Real Analysis - I
Dr. Jaikrishnan J
Department of Mathematics
Indian Institute of Technology, Palakkad

Lecture - 2.5
Functions and Relations

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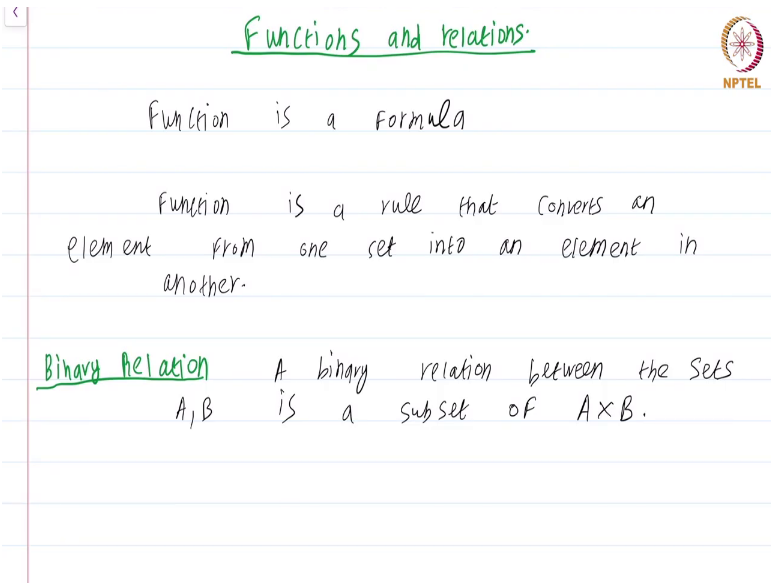

The idea of a function is ubiquitous in mathematics. It is somewhat surprising that this notion has had a modern form only about 100 to 150 years ago. The naive idea is that a function is a formula. This idea though it works admirably in most scenarios is inadequate. The best way to think of a function is that a function is a rule that converts an element from one set into an element in another.

So, the best way to think of the function is as a rule. So, one dynamic way to think of it is a function has an input and it transforms the input into the output. The question is how do you make this mathematically precise? How do you make this idea precise using set theory specifically?

To do that first, let us take a more general concept of a relation . This more precisely the notion of a binary relation. A binary relation, between the sets A, B is a subset of A cross B.

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$R \subseteq A \times B := \{(a,b) : a \in A,\ b \in B\}$.

$(a,b) = (c,d)$ iff $a=b$, $c=d$.

Function :- A function $F: A \to B$ is a binary relation R such that if (a,b), $(a,c) \in R$, then $b=c$. In addition, we require that for each $a \in A$, there is atleast one $b \in B$ satisfying $(a,b) \in R$.

Now, I have not told you what A cross B is. A cross B is by definition, the collection of all ordered pairs a comma b such that a comes from the set A and b comes from the set B. Now, the question arises what is an ordered pair? Well, you will have to wait till the module on axiomatic set theory to see the definition of an ordered pair; but let me just say that the ordered pair is characterized by the following property that (a,b) = (c,d) if and only if a = b and c = d.

A set if you recall has no ordering within it, there is no notion of first element or a second element; whereas, ordered pairs have the notion of a first coordinate and a second coordinate. Two ordered pairs are equal only if their corresponding first and second coordinates are also equal. So, the binary relation between the sets A and B is just a subset of A cross B.

What is a function? Well, a function F from A to B is a binary relation R such that if (a,b). So, let us call this binary relation R. So, here also I will write R subset of A cross B. Binary relation, remember is also going to be a set. It is a subset of A cross B. So, a function F to be, F from A to B is a binary relation R such that if (a,b) and (a,c) are both in R, then b = c.

In addition, we require that for each a in A, there is at least one b in B satisfying a comma b is an element of R. So, a function is a special type of relation between the sets A and B. The characteristic property of a function is that every element of A is related to some element of B and it every element of A is related to exactly one element of B.

