Algebra - I

Prof. S. Viswanath & Prof. Amritanshu Prasad

Department of Mathematics

Indian Institute of Technology, Madras

Lecture – 50

Tensor and Exterior Algebras

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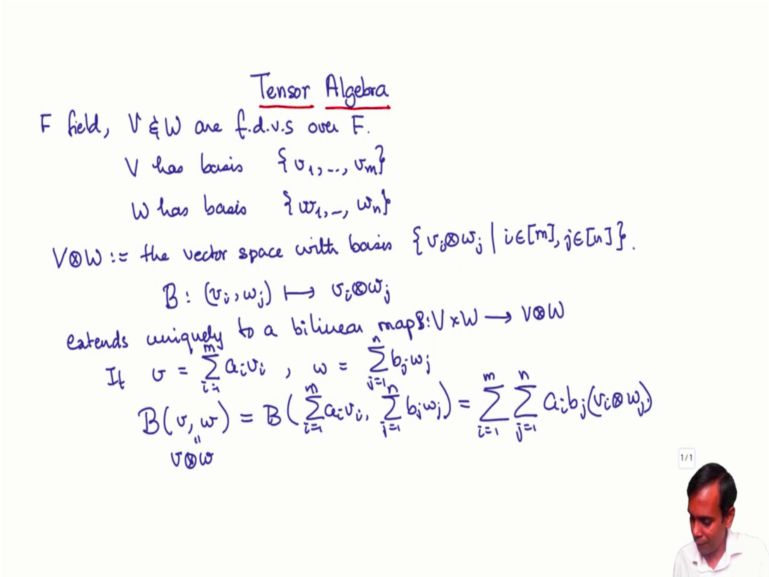

While studying linear algebra, you may have come across the notion of tensor products and tensor products lead to the construction of a very interesting algebra called the Tensor Algebra which is what I am going to talk about in this lecture. But before we talk about the tensor algebra let me recall for you what a tensor product is.

So, we are going to fix a field F and all our vector spaces are going to be over F and let us say V and W are vector spaces. Let us say finite dimensional vector spaces over F. And say V has a basis, v 1 dot dot dot v m; W has basis w 1 dot dot dot w n.

Then the tensor product of V and W is can be defined to be the vector space. Let us define it, the vector space F vector space of course, with basis and I will just give some symbols. I will write down some symbols v i tensor w j, where i goes from 1 to m and j goes from 1 to n. So, if V is an n dimensional m dimensional vector space and W is an n dimensional vector space. Then by definition v tensor w is an m times n dimensional vector space.

Now, there is something slightly unsettling about this definition, it is that the tensor product of V and W seems to depend on the choice of basis of the vector spaces V and W. So, this is of course, not something that should happen. The tensor product of V and W should be independent of the choice of basis of V and W. If you change the base you should not get a different tensor product for these vector spaces.

So, let me address that issue and see what happens when you change vector spaces, when you change basis of these vector spaces you get a different maybe a priori different notion of a tensor product, but these two edge turn out to be essentially the same thing, we can identify them.

So, one important thing to note, before we go on to identify them is that you have this you can define a map on basis. Let us say B which takes v i comma w j. So, you take this basis cross this basis and send it to v i tensor w j. This extends uniquely, to a bilinear map from V cross W to V tensor W.

So, v cross w is just the Cartesian product of the vector spaces V and W and we. So, we think of it as pairs one element in V and one element in W. And how do you define B from v cross w to be tensor w? So, what is a typical element of v cross w? So, if we have a typical element of v cross w is of the form v comma w where v is in v and w is in w.

So, if v is a vector capital V then v can be written as summation a i v i, i goes from 1 to m expanded in terms of our basis, w is b j w j, j goes from 1 to n. And then we can define B of v comma w to be. So, it is B of summation i equals 1 to n a i v i comma b j w ,j j goes from 1 to oops, this is m j goes from 1 to n. And using bilinearity you see that this is forced to be, summation double summation i goes from 1 to m j goes from 1 to n a i b j v i tensor w j.

So, this is the map B from v cross w to v tensor w. This is not a linear map, it is a bilinear map. And it is also not a subjective map not every vector in v tensor w is of the form B of v comma w and this B of v comma w is usually denoted by v tensor w and this is exactly what it is, ok.

So, what I am saying is that not every vector in v tensor w is of the form little b tensor little w for some vector v in little v in b big V and some vector little w in big W. I will leave it as an exercise for you to check that, when v and w are not say let us say they are two dimensional vector spaces ok. So, that is the definition of tensor product, but what happens if we start with a different base?

