

Complex Analysis
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Lecture No – 31
Schwarz Reflection Principle

In Real Analysis we have seen many instances where we had to extend our given function which is defined on an interval to a larger interval. So a function f defined on I , an interval on \mathbb{R} had to be extended to a function on J , a larger interval containing I . In fact, if we start off with a continuous function we would have demanded many times to get hold of an extension which was also continuous. If we started off with a smooth function on I , the requirement was that we get a smooth extension to J . So the regularity had to be preserved.

In this lecture, we will address a similar problem for holomorphic functions. Holomorphic functions, as we have already seen by now, is a very rigid class of functions. It is not very easy to answer the question of extension many times. There would not be an extension possible at all. For example, on the unit disk, if we take the function $f(z) = \frac{1}{z-1}$, then it does not extend past the boundary at all.

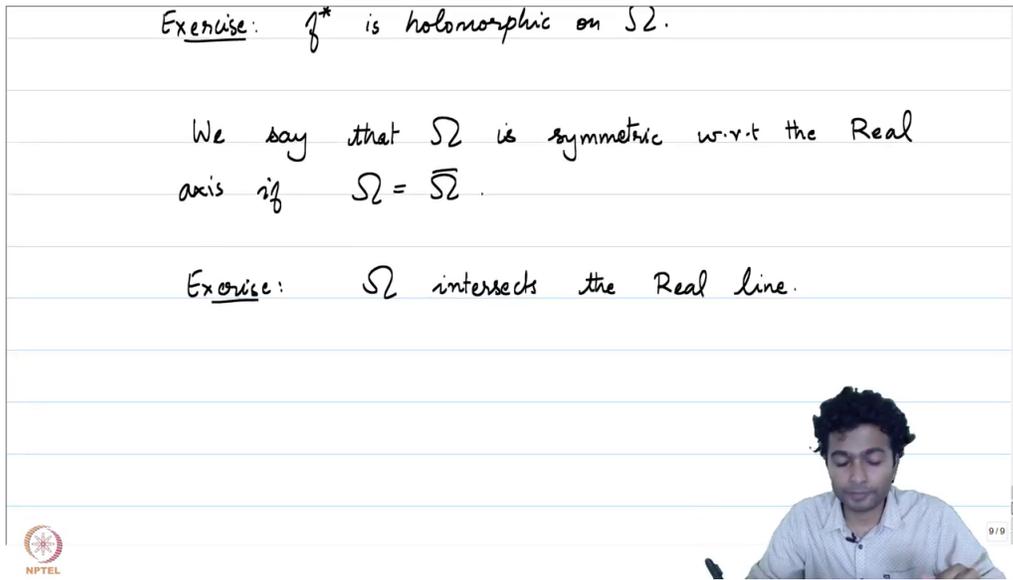
Here we will prove a theorem called the Schwarz Reflection Principle which tells us certain special circumstances when we will be able to talk about the extension of a holomorphic function. Let us try to set up the circumstances where we can indeed talk about such an extension.

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Exercise: f^* is holomorphic on Ω .

We say that Ω is symmetric w.r.t the Real axis if $\Omega = \overline{\Omega}$.

Exercise: Ω intersects the Real line.



Let Ω be an open connected set. Define $\overline{\Omega} := \{z \in \mathbb{C} : \bar{z} \in \Omega\}$. Suppose $f : \Omega \rightarrow \mathbb{C}$ be a holomorphic function. Then define $f^* : \overline{\Omega} \rightarrow \mathbb{C}$ by $f^*(z) = \overline{f(\bar{z})}$.

EXERCISE 1. f^* is holomorphic on $\overline{\Omega}$.

We say that Ω is symmetric with respect to \mathbb{R} if $\Omega = \overline{\Omega}$.

EXERCISE 2. If Ω is symmetric with respect to \mathbb{R} , then Ω intersects the real line.

We shall now focus on open connected sets Ω which are symmetric with respect to \mathbb{R} . Let f be a holomorphic function on Ω . Since $\Omega = \overline{\Omega}$, the domain of definition of both f and f^* are the same and are holomorphic on Ω .

Define $g : \Omega \rightarrow \mathbb{C}$ given by

$$g(z) = f(z) - f^*(z).$$

Then g is holomorphic on Ω .