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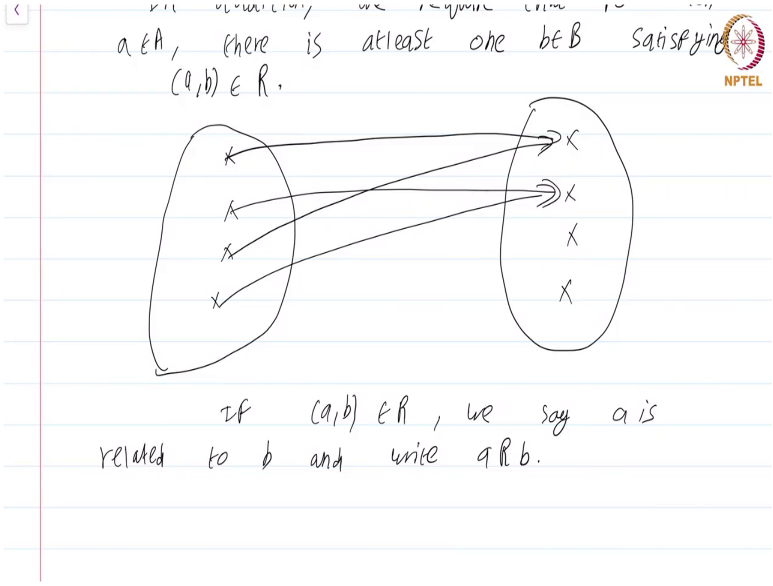

So, let us illustrate this by a very bad picture. Here you have the set A, here you have the set B. What this says is each element of A is connected to exactly one element of B; but it can happen that multiple elements of A are connected to the same element of B, that is very much possible. All that is required is for each element in A, there is exactly one element in B that corresponds to it or is related to it .

So, implicit in my discussion above was this talk about an element being related to another element. So, what this means is if a comma b is in the relation R, we say a is related to b, related to b and write a R b instead. So, we are using sort of infix notation instead of ordered pair notation to say that a is related to b is same as saying (a,b) is an element of R.

A function would have a another special property that if a comma b is there in the relation and a comma c is also there in the relation, then b has got to be equal to c. So, the function is written F from a to b. So, we will adopt a different notation for functions.

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or $(a,b) \in R$, we say a is related to b and write aRb.

We will write $f(a) = b$ to indicate aRb or $(a,b) \in R$ when R is a function.

A function is usually described by a formula.

$f: \mathbb{R} \to \mathbb{R}$, $f(x) = x$.

identity fn.

We will write F of a equal to b to indicate a R b or a comma b is an element of R, when R is a function. So, when a function, when we have a relation that is a function, we throw away the relation notation and simply write F of a equal to b. The idea is that F acts on a and somehow transforms it into the element b.

So, how do you specify functions? There are several ways. The most common ways of course, a formula. A function is usually described by a formula. For example, you can say F from R to R and the function is F of x equal to x. This is called the identity function. The transformation does nothing. It takes the input and just leaves it as it is without touching it, that is one of the most common functions. In fact, you can define this identity function for any set.

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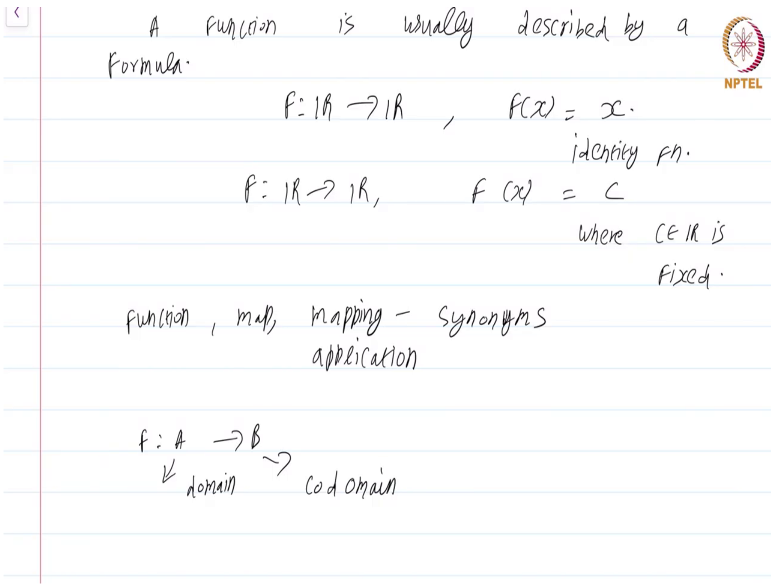

Second type of function is F from R to R, you can just define F of x equal to some C; where, C in R is fixed . You map every element of the real numbers to this constant C. We will of course, use the adjective mapping the terminology related to functions is not fixed. We often say function acts on A, function maps A, function takes A to the corresponding element B. So, you must be familiar with all this common terminology. For the purpose of this course, function map and mapping are synonyms.

