Complex Analysis

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Lecture No – 21

Homotopy of Curves

We will soon discuss one of the fundamental theorems in complex analysis called Cauchy's theorem. Very broadly speaking Cauchy's theorem says that if we start off with a domain Ω and if γ_0 and γ_1 are two curves in Ω such that γ_0 can be continuously deformed to the curve γ_1 then, $\int_{\gamma_0} f(z)dz = \int_{\gamma_1} f(z)dz$ if f is a holomorphic function on Ω. The notion of continuously deforming a curve γ_0 to obtain a curve γ_1 is made precise by the topological notion called homotopy.

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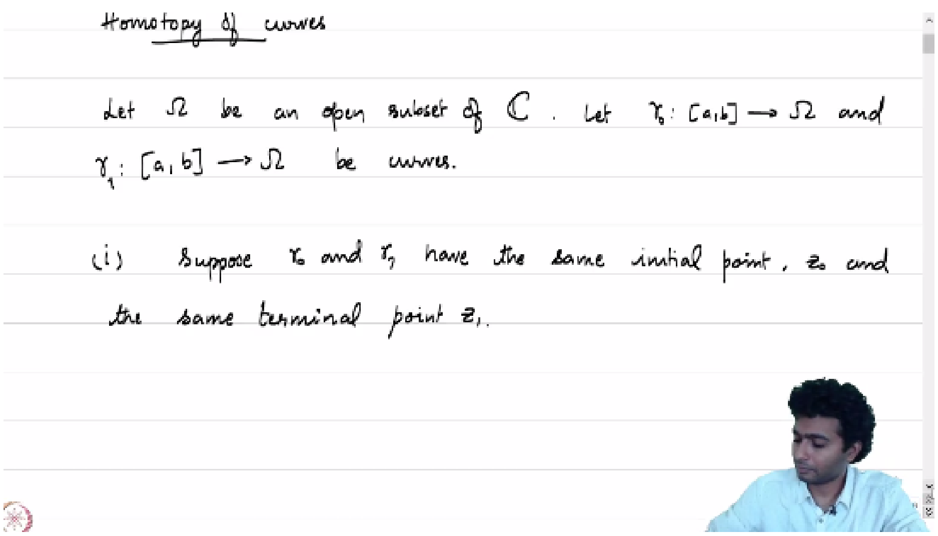

Let Ω be an open subset of $\mathbb{C}$. Let $\gamma_0 : [a,b] \longrightarrow \Omega$ and $\gamma_1 : [a,b] \longrightarrow \Omega$ be curves.

(1) (**Homotopy of two curves with fixed end points**) Suppose γ_0 and γ_1 have the same initial point, z_0 and the same terminal point, z_1. We say that γ_0 is homotopic to γ_1 with fixed end-point if there exists a continuous map

$H : [0,1] \times [a,b] \longrightarrow \Omega$ such that $\forall\, t \in [a,b]$,

$$H(0,t) = \gamma_0(t)$$

$$H(1,t) = \gamma_1(t)$$

and $\forall\, s \in [0,1]$,

$$H(s,a) = z_0$$

$$H(s,b) = z_1.$$

Let us try to understand what this definition actually means. H is a function defined on $[0,1] \times [a,b]$, where $[a,b]$ is common domain of both γ_0 and γ_1. Now we will think of first parameter from $[0,1]$ as the stage at which we are in. For example, if

$$H(s,t) = \gamma_s(t)$$

then γ_s is a continuous map from z_0 to z_1.

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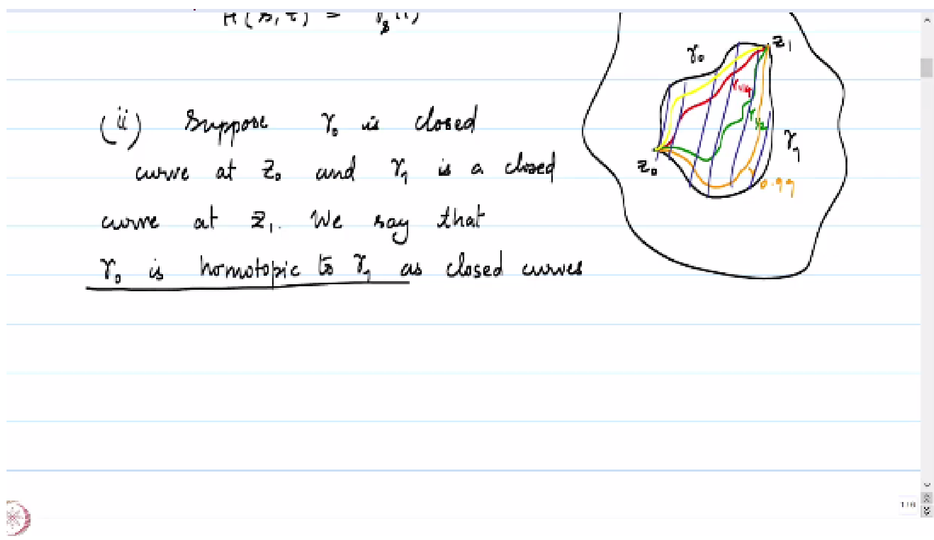

