Complex Analysis

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Lecture No – 37

Problem Session

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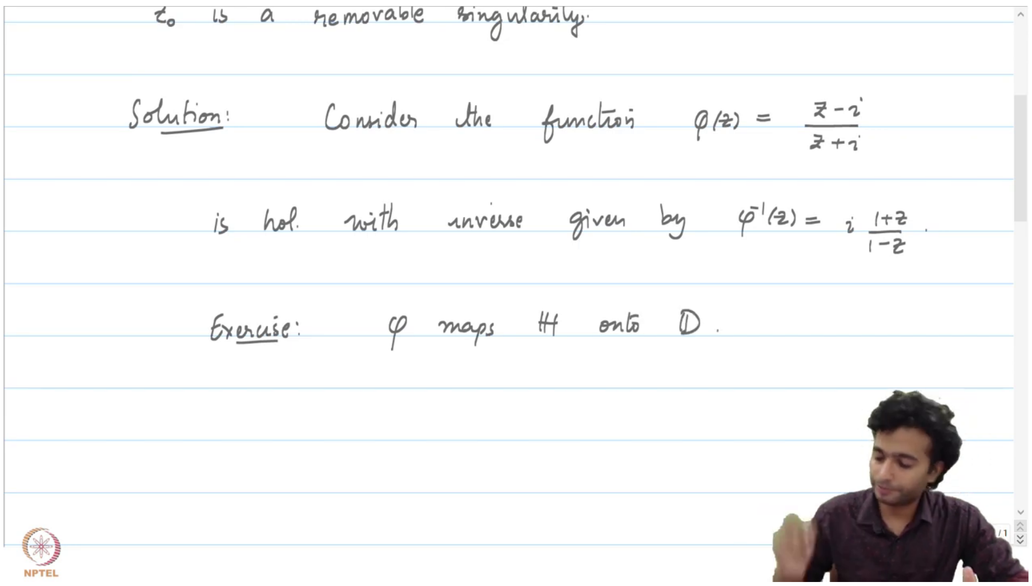

PROBLEM 1. Let $f : D(z_0, r)\backslash\{z_0\} \longrightarrow \mathbb{C}$ be a holomorphic function such that $\mathfrak{Im}(f(z)) > 0$ for every $z \in D(z_0, r) \setminus \{z_0\}$. Then prove that z_0 is a removable singularity.

SOLUTION 1. Consider the function $\phi(z) = \dfrac{z-i}{z+i}$. Then ϕ is holomorphic with inverse given by $\phi^{-1}(z) = i\dfrac{1+z}{1-z}$.

Now the reader should verify that ϕ maps the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \mathfrak{Im}(z) > 0\}$ onto the unit disc $\mathbb{D}$ centred at the origin. Since we are given that $\mathfrak{Im}(f(z)) > 0$ for every $z \in D(z_0, r) \setminus \{z_0\}$, we have, $f : D(z_0, r) \setminus \{z_0\} \longrightarrow \mathbb{H}$.

By composition with ϕ, we have $\phi \circ f : D(z_0, r) \setminus \{z_0\} \longrightarrow \mathbb{D}$. Hence $\phi \circ f$ is bounded and by Riemann removable singularity theorem, there exists a holomorphic function g

on $D(z_0, r)$ such that

$$\phi \circ f(z) = g(z) \qquad \text{on } D(z_0, r) \setminus \{z_0\}.$$

By open mapping theorem, we have $g : D(z_0, r) \longrightarrow \mathbb{D}$ and $|g(z_0)| < 1$. Since ϕ is invertible, we can compose g with ϕ^{-1} and we have

$$f(z) = \phi^{-1} \circ g(z) \qquad \text{on } D(z_0, r) \setminus \{z_0\}$$

and $\phi^{-1} \circ g$ is holomorphic on $D(z_0, r_)$. Therefore z_0 is a removable singularity of f.

PROBLEM 2. Let z_0 be a pole of a function $f : D(z_0, r) \setminus \{z_0\} \longrightarrow \mathbb{C}$. Then prove that there exists $R > 0$ such that

$$\{z \in \mathbb{C} : |z| > R\} \subseteq f\left(D(z_0, r) \setminus \{z_0\}\right).$$

PROOF. Let z_0 be a pole of order m. Let us assume that $f(z) \neq 0$ on $D(z_0, r) \setminus \{z_0\}$. Then there exists a holomorphic function $g : D(z_0, r) \longrightarrow \mathbb{C}$ with $g(z) \neq 0$ on $D(z_0, r)$ such that on $D(z_0, r) \setminus \{z_0\}$, we have $f(z) = \dfrac{g(z)}{(z - z_0)^m}$.