I will use all of these three terms interchangeably. In French, functions are also called application. The reason being that is that you apply the function to the element a to get the element b. So, it sort of applies on elements of the set a ok. So, some more terminology regarding functions when you have a function F from A to B, the set A is called the domain, the set B is called the codomain.

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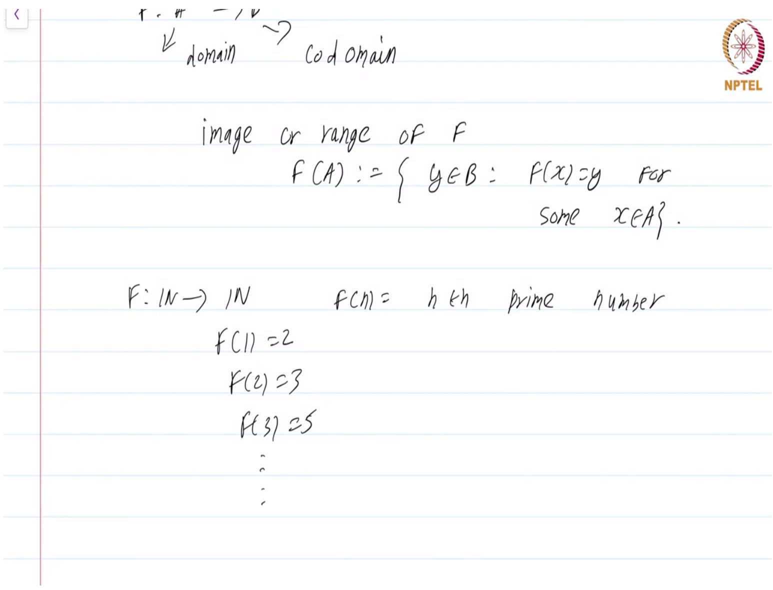

There is a third set, this is called the image or range of F. This is the set F of A which is defined to be the collection of all y in B such that F of x equal to y for some x in A.

Those elements of B that are mapped by F are to said to be in the image or range of F. Note range and co-domain are not the same. The co-domain of a function is the set on which the relation is defined the set B, in this definition A cross B; whereas, the range is a subset of the co-domain .

What are the other ways of describing functions other than just using a formula? Well, you can use things like this. You can use English language. For instance, you can define a function F from N to N, that says F of n is the nth prime number ok. This merely says that F of 1 is 2, F of 2 is 3, F of 3 is 5 and so on .

Note that, there is no formula for finding the nth prime number. In fact, if you can find a nice formula for the nth prime number, you will probably get a fields medal. A fields medal is the highest honour that a mathematician can get. There is no formula for finding out the nth prime number.

There are of course, algorithms for doing it. An algorithm is different from a formula. An algorithm is a recipe or a set of steps for figuring out what the nth prime number is. You can write an algorithm for finding the nth prime number, but that does not give you a formula. So, this is also a very well good function that remember in the beginning of this module, I said that the notion of functions as formulas is inadequate.

Here is an example of a very good function to consider which would not be a function, if you demand that functions have to be formulas .

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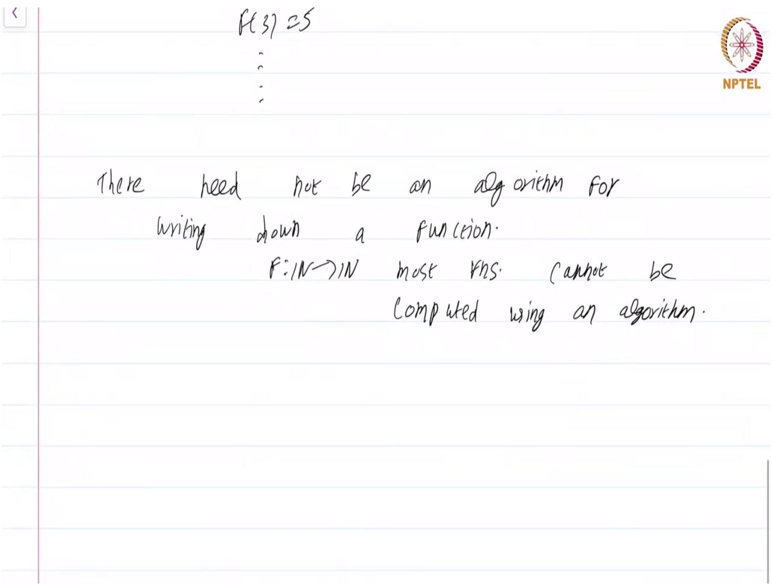

Functions need not be algorithmic also. They are need not be an algorithm for writing down a function. There is no necessity that the function, when you describe the function, you need to give an algorithm. There is no necessity for that. An example of that is quite involved.

