Complex Analysis

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Lecture No – 3.4

Cauchy Riemann Equations

Up till now, we have seen that complex differentiability is not very different from the notion of differentiability which was developed in Real Analysis course. The definitions were similar and some of the properties of like the laws of Calculus pertains to this lecture are satisfied by the differentiable functions and also the properties which we explore in complex differentiability. In this lecture we will see how complex differentiable function different from real differential function. We will prove that Complex differentiable functions are real differentiable functions which have some rigidity conditions.

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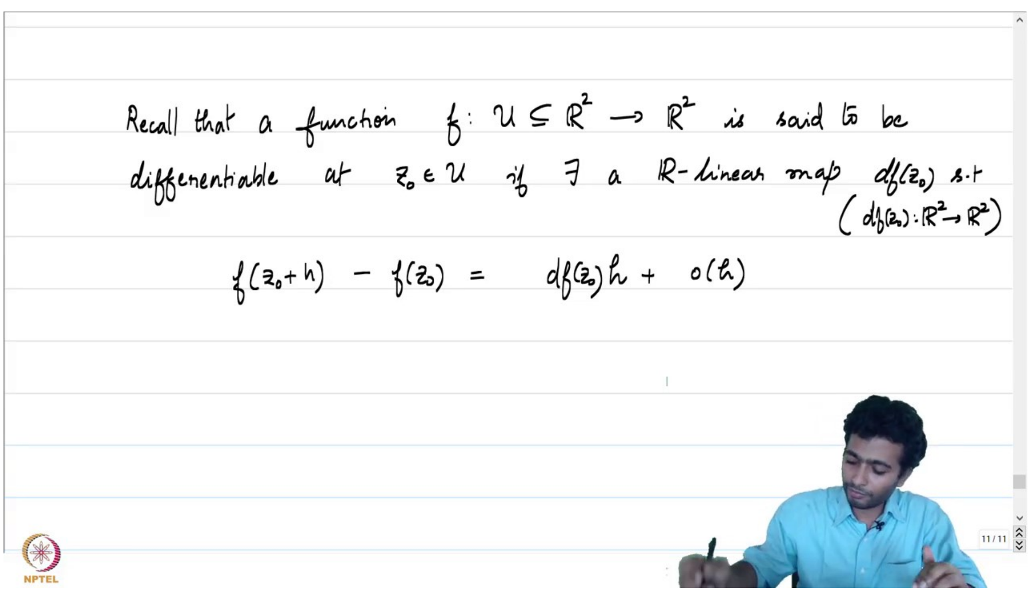

Recall that a function $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ is said to be differentiable at $z_0 \in U$ if $\exists$ a $\mathbb{R}$ linear map $df(z_0)$ such that $f(z_0 + h) - f(z_0) = df(z_0)h + o(h)\,, h \in \mathbb{R}^2$.

The function is said to be complex differentiable at $z_0 \in U$ if $\exists$ a complex number $f'(z_0)$ such that $f(z_0+h) - f(z_0) = f'(z_0)h + o(h)$.

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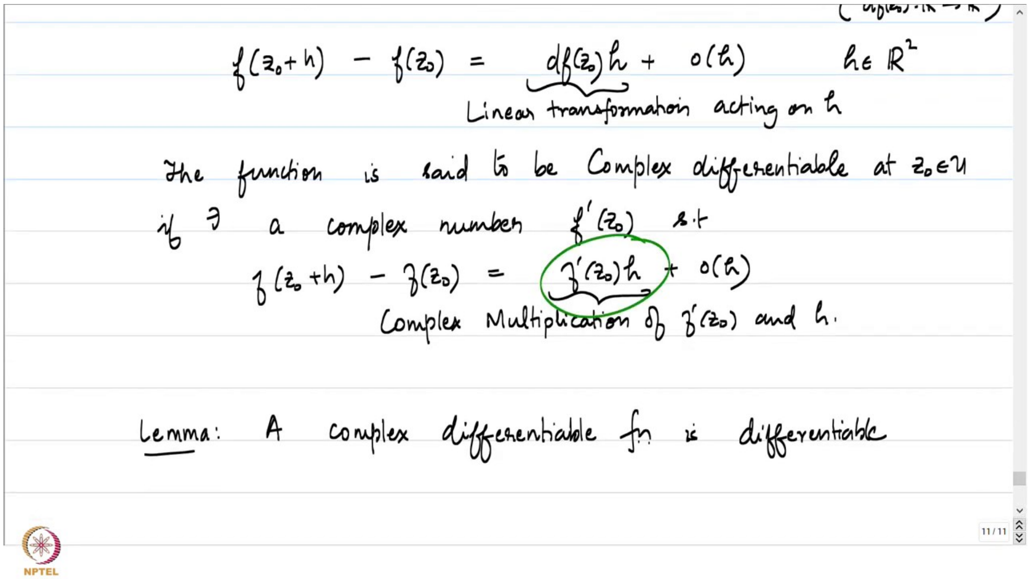