On $D(z_0, r) \setminus \{z_0\}$, let us define h_1 to be,

$$h_1(z) = \frac{1}{f(z)} = \frac{(z - z_0)^m}{g(z)}.$$

Then h_1 has a removable singularity at z_0 and that $h(z_0) = 0$ where h extends h_1 to $D(z_0, r)$. By the open mapping theorem, since h is not a constant function, there exists $\epsilon > 0$ such that $D(0, \epsilon) \subseteq h(D(z_0, r))$. If $w \in \mathbb{C}$ be such that $|w| > \frac{1}{\epsilon} = R$, then $\frac{1}{w} \in D(0, \epsilon)$ and since $w \neq 0$, there exists $z \in D(z_0, r) \setminus \{z_0\}$ such that

$$\frac{1}{w} = h(z) = h_1(z) = \frac{1}{f(z)}.$$

Hence $f(z) = w$ and $\{w \in \mathbb{C} : |w| > R\} \subseteq f(D(z_0, r) \setminus \{z_0\})$. □

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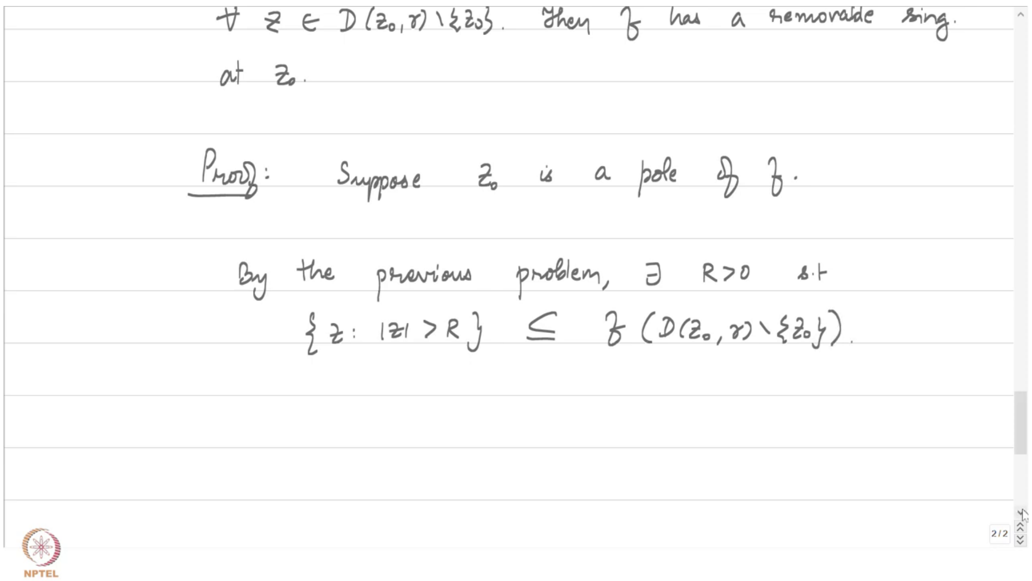

PROBLEM 3. Let $f : D(z_0, r) \setminus \{z_0\} \longrightarrow \mathbb{C}$ be a holomorphic function such that the real part of f, $\mathfrak{Re}\left(f(z)\right) < M$ for some real number M and for every $z \in D(z_0, r) \setminus \{z_0\}$. Then f has a removable singularity at z_0.

SOLUTION 2. Suppose z_0 is a pole of f. Then by Problem 2, there exists $R > 0$ such that

$$\{z \in \mathbb{C} : |z| > R\} \subseteq f\left(D(z_0, r) \setminus \{z_0\}\right). \tag{1}$$

Let $w \in \mathbb{C}$ such that $|w| > R$ and such that $\mathfrak{Re}(w) > M$. By (1), there exists $z \in D(z_0, r) \setminus \{z_0\}$ such that $f(z) = w$. Then $\mathfrak{Re}\left(f(z)\right) = \mathfrak{Re}(w) > M$ for $z \in D(z_0, r) \setminus \{z_0\}$, which is a contradiction. Hence z_0 cannot be a pole.