I just content to say that there are many functions. In fact, most functions cannot be described by algorithms. Functions from natural numbers are natural numbers itself, not just functions from R to R. If you take F from N to N. Most functions most functions do not or rather let me write cannot be computed using an algorithm. Let alone a formula using an algorithm.

So, the definition of a function, it should be well defined. What that well defined is exactly is immaterial. Now, for this purposes of this course, every function that we consider will obviously, be well defined . What other ways are there? Well, there is another simple way, you can just describe functions by a table suppose you have a finite set.

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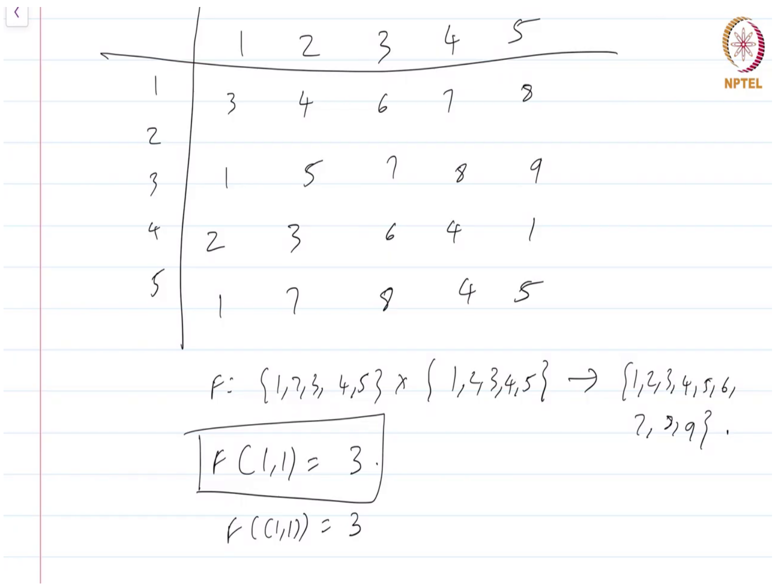

Let us say just 1 2 3 4 5 and you want to write down a function from 1 2 3 4 5 to 1 2 3 4 5, you can just write down in sort of in a tabular form. So, this sort of table will actually ok, since I started, let me define the function here its actually more general function than what I was thinking of. Let us write 3 4 6 7 8, 1 5 7 8 9, 2 3 6 4 1, 1 7 8 4 5. Now, what is this table saying? It seems a bit convoluted. Well, this is not originally I was going for a function from 1 2 3 4 5 to 1 2 3 4 5, but what I have written down actually is a bit more general.

I have constructed a function from 1 2 3 4 5 cross 1 2 3 4 5 from the Cartesian product to 1 2 3 4 5 6 7 8 9 . To find out what F(1,1) is look at the first row, first column and see what it is? So, it is clear that F(1,1) is actually 3. You notice there is an abuse of notation happening.

An abuse of notation is when you take a notation and do not use the full correct; I mean you do not use the notation correctly for the sake of brevity. Here technically, I should be writing F of 1 comma 1 equal to 3 right that is the way functional notation was defined.

You are writing F(1) = 2, F (2) = 3. Here the element on which the function is defined itself is an ordered pair. So, you must write F(1,1) in within brackets is 3, but that notation quickly becomes very tedious. So, we do not use such precise notation all the time and such things are called abuse of notation.

So, you can define functions using tables. What other ways of defining functions are there? We have already seen defining function using formula, defining function using a description, defining functions using an algorithm, defining functions using a table, there is also one other way of defining a function, this is a sort of mix of both.