(2) (**Homotopy of closed curves**) Suppose γ_0 is closed curve at z_0 and γ_1 is a closed curve at z_1. We say that γ_0 is homotopic to γ_1 as closed curves if there exists a

continuous map $H : [0,1] \times [a,b] \longrightarrow \Omega$ such that $\forall\, t \in [a,b]$,

$$H(0,t) = \gamma_0(t)$$

$$H(1,t) = \gamma_1(t)$$

and

$$H(s,a) = H(s,b) \qquad \forall\, s \in [0,1].$$

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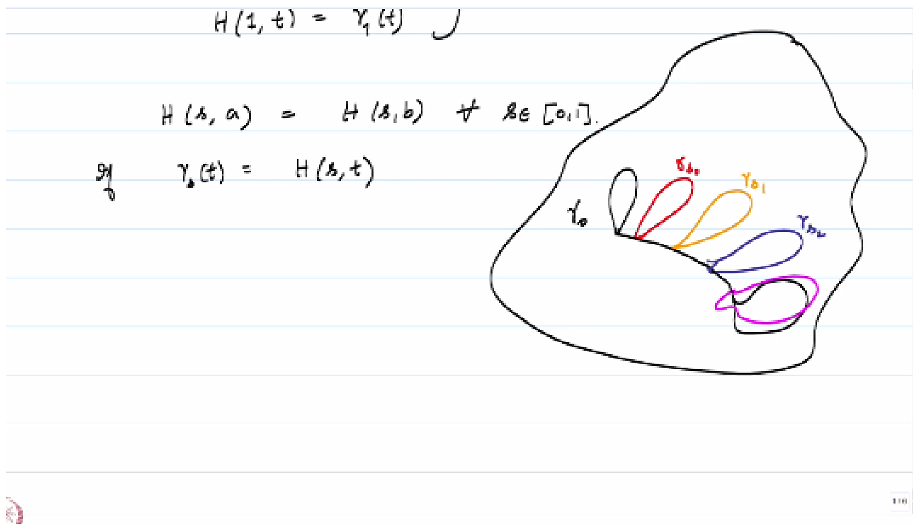

If $\gamma_s(t) = H(s,t)$, then γ_s is a closed curve.

In either case, the map H is called a homotopy from γ_0 to γ_1.

EXAMPLE 1. Let Ω be a convex domain (i.e., if $z_0, z_1 \in \Omega$, then $(1-t)z_0 + tz_1 \in \Omega\ \forall\, t \in [0,1]$). Suppose $\gamma_0 : [a,b] \longrightarrow \Omega$ and $\gamma_1 : [a,b] \longrightarrow \Omega$ be any two curves from z_0 to z_1.

Define $H : [0,1] \times [a,b] \longrightarrow \Omega$ by

$$H(s,t) = (1-s)\gamma_0(t) + s\gamma_1(t).$$

Then H is a homotopy from γ_0 to γ_1.

Properties

Let $\Omega \subseteq \mathbb{C}$ be an open subset.

(1) The relation of γ_0 being homotopic to γ_1 with fixed end points is an equivalence relation.

PROOF.

(a) Let $\gamma_0 : [a,b] \longrightarrow \Omega$. Consider $H : [0,1] \times [a,b] \longrightarrow \Omega$ given by

$$H(s,t) := \gamma_0(t).$$

Then H is a homotopy from γ_0 to itself with fixed end points.

(b) Let $\gamma_0 : [a,b] \longrightarrow \Omega$ and $\gamma_1 : [a,b] \longrightarrow \Omega$ be such that γ_0 is homotopic to γ_1 with fixed end points. Then $H : [0,1] \times [a,b] \longrightarrow \Omega$ be a homotopy from γ_0 to γ_1.

Define $H_1 : [0,1] \times [a,b] \longrightarrow \Omega$ given by

$$H_1(s,t) = H(1-s,t).$$

Then H_1 is a homotopy with fixed end points from γ_1 to γ_0.

