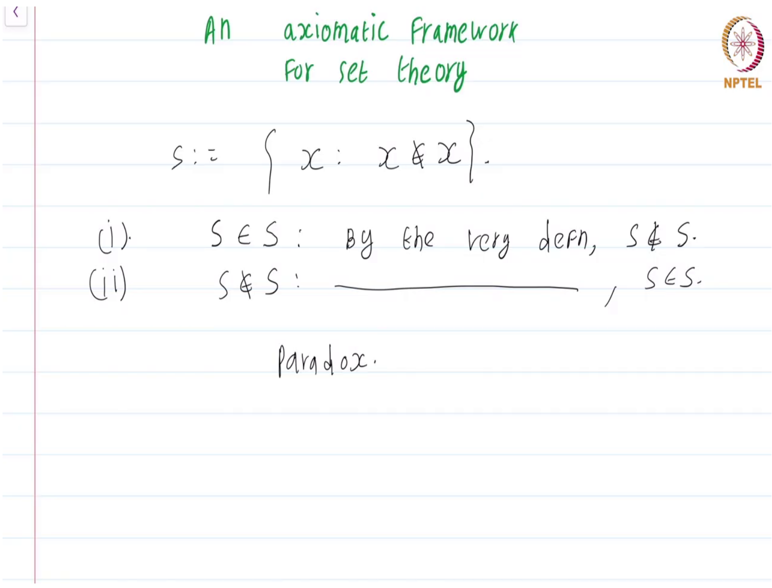

At last we come to the module where we present the Axioms for Set Theory. Before I present the axioms of set theory, I would like to first present a very very famous paradox by the double Nobel Laureate Bertrand Russell.

Consider the set $S := \{x : x \notin S\}$.

Just because I have written down a set does not mean that it is indeed a set. Let us see what goes wrong with S. There are only two possibilities.

First possibility is that $S \in S$. This cannot happen simply because by the very definition of S. S would not be an element of S.So, this is ruled out. Thus other possibility is $S \notin S$. Well, then again by the very definition of S, $S \in S$, we end up with a neat little paradox.

So, a paradox is a situation in which both a statement and its negation have some sort of argument supporting them. This paradox shows that if we are not careful about how we define sets, it could lead up to major issues.

There is an amusing version of this paradox. There is a barber who shaves all those and only those who do not shave themselves. The question is does the barber shave himself? Answering this question results in a contradiction, the barber cannot shave himself as he only shaves those who do not shave themselves. Thus if he shaves himself, then he must not shave himself.

On the other hand if the barber does not shave himself, then he fits into the group of people who would be shaved by the barber and thus he must shave himself. So, this is an amusing version of the paradox which is somewhat related. Now, the paradox given by Bertrand Russell led to the formulation of modern set theoretic axioms.

This was pioneered by Ernst Zermelo in the early 20th century. A bit later Abraham Fraenkel and Thoralf Skolem independently propose some modifications to the axioms of Zermelo. Together with the controversial axiom of choice this axiom system today is known as ZFC.

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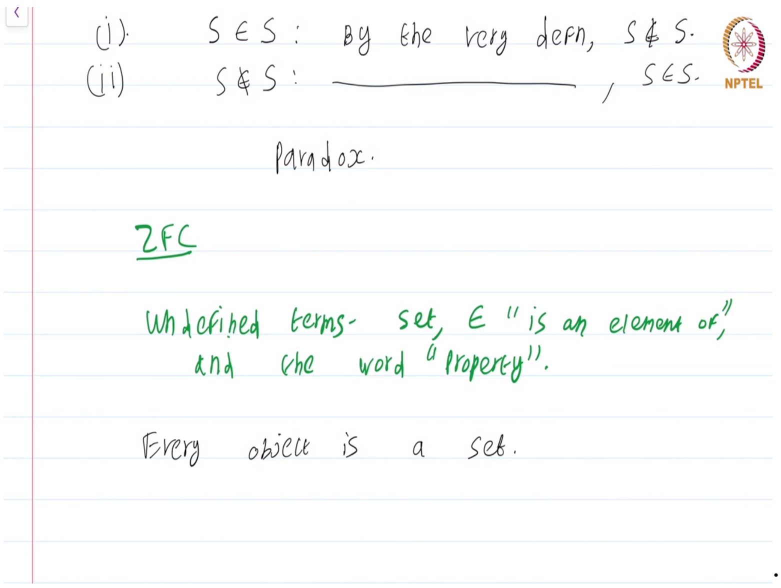