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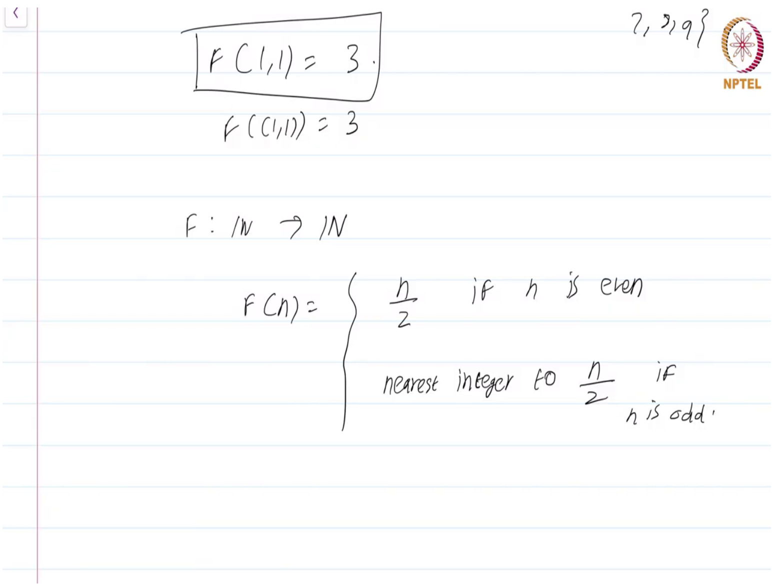

You can suppose you want to define a function like in the following way. You want to define function from N to N that divides by 2; but there is a problem because 3 divided by 2 is no longer a natural number. So, what you do is the following. You define it like this, you define F of n and you write this flower brackets big flower brackets and write n by 2, if n is even ok, then you put nearest integer to n by 2 if n is odd .

Now, this is an English description, but its ambiguous because whenever you divide whenever you divide an odd number by 2; let us say 3 by 2, you get 1.5 which is the nearest integer 1 or 2, both are equally near. So, we have to make a choice here is an example of a function that is not well defined. Given a element in the domain, it is not clear where that point is going to where that element is getting mapped to. So, we can improve this description and just write n plus 1 by 2.

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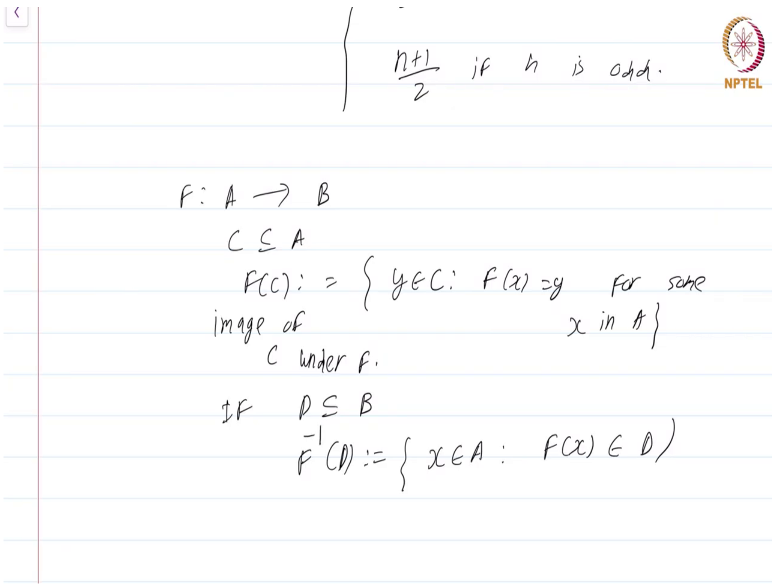

If n is odd, this is a proper function. In fact, it rounds up 1.5 to 2. So, 3 will get mapped to 2. So, this is yet another way of defining functions in parts. So, you will probably encounter a mixture of all four in this course; sometimes by table, sometimes by formula, sometimes as a parts, one subset you define using one formula, another subset you define using another formula, using an algorithm. You should be familiar with all of these. So, this is the basic terminology about functions and relations.

I am assuming that you are familiar with all this before. So, I am going fast. One more thing, I have not talked much about relations in a later module on equivalence relations. I will talk about the most important category of relations there. So, one more slide concept, we need to do in functions that is I had defined what the image or the range is here. I said image or range of F is F of A which is defined to be y in b such that F of x equal to y, so that you can do for any subset.

Suppose, you have a function F from A to B and you have a subset C of A, you can define F of C to be the collection of all elements y in B such that F of x equal to y for some x in A. This is called the image of C under F. Similarly, we have something called the pre image, if D is subset of B, then F inverse D is defined to be the collection of all x in A such that F of x is an element of D. This is written F with a minus 1 on top. Let me write that clearly F with the minus 1 on top.

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Pre-image of D

or inverse image of D.

A function $f: A \to B$ is said to be injective if $f(x) = f(y) \Rightarrow x = y$.

one-to-one $\forall x, y \in A$.