(c) Let $\gamma_0, \gamma_1, \gamma_2 : [a,b] \longrightarrow \Omega$ be such that γ_0 is homotopic with fixed end points to γ_1 and γ_1 is homotopic with fixed end points to γ_2.

That is, there exist $H_1 : [0,1] \times [a,b] \longrightarrow \Omega$ and $H_2 : [0,1] \times [a,b] \longrightarrow \Omega$ such that

$$H_1(0,t) = \gamma_0(t), \quad H_1(1,t) = \gamma_1(t)$$

and

$$H_2(0,t) = \gamma_1(t), \quad H_2(1,t) = \gamma_2(t).$$

Define $H : [0,1] \times [a,b] \longrightarrow \Omega$ given by

$$H(s,t) := \begin{cases} H_1(2s,t) & 0 \le s \le \frac{1}{2} \\ H_2(2s-1,t) & \frac{1}{2} \le s \le 1. \end{cases}$$

Then H is a homotopy with fixed end points from γ_0 to γ_2.

□

A very similar proof tells us that the relation of homotopy as closed curve is also an equivalence relation.

(2) Let $\gamma_0, \gamma_1 : [a,b] \longrightarrow \Omega$ be closed curves at z_0. Then $\gamma_0 \sim \gamma_1$ as closed if and only if there exists a closed curve γ_2 at z_0 such that γ_0 is homotopic with fixed end points to a reparametrization of $\gamma_2 + \gamma_1 + (-\gamma_2)$.

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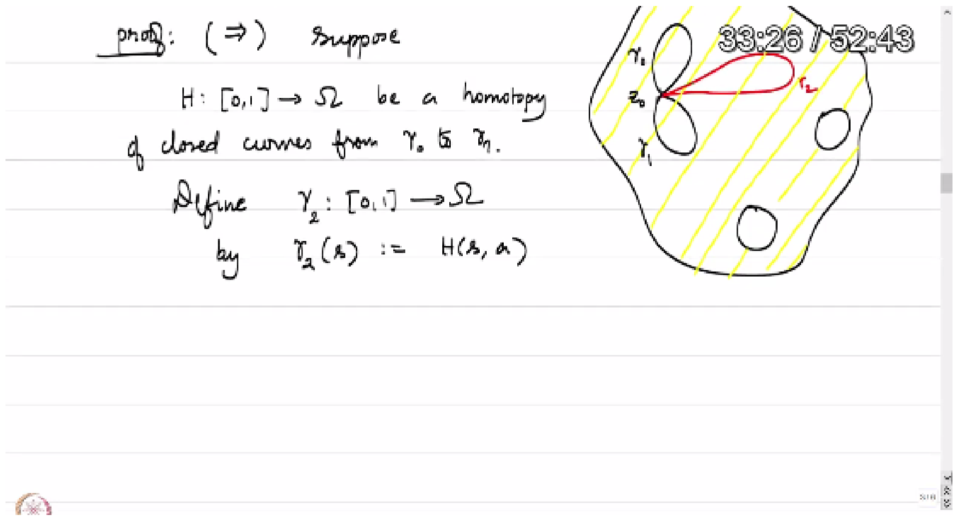

PROOF. (⇒) Suppose $H : [0,1] \times [a,b] \longrightarrow \Omega$ be a homotopy of closed curves from γ_0 to γ_1.

Define $\gamma_2 : [0,1] \longrightarrow \Omega$ by

$$\gamma_2(s) = H(s,a).$$

Claim: γ_0 is homotopic to a reparametrization of $\gamma_2 + \gamma_1 + (-\gamma_2)$ with fixed points.

Define

$$\sigma_s = \gamma_2 \restriction_{[0,s]} + \gamma_s + (-\gamma_2) \restriction_{[-s,0]}$$

where $\gamma_s(t) = H(s,t)$.

Then $\sigma_s : [0,(b-a)+2s] \longrightarrow \Omega$. Now reparametrize σ_s to obtain $\tilde{\sigma}_s$, $\tilde{\sigma}_s : [a,b] \longrightarrow \Omega$. This can be done by composing σ_s with a linear homeomorphism between $[0,(b-a)+2s]$ and $[a,b]$.

Define

$$\tilde{H}(s,t) := \tilde{\sigma}_s(t).$$

Now it is left as an exercise to the reader to verify that $\tilde{H}$ is continuous and also a homotopy with fixed end points from γ_0 to a reparametrization of $\gamma_2 + \gamma_1 + (-\gamma_2)$.

($\Leftarrow$) It is enough to check that γ_1 is homotopic as a closed curves to a reparametrization of $\gamma_2 + \gamma_1 + (-\gamma_2)$, because of the transitivity property of homotopy.