Suppose z_0 is an essential singularity. Then by Casorati-Weierstrass theorem, $f(D(z_0, r) \setminus \{z_0\})$ is dense in $\mathbb{C}$. Let $w \in \mathbb{C}$ be such that $\mathfrak{Re}(w) > M$ and let $\{z_n\}_{n \in \mathbb{N}}$ be a sequence in $D(z_0, r) \setminus \{z_0\}$ such that $z_n \longrightarrow z_0$ and $f(z_n) \longrightarrow w$. Then there exists $N \in \mathbb{N}$ such that for every $n > N$, we have $\mathfrak{Re}\left(f(z_n)\right) > M$, which is again a contradiction. Hence z_0 cannot be an essential singularity.

PROBLEM 4. Let z_0 be an isolated singularity of f. Then e^f does not have a pole at z_0.

SOLUTION 3. We will consider the all three cases of z_0 being an isolated singularity of f.

Case-I: Let z_0 be a removable singularity of f. That is, there exists a holomorphic function $g : D(z_0, r) \longrightarrow \mathbb{C}$ such that $f(z) = g(z)$ on $D(z_0, r) \setminus \{z_0\}$. Hence, we have $e^{f(z)} = e^{g(z)}$ for each $z \in D(z_0, r) \setminus \{z_0\}$. Since e^g is a holomorphic function on $D(z_0, r)$, e^f has a removable singularity at z_0.

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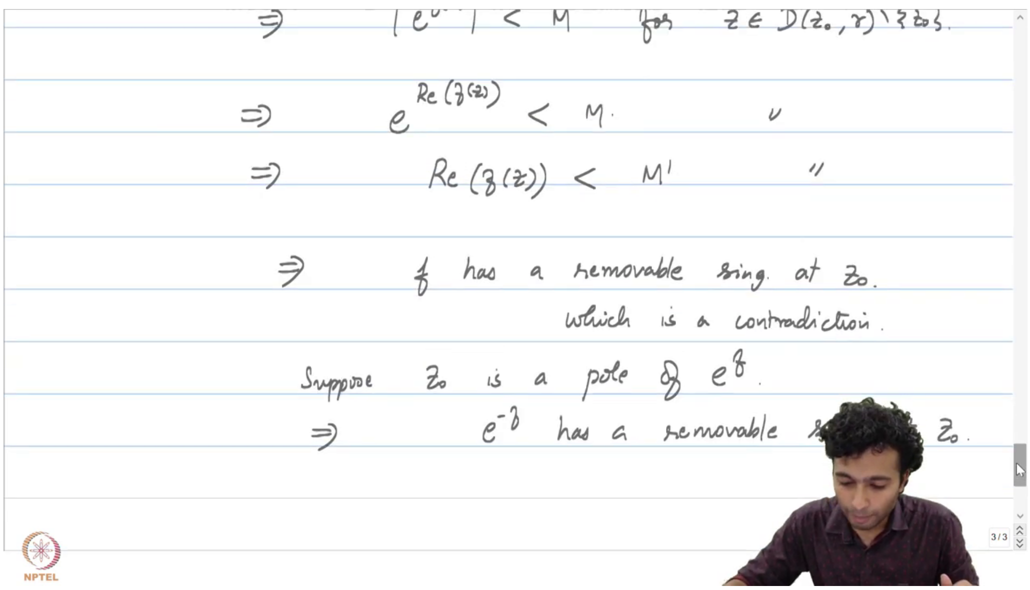

Case-II: Let z_0 be a pole of f. Suppose z_0 is a removable singularity of e^f. Then,

$$\begin{aligned} \left|e^{f(z)}\right| &< M && \text{for } z \in D(z_0, r) \setminus \{z_0\} \\ \Longrightarrow e^{\mathfrak{Re}(f(z))} &< M && \text{for } z \in D(z_0, r) \setminus \{z_0\} \\ \Longrightarrow \mathfrak{Re}\big(f(z)\big) &< M' && \text{for } z \in D(z_0, r) \setminus \{z_0\} \text{ and for some } M'. \end{aligned}$$

Hence by Problem 3, f has a removable singularity at z_0 which is a contradiction to our assumption that z_0 is a pole of f. Therefore z_0 cannot be a removable singularity of e^f.