$f: \mathbb{N} \to \mathbb{N}$ $\quad f(n) = n^2$.

$f: \mathbb{Z} \to \mathbb{N}$ $\quad f(n) = n^2$.

$n \mapsto n^2$ $\quad$ n maps to or goes to n^2.

$n \rightsquigarrow n^2$.

And its read pre image of D or inverse image of D . Now, this notation is a bit confusing because this inverse notation is related to the notion of a inverse function. What is the inverse function? Well, I need to define two notions.

A function F from A to B is said to be injective, if F of x equals F of y implies x equal to y. So, for all x y in A. ,That means, if you take two elements of the set A and if it happens that F of x equal to F of y, that means, x and y are the same.

This is just a fancy way of saying that every element of A gets mapped to a unique element of B and two different elements of A cannot get mapped to the same element of B. The first part comes from the definition of a function, the second part comes from the definition of injective. So, injective functions are those for which each point in A has a unique element in B and this element is not shared with any other element of A. Injective functions are also called 1 to 1 ok.

So, if you consider the function F from N to N , F of n equals n square ok. I am defining the function by a formula; then this function is clearly injective. But the function F from Z to N F of n equals n square, the very same function squaring, but taken on the integers is no longer injective.

Sometimes to highlight that functions are maps that transform, Instead of writing F from N to N, F of n equal to n squared, we also use this notation n goes to n square. This is to be read as n maps to or goes to n square. This is yet another way of just writing down the function; some textbooks also use this n goes to n square. It will be self evident. What the meaning is. It is just function getting transformed . So, function is said to be injective if it has this property.

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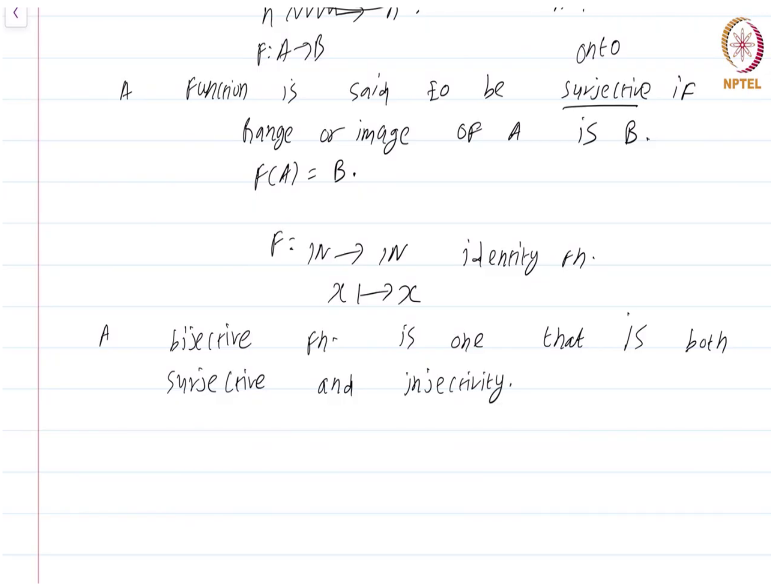

A function F from A to B is said to be surjective if range or image of A is B . A surjective function is one in which every image of B, every point in B is an image point. That means, there is some element in A that gets mapped to it. In short, F of A equal to B .

So, for an example of surjective function look no further than the function x goes to x, the identity function. This function is certainly going to be surjective ok. Now, a bijective function. One more thing surjective functions are also called onto, onto functions. A bijective function, as you could have guessed is one that is both surjective and injective that is both surjective and injective ok.

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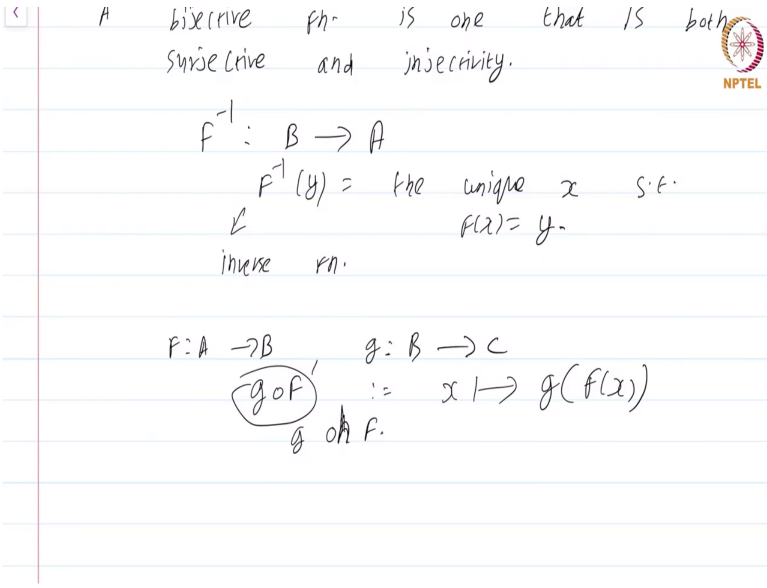