LEMMA 1. *A complex differentiable function is differentiable.*

PROOF. Let f be complex differentiable. If $f'(z_0) = a + ib$ and $h = x + iy$, then

$$df(z_0)\begin{pmatrix} x \\ y \end{pmatrix} := (a+ib)(x+iy) = \begin{pmatrix} ax - by \\ ay + bx \end{pmatrix}.$$

Now, with $df(z_0)h := f'(z_0)h$, the function is differentiable. □

Notice that a function $T : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$ linear then $T(z) = T(z \cdot 1) = zT(1) = \alpha z$, where $\alpha = T(1)$.

If $df(z_0)$ is a $\mathbb{C}-$ linear map, then $\exists$ a complex number $f'(z_0)$ such that

$$df(z_0)h = f'(z_0)h$$

where $h \in \mathbb{C}$.

THEOREM 2. *Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be a differentiable function. Then f is complex differentiable if and only if $df(z_0)$ is $\mathbb{C}$ linear.*

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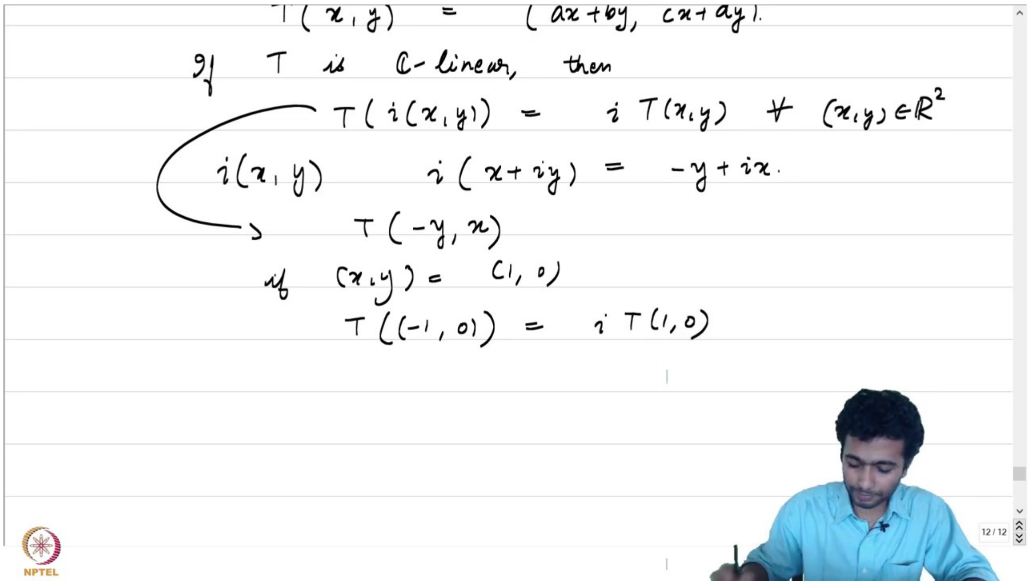

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be $\mathbb{R}$ linear. Suppose T is also $\mathbb{C}$ linear as a function from $\mathbb{C} \to \mathbb{C}$. We have $a, b, c, d \in \mathbb{R}$ such that $T(x, y) = (ax + by, cx + dy) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$.