ZFC acts as a foundation for much of modern mathematics, virtually every set that you encounter is a set in ZFC. There are other formulations of set theory and there are other foundations for mathematics as well, but ZFC is what is used by a vast majority of mathematicians. We shall be very brief in our discussion, we will refer to the text of Halmos for more details.

In this course, we will not be too worried about set theoretic issues. We will rather concentrate only on naive set theory which is what we have seen so far in this course, but it is culturally good to know the axioms of modern set theory just for cultural purposes. So, the undefined terms in this axiomatic framework are the terms set, the relation is an element of and property and the word property. Now the word property is an undefined term only in this course. Remember I just told that Ernst Zermelo's axioms were modified by Thoralf Skolem and Abraham Frankel. Their work is primarily to make what this property means more precise using mathematical logic.

So, in a course that deals more with mathematical logic, you will see a precise definition of what a property is. With that preliminary set let us begin with our axiomatic framework. The first remark I want to make is that everything in this framework is a set, every object is a set. This means that whatever construction which we do functions, relations even more complicated objects like manifolds, which you will see in a future course no doubt, must all boil down to be sets.

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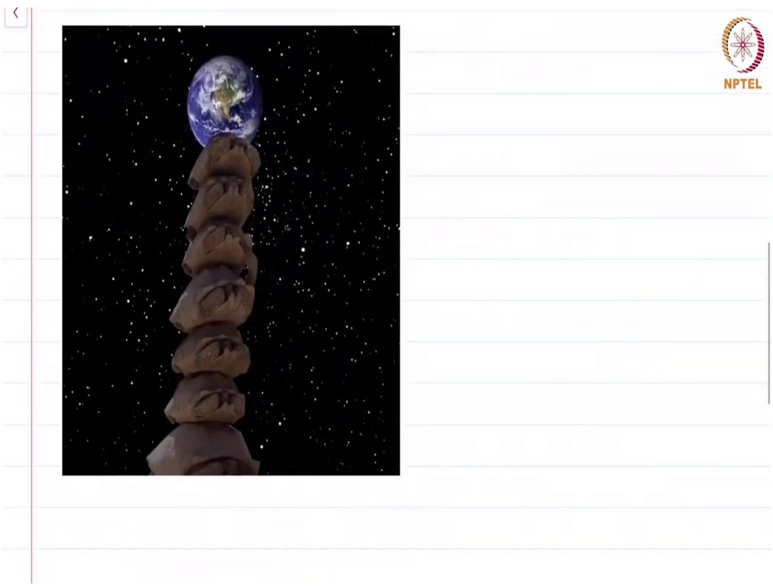

In this regard we have this famous picture of a mythological tale that says that the earth is supported by a turtle and the question arises what is this turtle supported on? Well obviously it is a bigger turtle and then so on. This is a classical illustration of what is known as infinite regress, but this issue of infinite regress won't happen in set theory which we will see in a moment.

So, we have talked about the undefined terms in this framework. Please remember that they are the term set, is an element of, and property. Let us list down the axioms. So, the axioms of set theory.