Once, you have such a function, then you can define an inverse function. Inverse function F inverse is a function from B to A and its defined like this. F inverse of y is the unique x such that F of x equal to y. Why is this x unique and why does such an x exist?

Well, the uniqueness comes from injectivity and the fact that there is such an x comes from surjectivity. Every element in the set B in particular, the element y will have at least one pre image by surjectivity and this pre image has to be unique by the fact that F is injective. So, this is called the inverse function; this is called the inverse function.

It is also an inverse in a different sense, not a different sense; in the exact sense that I am going to describe now. Given a function F from A to B and a function g from B to C, we can compose these functions. Written g circle F which we read as g of F; g o F ok, we read it as g o F. I mean I am just writing o as oh, I do not know how to spell o. So, we read it as g o F or g circle F or g composed with F. This is just the function it takes x to g of F of x. You first apply F and then, you apply g .

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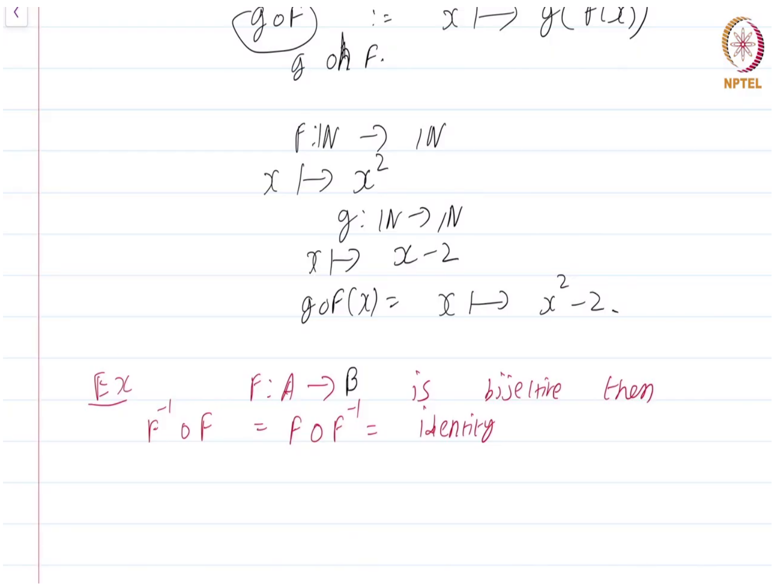

So, an example of this is suppose you have a function F from n to F that is given by x square and a function g from N to N given by x minus 2 . So, this is x goes to x square; this is x goes to x minus 2, then goF of x is nothing but the function x maps to x squared minus 2 . You first apply the function F and then, apply the function g .

Now, exercise for you if F from A to A is bijective, then F composed with F inverse is identity. Same is true of F inverse composed with F. In fact, there is no reason, you can always say F from a to B itself, except now you have to check you cannot write F inverse F equal to F o F inverse equal to identity, you will have to be a bit more careful.

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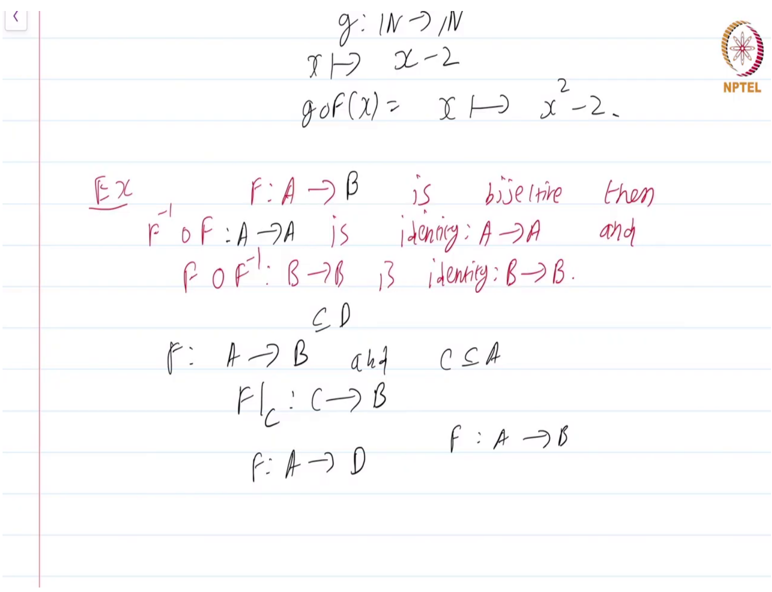