Now let us look into the case that our function f satisfies the condition that, $f(x) \in \mathbb{R}$ whenever $x \in \mathbb{R}$. Then, for $x \in \Omega \cap \mathbb{R}$, we have $f(x) = f^*(x)$. Hence $g(x) = 0$. Since Ω is open, there exist $a, b \in \mathbb{R}$ such that $[a, b] \subseteq \Omega$ and $g(x) = 0$ for $x \in [a, b]$. But by

the identity theorem, we know that zeroes of a non-constant holomorphic function are isolated. Hence $g \equiv 0$ on Ω . That is, $f(z) = f^*(z)$ on Ω .

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Let Ω be an open connected set which is symmetric w.r.t \mathbb{R} . Then set

$$\Omega_+ = \{z \in \Omega : \Im(z) > 0\}$$

$$\Omega_- = \{z \in \Omega : \Im(z) < 0\}$$

$$I = \{z \in \Omega : \Im(z) = 0\}$$

Hence $\Omega = \Omega_+ \cup I \cup \Omega_-$.

Theorem: Let Ω be as above. Suppose $f: \Omega_+ \cup I \rightarrow \mathbb{C}$ which is cont. on $\Omega_+ \cup I$ and holomorphic on Ω_+ . Suppose $f(x) \in \mathbb{R} \forall x \in I$, then $\exists g: \Omega \rightarrow \mathbb{C}$ hol. on Ω s.t. $g(z) = \overline{f(\bar{z})}$ for $z \in \Omega_+ \cup I$.



We will be using the following notations:

Let Ω be an open connected set which is symmetric with respect to \mathbb{R} . Then

$$\Omega_+ = \{z \in \Omega : \Im(z) > 0\}$$

$$\Omega_- = \{z \in \Omega : \Im(z) < 0\}$$

$$I = \{z \in \Omega : \Im(z) = 0\}.$$

Hence $\Omega = \Omega_+ \cup I \cup \Omega_-$.

THEOREM 3 (Schwarz Reflection Principle). *Let Ω be symmetric with respect to \mathbb{R} . Suppose $f: \Omega_+ \cup I \rightarrow \mathbb{C}$ be a function which is continuous on $\Omega_+ \cup I$ and holomorphic on Ω_+ . Suppose $f(x) \in \mathbb{R}$ for every $x \in I$, then there exists $g: \Omega \rightarrow \mathbb{C}$ holomorphic on Ω such that $g(z) = f(z)$ for each $z \in \Omega_+ \cup I$.*

PROOF. Define the function g as,

$$g(z) = \begin{cases} f^*(z) & \text{for } z \in \Omega_- \\ f(z) & \text{for } z \in \Omega_+ \cup I. \end{cases}$$

Now it is left as an exercise for the reader to check that the function g is continuous on Ω .

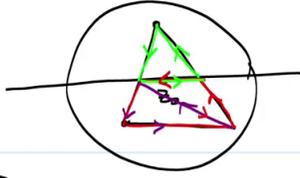
We shall prove that g is holomorphic on Ω . Since f is holomorphic on Ω_+ , it is easy to verify that g is holomorphic on $\Omega_+ \cup \Omega_-$. Hence to establish that g is holomorphic on Ω , it is enough to check complex differentiability at points on I .

Let $z_0 \in I$ and let $r > 0$ be such that $\overline{D(z_0, r)} \subseteq \Omega$. One can verify that by Morera's theorem, for proving g is holomorphic, it is enough to check $\int_T g = 0$ for every triangle T in $D(z_0, r)$.

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Exercise: By Morera's theorem, it is enough to check $\int_T g = 0$ for every triangle T in $D(z_0, r)$.

Enough to show that $\int_T g = 0$ for $T \subseteq \Omega_+ \cup I$ or $T \subseteq \Omega_- \cup I$.



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For this, it is enough to show that

$$\int_T g = 0 \quad \text{for } T \subseteq \Omega_+ \cup I \text{ or } T \subseteq \Omega_- \cup I.$$

We will establish this statement by proving that

$$\int_T g = 0$$

where T is the triangular path $\gamma_{z_1 \rightarrow z_2 \rightarrow z_3 \rightarrow z_1}$ where $z_1, z_2, z_3 \in D(z_0, r)$ such that $z_2, z_3 \in I$. We know that the convex hull \hat{T} is a compact set and g is uniformly continuous on \hat{T} .