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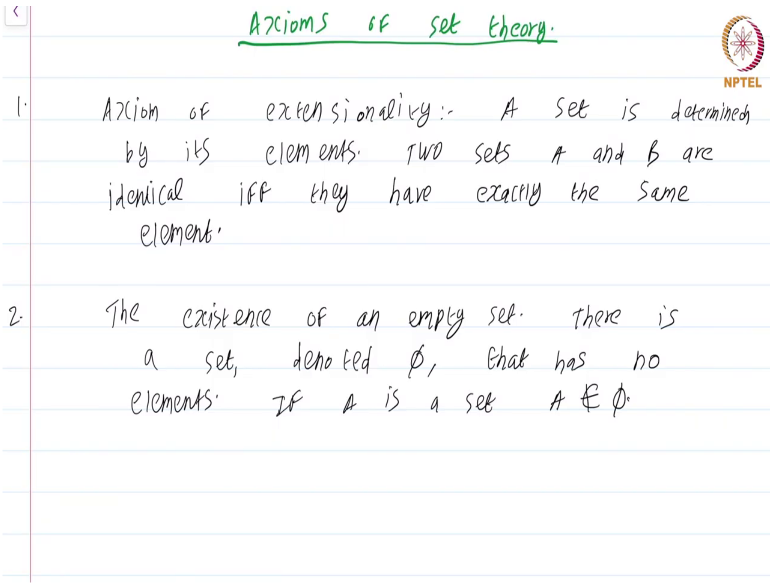

Now, different formulations involve different axioms. What I follow is more or less standard. The first axiom is the axiom of extensionality; this says a set is determined by its elements. In other words, two sets A and B are identical, recall in this course identical always means logically identical, are identical if and only if they have exactly the same elements.

Any framework for a mathematical study should tell you when two objects in that study are logically identical and this axiom plays that role. A simple corollary of this axiom is the fact that to show that two sets A and B are equal all one has to do is to show that $A \subset B$ and $B \subset A$.

The second axiom which is related to the remark I make about infinite regress not being an issue in set theory. This is the existence of an empty set. This says there is a set denoted ϕ that has no elements. In other words, if A is a set; $A \notin \phi$ irrespective of what this A is.

Note that I do not use the definite article the, I say there is a set that it is not a unique set. Well that is not an issue. I leave it as an exercise to prove that there cannot be two different empty sets. This will follow from the axiom of extensionality.

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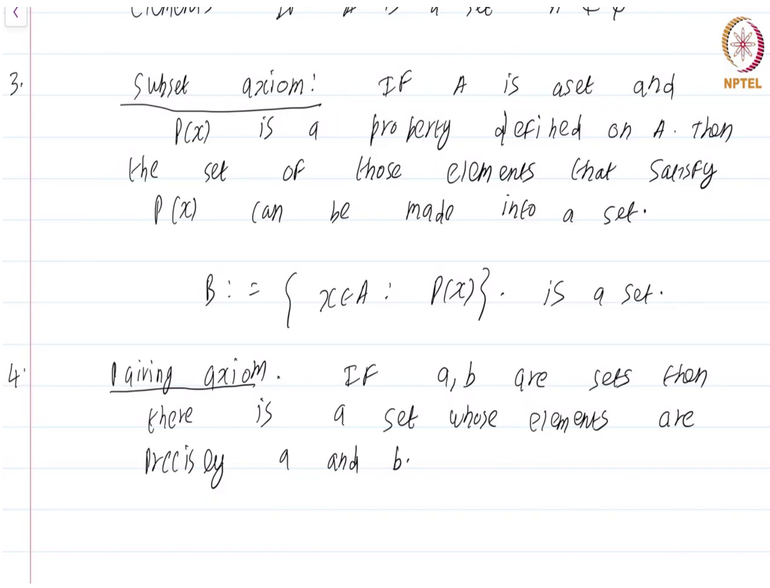

Three, subset axiom, this axiom is the most common way to construct subsets of a given set. If A is a set and $P(x)$ is a property over A or better terminology will be defined on A, then the set of those elements that satisfy $P(x)$ can be made into a set.

In brief, if you define $B = \{x \in A : P(x)\}$, is a set, B is a set. So, most sets that we define, we are essentially using a subset axiom and the undefined term property to define them. For instance the definition of odd numbers, even numbers, prime numbers that we have already seen.

The next axiom is vitally important especially for this course because this course at the end of the day is about functions. This is the pairing axiom; this is the pairing axiom, this says the following. If a, b are sets, then there is a set of elements rather than saying of elements whose elements is a set whose elements are precisely a and b. In other words $\{a, b\}$ is a set.