Suppose z_0 is a pole of e^f. Then e^{-f} has a removable singularity at z_0. By the argument above, we have z_0 is a removable singularity of $-f$ and hence is a removable singularity of f, again a contradiction. Thus z_0 is an essential singularity of e^f.

Case-III- Let z_0 be an essential singularity of f. By Casorati-Weierstrass theorem, we have $\overline{f(D(z_0,r)\setminus\{z_0\})} = \mathbb{C}$. Since exponential map is continuous, we have,

$$\exp\left(\overline{f(D(z_0,r)\setminus\{z_0\})}\right) \subseteq \overline{\exp\left(f(D(z_0,r)\setminus\{z_0\})\right)}.$$

Hence we have

$$\mathbb{C}\setminus\{0\} \subseteq \overline{\exp\left(f(D(z_0,r)\setminus\{z_0\})\right)}.$$

Therefore $\overline{\exp\left(f(D(z_0,r)\setminus\{z_0\})\right)} = \mathbb{C}$ as it is closed and contains $\mathbb{C}\setminus\{0\}$ implies that z_0 is an essential singularity of e^f.

PROBLEM 5. Let $f : \Omega\setminus\{z_0, z_1, z_2 \ldots\} \longrightarrow \mathbb{C}$ be a holomorphic function, where Ω is an open connected subset of $\mathbb{C}$ and z_n be a sequence of distinct points such that $z_n \longrightarrow z_0$ in Ω. Let $z_1, z_2, \ldots$ be poles of f. Then given $\epsilon > 0$, prove that $f(D(z_0,\epsilon)\setminus\{z_0, z_1, z_2, \ldots\})$ is dense in $\mathbb{C}$.

SOLUTION 4. Consider $\alpha \in \mathbb{C}$ such that $\alpha \notin \overline{f(D(z_0,\epsilon)\setminus\{z_0, z_1, \ldots\})}$. Then there exists $\delta > 0$ such that $D(\alpha,\delta) \cap \overline{f(D(z_0,\epsilon)\setminus\{z_0, z_1, \ldots\})} = \varnothing$.

On $\Omega\setminus\{z_0, z_1, z_2 \ldots\}$, define $g(z) = \dfrac{1}{f(z)-\alpha}$. Then,

$$|g(z)| = \frac{1}{\left|f(z)-\alpha\right|} < \frac{1}{\alpha} \qquad \text{on } \Omega\setminus\{z_0, z_1, z_2 \ldots\}.$$

Since z_n for $n > 0$ are isolated singularities of f and g is bounded by $\frac{1}{\delta}$ on $\Omega\setminus\{z_0, z_1, z_2 \ldots\}$, g has a removable singularity at z_n for every $n > 0$.

Since z_n is a pole of f, we have $g(z_n) = 0$ and therefore $g : D(z_0, r)\setminus\{z_0\} \longrightarrow \mathbb{C}$ has a removable singularity at z_0 and moreover by the continuity of g, we have $g(z_0) = 0$.

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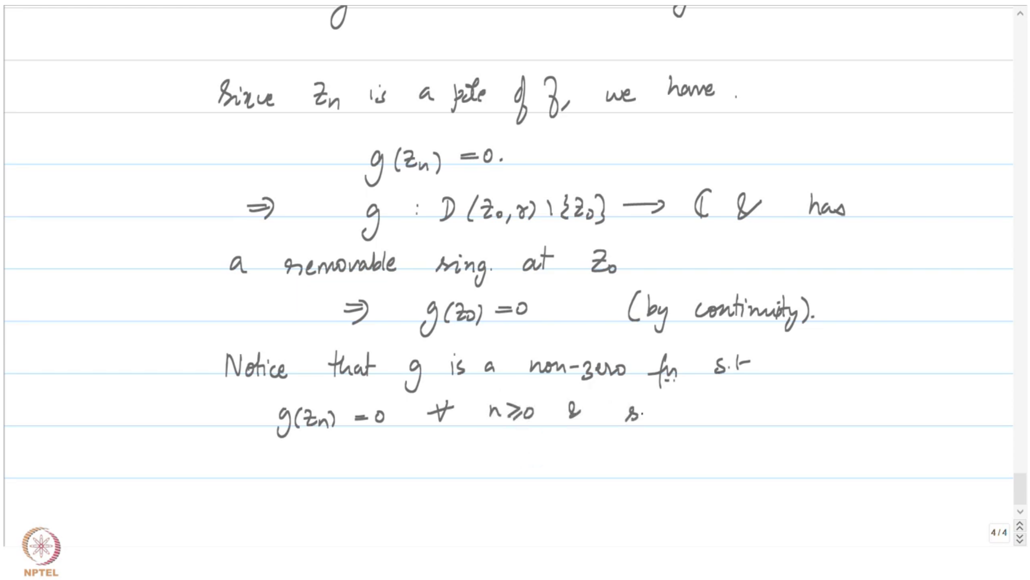