If T is $\mathbb{C}-$ linear, then

$$T(i(x, y)) = i\,T(x, y) \quad \forall (x, y) \in \mathbb{R}^2.$$

Note that (x, y) can be identified as $x + iy$, with this identification, $i(x, y)$ corresponds to $i(x + iy) = -y + ix$. Hence, $T(i(x, y)) = T(-y, x) = i\,T(x, y)$.
If $(x, y) = (1, 0)$; $T((0, 1)) = i\,T((1, 0))$.
Thus $(b, d) = i(a, c) = (-c, a) \implies b = -c,\ d = a \to (*)$.
Thus, from here we can observe that any linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(x, y) = (ax + by, -bx + ay)$$

is $\mathbb{C}-$ linear.
Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ be a differentiable function and $f = u + iv$. Then

$$df(z_0) = \begin{pmatrix} u_x(z_0) & u_y(z_0) \\ v_x(z_0) & v_y(z_0) \end{pmatrix}$$

where $u_x(z_0) = \dfrac{\partial u}{\partial x}(z_0)$ and other quantities defined similarly.

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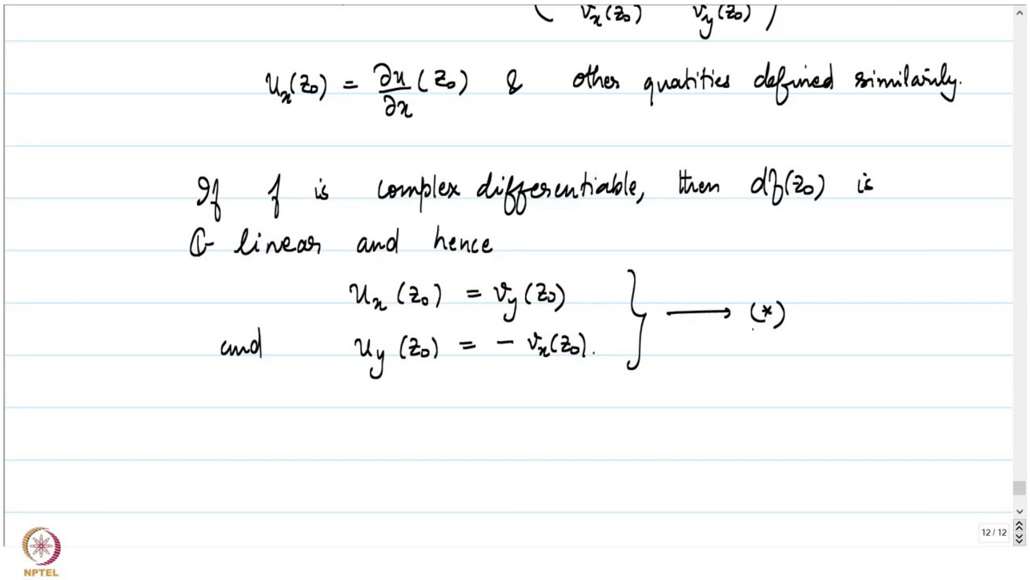

THEOREM 3. *Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be differentiable at $z_0 \in U$. Then f is complex differentiable if and only if, for $f = u + iv$, we have*

$$\left.\begin{aligned} u_x(z_0) &= v_y(z_0) \\ u_y(z_0) &= -v_x(z_0) \end{aligned}\right\} \to (*)$$

The equations $(*)$ are called the Cauchy- Riemann equations.

COROLLARY 4. *Let $f : U \subseteq \mathbb{C} \to \mathbb{C}$ be a differentiable function on U. Then f is holomorphic on U if and only if f satisfies the Cauchy-Riemann equations at every point of U.*