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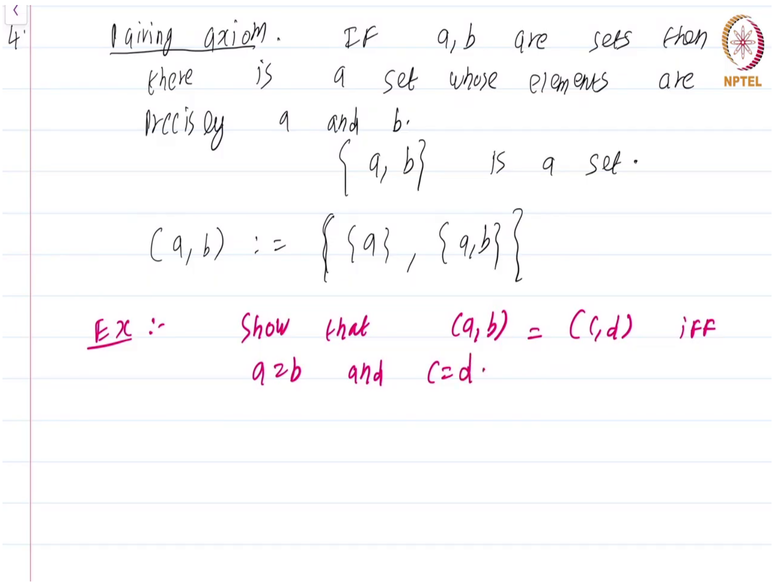

Now, why is this so useful? Well, you can use it to construct an ordered pair; you can construct an ordered pair, this construction is due to Kuratowski. You consider the ordered pair $(a, b) := \{\{a\}, \{a, b\}\}$. So, it is a set of two elements. The first element is a, I really should not say first element, one of the elements is just $\{\}$ and the other element is the $\{a, b\}$.

Now, why does this define the ordered pair? Well, it is an exercise.

Show that $(a, b) = (c, d)$ if and only if $a = c$ and andnd nd

Recall from the module on functions and relations that we need ordered pairs to define a relation in turn to define a function.

So, this is in some sense one of the most important axioms because this allows us to define functions. The 5th axiom is fairly straightforward, it is the union axiom. It will be stated in a really weird way because everything for us is a set. It says the following.

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5. Union axiom :- Given a set $\mathcal{F}$, there is a set U whose elements are precisely those that belong to at least one element of $\mathcal{F}$.

$$A \cup B = \{x : \underbrace{x \in A \text{ or } x \in B}\}.$$

$$\bigcup \{A, B\}$$

6. Power set axiom: Given a set X, there is a set $\wp(X)$ whose elements are precisely the subsets of X.

Given a set given a set F there is a set capital U whose elements are precisely those that belong to at least one element of F. Recall from our naive discussion of unions and intersections that I had defined the union of two sets

$$A \cup B = \{x : x \in A \ or \ x \in B\}.$$

But this definition has a problem simply because the property that I am using to define the union, it is not clear what set it is defined on. Property should always be defined on a set and it is not clear which set this is defined on.

This is not really a problem because the union axiom just says that whatever set you take, you can consider another set whose elements are precisely the elements of the elements of F, ok. So, that is a bit of a word full. So, let me just write that, you can construct the set $A \cup B$, all you have to do is consider the set A, B which is a set by the pairing axiom and then just take the union.

So, it is a bit of a convoluted way of just defining the union of two sets, but this axiom is far more general than just the union of two sets. You can take a union of any number of sets.

6 - The power set axiom. Well as the name suggests this axiom is fairly straightforward. Given a set X, there is a set; the power set of X whose elements are precisely the subsets of X, ok. So, why is this axiom quite interesting? The subset axiom says that given a set and a property defined on the set, you can extract the subset from that property but it is not clear whether you can extract every single subset this way. The power set axiom says that there is

a set whose elements are precisely the subsets of X. So, in particular every subset of X is in fact a set.

The next axiom is very subtle, it is the axiom of infinity; axiom of infinity.