F inverse composed with F will be a function from A to A. This function is identity from A to A and F o F inverse is a function from B to B is identity from B to B . So, in this sense, it is an inverse. When you compose a function with its inverse, you get back the identity function except you will have to be careful which set it is the identity function on ok.

Now, given a function F from A to B, you can do one thing. C is a subset of A, then you can restrict the function F to the set C ok. This is the just the same function except the domain of the function is the smaller set C. You are using the same rule specified by F that same rule you are applying to a subset.

Similarly, if you have a function from F from A to B and B is subset of from some D, then you can also consider F as a function from A to D. You can also expand the range ok. So, technically a function F has three components F from A to B has three components which we can illustrate by a picture.

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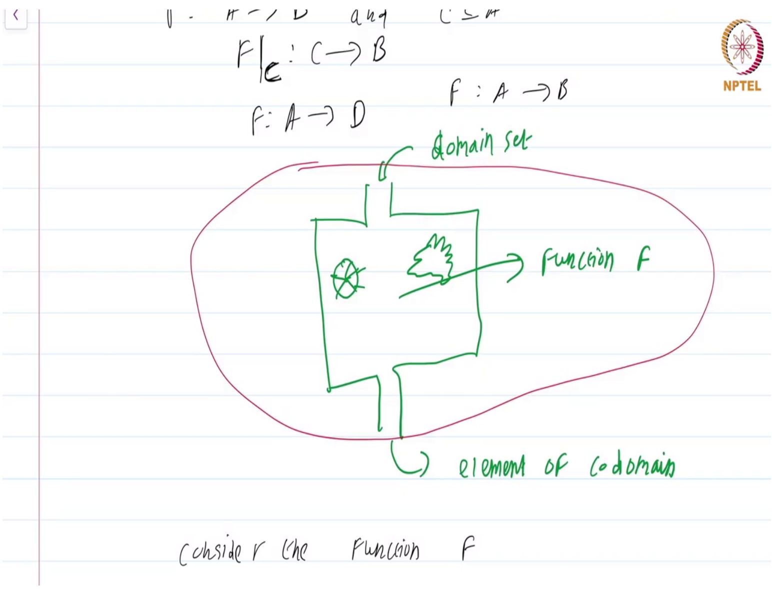

We have a function apparatus which is some function F which you can think of as some machinery what it exactly does nobody knows. Actually, you have to specify precisely what it does for it to be a machine. So, I am just drawing some parts that supposedly looks like gears something, there is something inside. You drop an elements of a set, this is the domain set and as output you take an element of the codomain .

A function technically has three parts it; there is this machine, there is this domain set and there is this co-domain set. You have to specify all three to specify a function fully. But again, abuse of notation more often than not, we will just read the function as this middle part which is this apparatus.

This apparatus is usually a formula or a description or an algorithm or table or something and from that description or table or whatever, it will be possible to guess what is the valid domain of this function and what is the codomain .

So, from this guess as in its not uniquely specified, but you can sort of find the domain to be the largest set on which the function F is defined, naturally defined and the codomain to be the largest set on which the points of the function map into . So, typically we abuse notation and just consider the middle part as the function.

So, instead of writing consider the function F from A to B, we will just write consider the function F defined by something without specifying what the domain is or the codomain is that is common in mathematics; but please keep in mind that you can write restrictions and this is the notation you put a vertical line and put the set on which you are restricting to.

There is no notation when you enlarge the codomain there is no shortcut notation for that. So, this is all we need for the time being about functions and relations. Again, I urge you to look back through your high school textbooks and have these concepts thorough. Sure, you have spent a lot of time on it already, just spend some more. So, this concludes the module on relations and functions. This is the course on Real Analysis and you have just watched the module on functions and relations.