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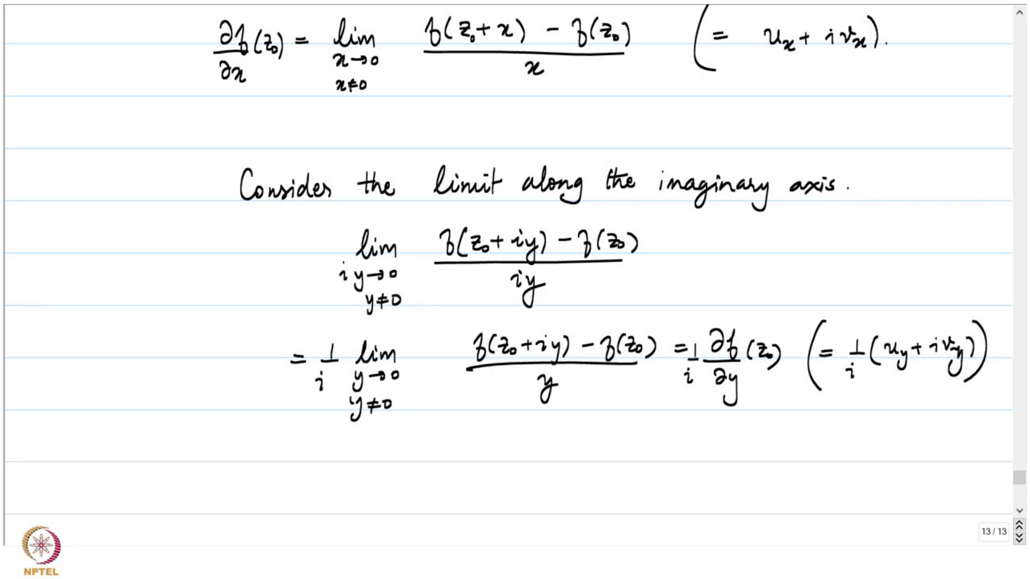

Recall that f is $\mathbb{C}-$ differentiable at z_0 if $\lim\limits_{\substack{z\to z_0 \\ z\in U\setminus\{z_0\}}} \frac{f(z)-f(z_0)}{z-z_0}$ exists.

Consider the limit along the direction parallel to $x-$ axis,

$$\frac{\partial f}{\partial x}(z_0) = \lim_{\substack{x\to 0 \\ x\neq 0}} \frac{f(z_0+x)-f(z_0)}{x} = u_x + i v_x$$

Consider the along the imaginary axis,

$$\lim_{\substack{iy\to 0 \\ y\neq 0}} \frac{f(z_0+iy)-f(z_0)}{iy} = \frac{1}{i}\lim_{\substack{y\to 0 \\ y\neq 0}} \frac{f(z_0+iy)-f(z_0)}{y}$$
$$= \frac{1}{i}\frac{\partial f}{\partial y}(z_0) = \frac{1}{i}(u_y + i v_y)$$

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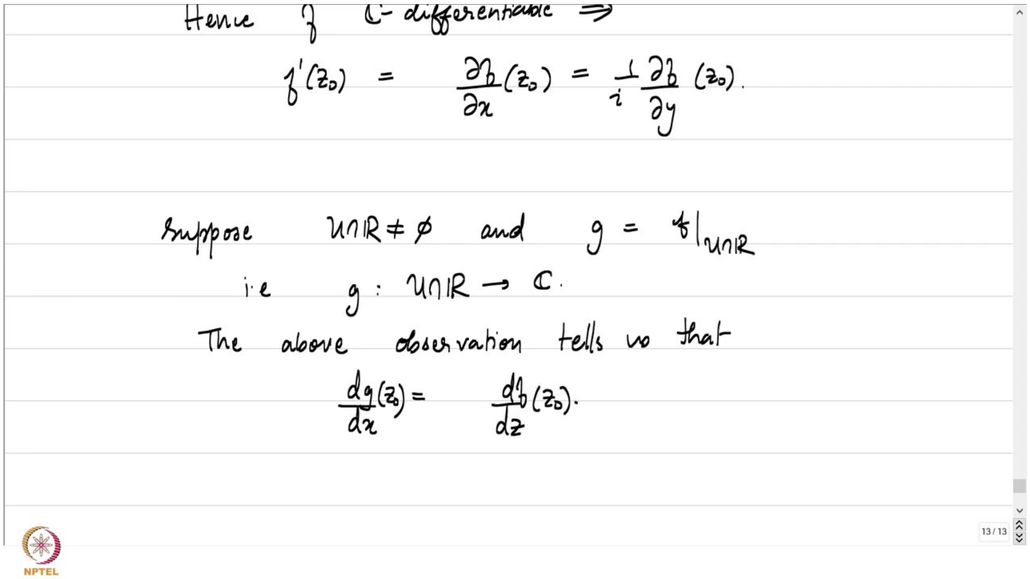