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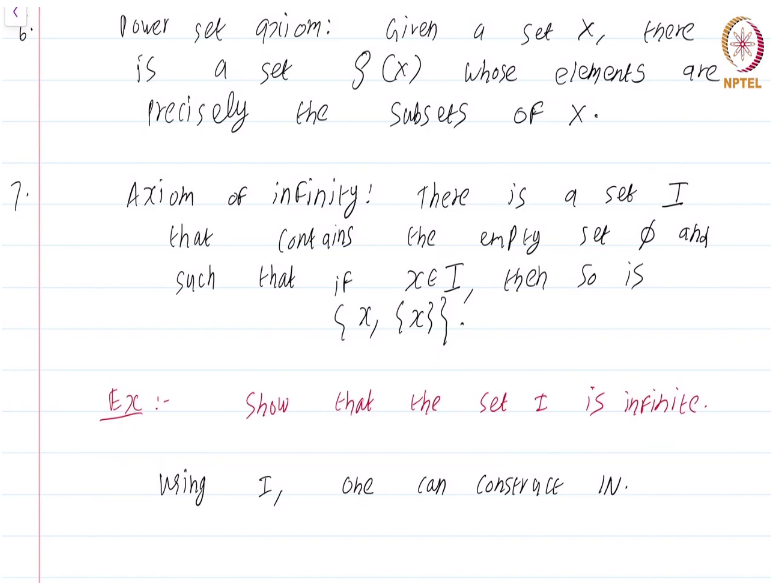

Now, it is a good idea now to pause the video. Look through the first six axioms to conclude that there is no way to produce an infinite set just using these 6 axioms. If you consider only these 6 axioms, the only sets that will exist are finite sets.

The axiom of infinity says there is a set I that contains the empty set ϕ and such that if $x \in I$ then so is $\{x, \{x\}\}$.

Now, why does this produce an infinite set? Well, again I leave it as an exercise.

Show that the set I is infinite.

If you are not familiar with finite and infinite sets, just a few modules down the line we will see cardinality. Solve this exercise after I defined rigorously what an infinite set is.

So, this axiom guarantees that there are infinite sets. In fact, using I one can construct the natural numbers $\mathbb{N}$. Once you construct the natural numbers $\mathbb{N}$, you can construct all the sets needed in most of mathematics. As the famous quote of Kronecker goes, God created the natural numbers, everything else is the work of man ok.

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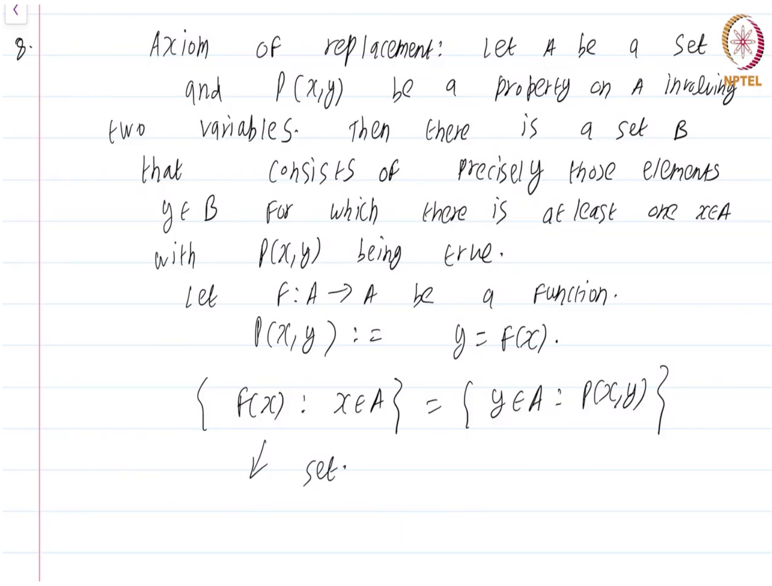

Now, we come to the final few axioms, the Axiom of replacement. As we go deeper and deeper into this, the axioms become more and more complicated and less and less relevant to this course. This axiom I am not even sure whether I will ever use this axiom in this course, but many many sets I will define and I will be tacitly using them. I will not be using it with my knowledge.