Let $M = \sup_{z \in \hat{T}} |g(z)|$ and $d = \sup_{z, w \in \hat{T}} |z - w|$. By uniform continuity, given an $\epsilon > 0$, there exists $\delta > 0$ such that $|g(z) - g(w)| < \epsilon$ whenever $|z - w| < \delta$ on \hat{T} . Pick w_2 and w_3 on $\gamma_{z_1 \rightarrow z_2}$ and $\gamma_{z_1 \rightarrow z_3}$ respectively such that $|w_2 - z_2| < \delta$ and $|w_3 - z_3| < \delta$. Then

$$\int_T g(z) dz = \int_{\gamma_{z_1 \rightarrow w_2 \rightarrow w_3 \rightarrow z_1}} g(z) dz + \int_{\gamma_{w_3 \rightarrow w_2 \rightarrow z_2 \rightarrow z_3 \rightarrow w_3}} g(z) dz.$$

But by Cauchy's theorem,

$$\int_{\gamma_{z_1 \rightarrow w_2 \rightarrow w_3 \rightarrow z_1}} g(z) dz = 0$$

since $\gamma_{z_1 \rightarrow w_2 \rightarrow w_3 \rightarrow z_1} \subset \Omega_+$. Hence

$$\int_T g(z) dz = \int_{\gamma_{w_3 \rightarrow w_2 \rightarrow z_2 \rightarrow z_3 \rightarrow w_3}} g(z) dz$$

$$(1) \quad \int_T g(z) dz = \int_{\gamma_{z_2 \rightarrow z_3}} g(z) dz + \int_{\gamma_{z_3 \rightarrow w_3}} g(z) dz + \int_{\gamma_{w_3 \rightarrow w_2}} g(z) dz + \int_{\gamma_{w_2 \rightarrow z_2}} g(z) dz.$$

Now let us consider first and third terms of (1).

$$(2) \quad \int_{\gamma_{z_2 \rightarrow z_3}} g(z) dz + \int_{\gamma_{w_3 \rightarrow w_2}} g(z) dz = \int_{\gamma_{z_2 \rightarrow z_3}} g(z) dz - \int_{\gamma_{w_2 \rightarrow w_3}} g(z) dz.$$

Since $\gamma_{z_2 \rightarrow z_3} = (1-t)z_2 + tz_3$ and $\gamma_{w_2 \rightarrow w_3} = (1-t)w_2 + tw_3$. Then, (2) can be written as,

$$\begin{aligned} \int_{\gamma_{z_2 \rightarrow z_3}} g(z) dz + \int_{\gamma_{w_3 \rightarrow w_2}} g(z) dz &= (z_3 - z_2) \int_0^1 g((1-t)z_2 + tz_3) dt \\ &\quad - (w_3 - w_2) \int_0^1 g((1-t)w_2 + tw_3) dt \end{aligned}$$

Now if $|((1-t)z_2 + tz_3) - ((1-t)w_2 + tw_3)| < \delta$, we have,

$$\begin{aligned} \left| \int_{\gamma_{z_2 \rightarrow z_3}} g(z) dz - \int_{\gamma_{w_2 \rightarrow w_3}} g(z) dz \right| &\leq \left| (z_3 - z_2) \int_0^1 g((1-t)z_2 + tz_3) - g((1-t)w_2 + tw_3) dt \right| \\ &\quad + \left| ((w_3 - w_2) - (z_3 - z_2)) \int_0^1 g((1-t)w_2 + tw_3) dt \right| \\ &\leq |z_3 - z_2|\epsilon + (|w_3 - z_3| + |w_2 - z_2|)M \\ &\leq d\epsilon + 2\epsilon M = (d + 2M)\epsilon. \end{aligned}$$

Now let us consider second and fourth terms in (1).

$$\begin{aligned} \left| \int_{\gamma_{z_3 \rightarrow w_3}} g(z) dz \right| &= \left| (w_3 - z_3) \int_0^1 g((1-t)z_3 + tw_3) dt \right| \\ &\leq \epsilon M. \end{aligned}$$

Similarly,

$$\left| \int_{\gamma_{w_2 \rightarrow z_2}} g(z) dz \right| \leq M\epsilon.$$

Hence,

$$\left| \int_T g(z) dz \right| \leq c\epsilon$$

for some positive constant $c \in \mathbb{R}$.

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Hence $\int_T g = 0.$

$\Rightarrow g$ is hol. in $D(z_0, r).$

$\Rightarrow g$ is hol. on $\Omega.$



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Since $\epsilon > 0$ was arbitrary, we have,

$$\int_T g(z) dz = 0.$$

Hence g is holomorphic in $D(z_0, r).$

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