Notice that g is a non-zero function such that $g(z_n) = 0$ for every $n \geq 0$ and such that $\{z_n : n \in \mathbb{N}\}$ has a limit point, which is a contradiction to the identity theorem. Hence $\overline{f(D(z_0, \epsilon) \setminus \{z_0, z_1, \ldots\})} = \mathbb{C}$.

PROBLEM 6. Consider $f(z) = \dfrac{1}{z(z-1)}$. Find the Laurent series expansion of f in the following annuli:

(a) $D(0,1) \setminus \{0\}$

(b) $D(1,1) \setminus \{1\}$

(c) $\{z : |z| > 1\}$.

SOLUTION 5.

(a) On $D(0,1)$, we know that

$$\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n.$$

Then for $z \neq 0$, we have,

$$\frac{1}{z(z-1)} = \frac{1}{z}\left(-\sum_{n=0}^{\infty} z^n\right) = -\sum_{n=0}^{\infty} z^{n-1} = \frac{-1}{z} - \sum_{n=0}^{\infty} z^n.$$

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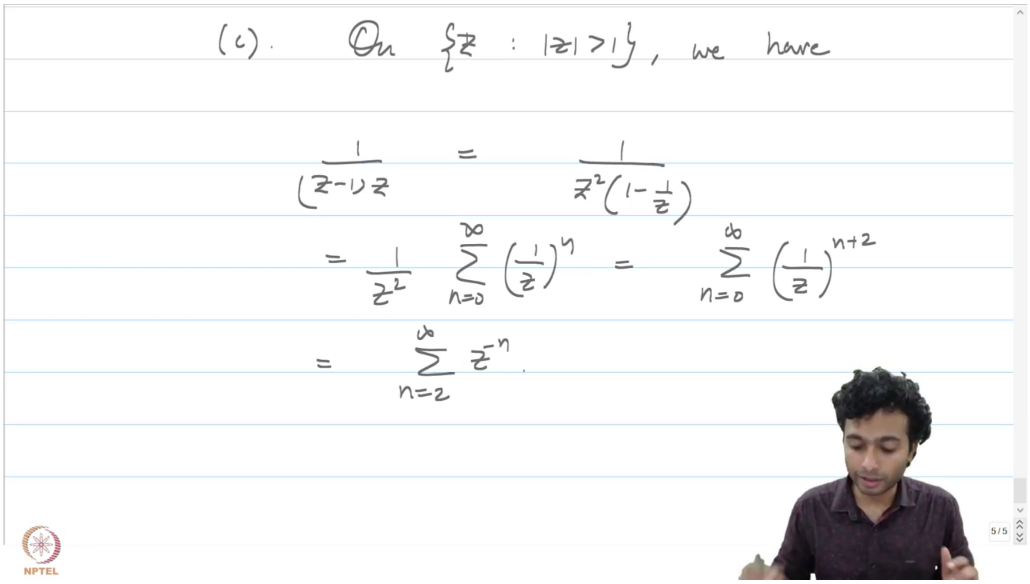

(b) On $D(1,1)$, we have

$$\frac{1}{z} = \frac{1}{1-(1-z)} = \sum_{n=0}^{\infty}(1-z)^n.$$

For $z \neq 1$,

$$\frac{1}{z(z-1)} = \frac{1}{(z-1)}\left(\sum_{n=0}^{\infty}(1-z)^n\right) = -\sum_{n=0}^{\infty}(1-z)^{n-1} = \frac{1}{z-1} - \sum_{n=0}^{\infty}(1-z)^n.$$

(c) On $\{z : |z| > 1\}$, we have,

$$\frac{1}{(z-1)z} = \frac{1}{z^2\left(1-\frac{1}{z}\right)} = \frac{1}{z^2}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+2} = \sum_{n=2}^{\infty} z^{-n}.$$