So, let A be a set and $P(x, y)$ be a property on A involving two variables, then there is a set B that consists of precisely those elements $y \in A$ for which there is at least one $x \in A$ with $P(x, y)$ being true.

That is indeed a long sentence and indeed a very very complicated axiom. This axiom is primarily used to construct many sorts of infinite sets. Paul Halmos has interpreted this axiom as saying anything intelligent one can do with a set yields a set.

Now, let me explain why this is called the axiom of replacement and in what way we will use it in our course. Of course, never explicitly mentioning, never even being conscious, always tacitly. Let $F : A \longrightarrow A$ be a function , then I defined the property

$$P(x, y) := \quad y = f(x).$$

This axiom asserts that $\{f(x) : x \in A\} = \{y \in A : P(x, y)\}$, this is a set.

So essentially what I have done is, I have replaced the variable x by its image variable y. So, in that sense this is called the Axiom of replacement. This also says that any function that I can apply to a set will still give me a set. That means the image of all functions are sets. So, I think now it should be clear what Halmos's quote is trying to say.

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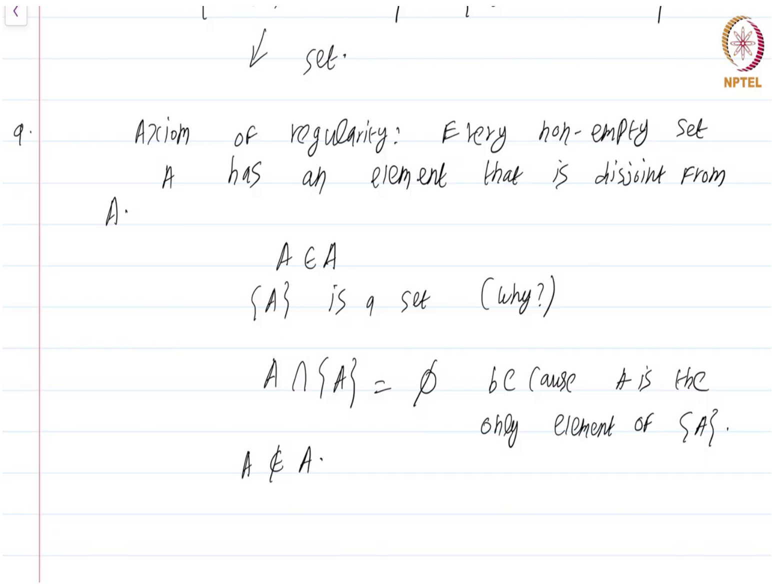

Now, the final two axioms, the axiom of regularity. This is there solely to rule out pathologies like what we saw with Russell's paradox. This says every non-empty set A has an element that is disjoint from A.

It would be a good idea to pause the video, get some pencil and paper and figure out what this axiom is trying to say.

This axiom rules out a pathological scenario in which we can have $A \in A$, a set being an element of itself. In fact, you can prove this from the axioms that we have developed. The pairing axiom says $\{A\}$ is a set. Why? Please check why this is true.

Now, what is the axiom of regularity? It says that $A \cap \{A\} = \phi$.

Why is this the case? Because A is the only element of $\{A\}$. From the fact that $A \cap \{A\} = \phi$; we can conclude that $A \notin A$. So, please pause the video and go through this proof again and make sure you understand what exactly is happening. It may seem like we are just manipulating symbols and that is true, but that is what rigorous checking of axioms, rigorous proofs from axioms actually entails.