Define

$$\sigma_s := \gamma_2 \restriction_{[1-s,1]} + \gamma_1 + (-\gamma_2) \restriction_{[-1,s-1]}$$

Let $\tilde{\sigma}_s : [a,b] \longrightarrow \Omega$ be a reparametrization of σ_s. Define

$$H : [0,1] \times [a,b] \longrightarrow \Omega$$

by

$$H(s,t) = \tilde{\sigma}_s(t).$$

Then H is a homotopy of closed curves from γ_1 to a reparametrization of $\gamma_2 + \gamma_1 + (-\gamma_2)$. □

(3) Let $\gamma_0 : [a,b] \longrightarrow \Omega$ be a curve from z_0 to z_1 and $\gamma_1 : [a,b] \longrightarrow \Omega$ be such that $\gamma_1(t) = z_0 \; \forall \; t \in [a,b]$. Then a reparametrization of $\gamma_0 + (-\gamma_0)$ is homotopic to γ_1.

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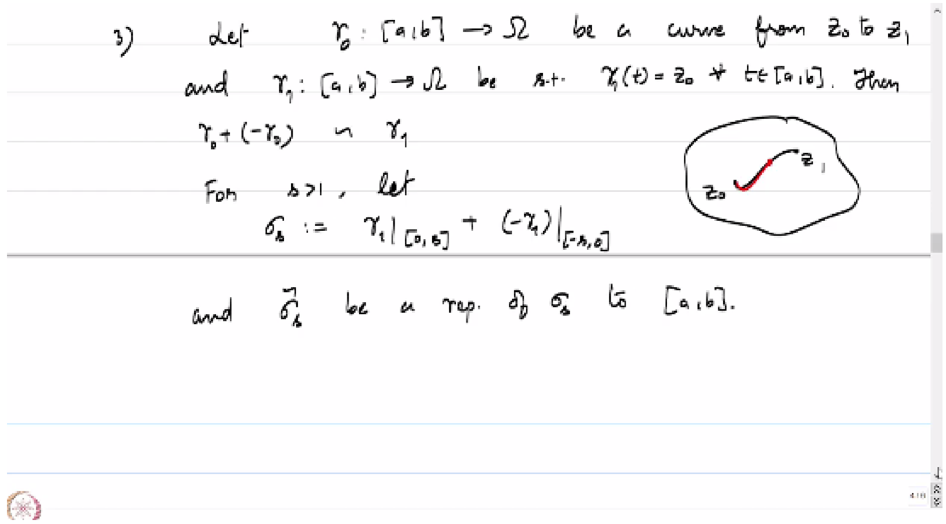

3) Let $\gamma_0 : [a,b] \to \Omega$ be a curve from z_0 to z_1 and $\gamma_1 : [a,b] \to \Omega$ be s.t. $\gamma_1(t) = z_0 \; \forall \, t \in [a,b]$. Then $\gamma_0 + (-\gamma_0) \sim \gamma_1$

For $s > 1$, let

$\sigma_s := \gamma_1|_{[0,s]} + (-\gamma_1)|_{[-s,0]}$

and $\tilde{\sigma}_s$ be a rep. of σ_s to $[a,b]$.

PROOF. For $s > 1$, let

$$\sigma_s := \gamma_0 \restriction_{[0,s]} + (-\gamma_0) \restriction_{[-s,0]}$$

and $\tilde{\sigma}_s$ be a reparametrization of σ_s to $[a,b]$. For $s = 0$, $\tilde{\sigma}_0(t) = z_0 \; \forall \, t \in [a,b]$.

Define

$$H(s,t) := \tilde{\sigma}_s(t).$$

Then H is a homotopy from γ_1 to a reparametrization of $\gamma_0 + (-\gamma_0)$. $\square$

EXERCISE 2. Let $\gamma_0, \tilde{\gamma}_0 : [a,b] \longrightarrow \Omega$ and $\gamma_1, \tilde{\gamma}_1 : [c,d] \longrightarrow \Omega$ be such that $\gamma_0 \sim \tilde{\gamma}_0$ and $\gamma_1 \sim \tilde{\gamma}_1$. Then

$$\gamma_0 + \gamma_1 \sim \tilde{\gamma}_0 + \tilde{\gamma}_1.$$

Definition. Let $\gamma_0 : [a,b] \longrightarrow \Omega$ be a closed curve at z_0. Then γ_0 is null-homotopic if γ_0 is homotopic to the constant curve where $\gamma_1 : [a,b] \longrightarrow \Omega$ given by $\gamma_1(t) = z_0$ as closed curves.