Hence f is $\mathbb{C}$ differentiable $\implies$

$$f'(z_0) = \frac{\partial f}{\partial x}(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}(z_0).$$

Suppose $U \cap \mathbb{R} \neq \varnothing$ and $g = f \restriction_{U\cap\mathbb{R}}$. i.e. $g : U \cap \mathbb{R} \to \mathbb{C}$. The above derivation tells us that

$$\frac{dg}{dx}(z_0) = \frac{df}{dz}(z_0).$$

Suppose $f : U \to \mathbb{C}$ be a differentiable function at $z_0 \in U$. Define the Wirtinger derivatives $\frac{\partial f}{\partial z}(z_0)$ and $\frac{\partial f}{\partial \bar{z}}(z_0)$ by the following formulae:

$$\frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) + \frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right)$$

$$\frac{\partial f}{\partial \bar{z}}(z_0) = \frac{1}{2}\left(\frac{\partial f}{\partial x}(z_0) - \frac{1}{i}\frac{\partial f}{\partial y}(z_0)\right).$$

EXERCISE 5. The function f satisfies the Cauchy-Riemann equations at $z_0 \in U$ if and only if $\frac{\partial f}{\partial \bar{z}}(z_0) = 0$.

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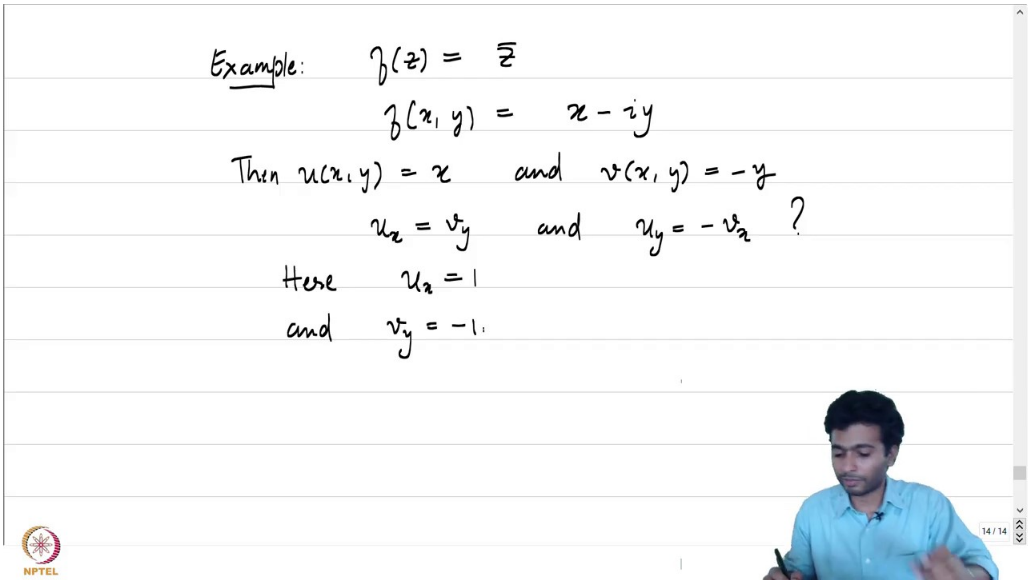

EXAMPLE 6.

- Consider the function $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \bar{z}$. Then $f(x, y) = x - iy \implies u(x, y) = x, v(x, y) = -y$. Here $u_x = 1$ and $v_y = -1$. Hence f does not satisfy the Cauchy-Riemann equations.
- $f(z) = \mathfrak{Re}(z)$. Here $u(x, y) = x$ and $v(x, y) = 0$.
- $f(z) = |z|^2 = x^2 + y^2$; $u(x, y) = x^2 + y^2$, $v(x, y) = 0$.