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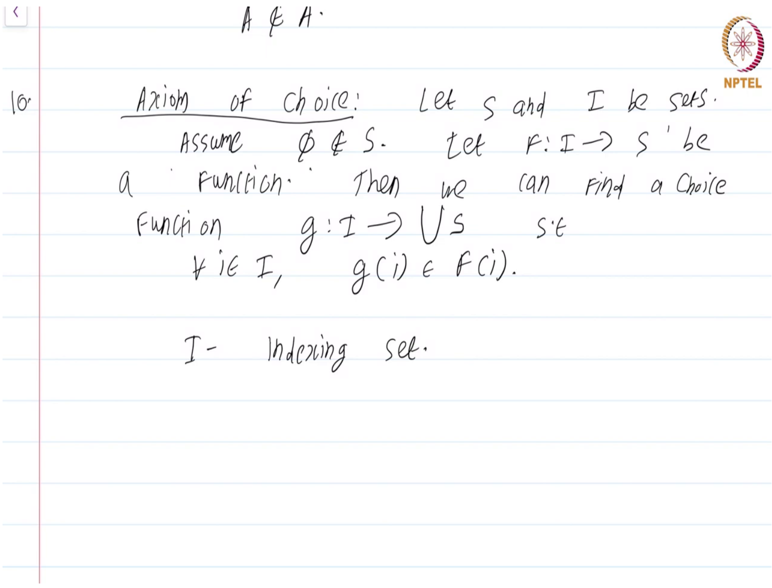

So, this axiom of regularity rules out the paradoxes of Bertrand Russell and finally, 10th. The C in ZFC is the axiom of choice. This is yet another axiom that I don't even know whether I am going to use. I might use it tacitly without my own knowledge. Let S and I be sets. Assume that $\phi \notin S$.

Then, we can find a choice function g from .oh I missed one hypothesis. Sorry about that. Let $F : I \longrightarrow S$ be a function, then we can find a choice function $g : I \longrightarrow S$ rather, not from I to S, from I to

Again a very very complicated axiom. Let me rephrase what this is saying in plain English. This I as you may have guessed is an indexing set. Essentially what we have done is, we have indexed some sets in the set S ok. Then what this axiom is saying is that you can find another set which is precisely one element from each of these index sets.

That is what essentially the function $g : I \longrightarrow \bigcup S$ says. It is an indexing of the elements of S. So, this axiom of choice is used in topology extensively. We will only briefly see topology as I say again I do not really know whether we will use this axiom, but I am sure I will use it without my own knowledge.

Let me make some more remarks about this, if the collection S has only finitely many elements, then this axiom is actually not needed. You can produce this choice function by hand; you can use mathematical induction to prove this. There is only one element of S, then saying that you can choose an element of that set is essentially saying that the set is non-empty. It is equivalent to saying that the set is non-empty. You have to exhibit an element to show that the set is non-empty. So, there is no arbitrary choice involved.

Not only that in many other cases also it will be easy to produce this choice function. For instance look at the set S. Let it be the power set of natural numbers. Later in this chapter on cardinality, you will see that this set is going to be a really vast set. There is no need to invoke the axiom of choice to choose one element from each of these sets.

Why is that? Well for each one of the subsets of the natural numbers, there is always a least element. So, there is a way to extract the element without using the axiom of choice. So, this concludes our presentation of the axiomatic framework of set theory. As a remark again, we will not explicitly cite any of these axioms in this course. In fact, this module is sort of semi-optional. This is just there for cultural purposes, but let me end by making one somewhat disassuring remark.

These axioms we have written down, we have used our intuition and a lot of work on what constitutes the idea of a set. It is clear that the paradox of Bertrand Russell will go away from these axioms, but it is not clear that there is some other paradox. How you prove that a particular set of axioms never lead to a contradiction is a very difficult topic in mathematical logic.

It is so far unknown whether these axioms are all consistent. The best result that I am aware of, note I am not a set theorist. The best I am aware of is it is known that if the first nine axioms do not lead to a contradiction, then the addition of the axiom of choice will also not lead to a contradiction, but something strange happens even if you add the negation of the axiom of choice, there will still be no contradiction. These are path breaking results of the mathematicians Cohen and Kurt Godel.

So, there are a lot of deep issues involved, since this is the first course on analysis, this itself is a bit too much. So, if this particular module is a bit difficult for you, do not worry this is just there for anticipating the future. If you are really, your interest is piqued by this module, please read the book of Halmos. It is an excellent book.

This is the course on real analysis and you have just watched the module on the axiomatic framework of set theory.

Thank you.