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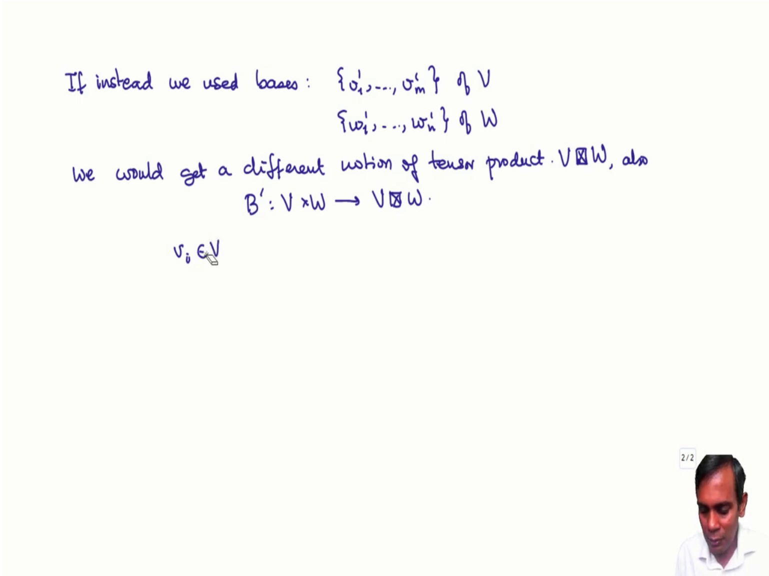

So, if instead we used a different basis. Let us say v 1 prime v m prime v of V and w 1 prime w n prime of W. Then we would get a different notion of tensor product, a priority different notion of tensor product. So, it would be let us call it v tensor w, but I will put a square bracket to say that this is with respect to square around the cross, to say that this is with respect to these new basis v prime and w prime.

And you would also get, bilinear map B prime from V cross W to V tensor W. So, this v square tensor w would be the vector space with basis v i prime tensor w j prime and b prime would be defined exactly how b is defined, except that we would use these other basis.

But what I want to say is that these two vector spaces are somehow these two notions of tensor product somehow the same. We can identify the vectors in 1 with the vectors in the other very naturally. So, to do this let us just firstly, take v i. So, this is an element of v and so this element can be expanded in terms of any basis of v.

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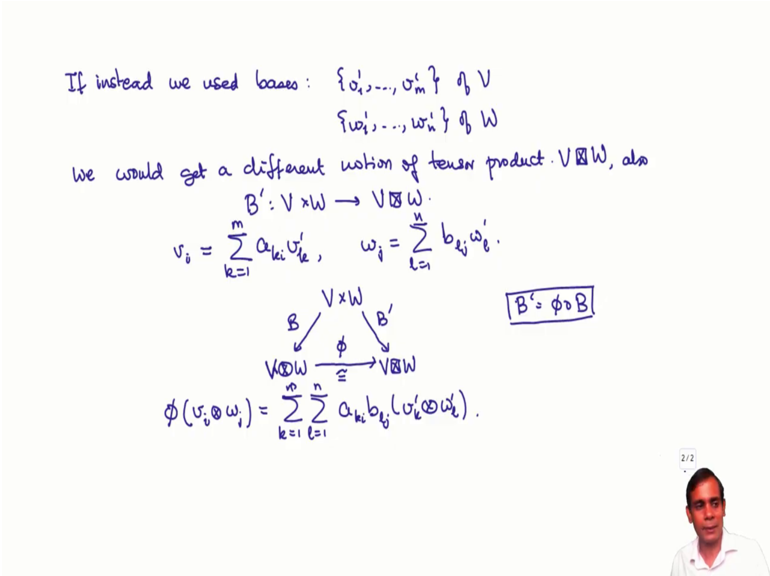

So, let us expand it in terms of this basis v 1 prime v 2 prime dot dot dot v m prime. So, this let us say is equal to k goes from 1 to m a k i p prime k and similarly let us say w j is summation l goes from 1 to n b l j w l prime, ok. So, now what do we have? We have I will draw it somewhat schematically, one thing we have in common for both these definition of tensor product V cross W that that is not changed.

And then we have these two notions of tensor product V round tensor V and then there is this bilinear map B and then we have this other B prime and we have V square tensor W. And I want to say that these two are related in the sense, I will give you an isomorphism phi from this vector space to this vector space which will make this diagram commute, in the sense that phi circle B is going to be B prime. And what is this v?

It is not difficult to write down, phi I will define it only on basis vectors of v i tensor w j and it is suggested by these expansions. It is just going to be summation k equals 1 to n, l equals k equals 1 to m, l equals 1 to n, a k i b l j v prime k tensor w l prime. This is sort of forced by requiring that B prime is phi circle B.

And so, this is a unique map from here to here and you can construct it is inverse in the same way. You use the expansion of the V prime basis and the W prime basis in terms of V and W basis respectively and construct a map going the other way and it is going to be an inverse for this map.

So, these two vector spaces turn out to be isomorphic, ok. We are not going to in any seriously serious way use these notions there is also something called the universal property of the tensor product, which I am will not go into right now. But in some sense tensor products are basis free, ok.

But for us it is enough to just think of the tensor product of two vector spaces in terms of basis you have a vector basis of these two vector spaces, then the tensor product is somehow a vector space, with basis sort of a Cartesian product of the basis of the two vector spaces. This notion of tensor product can be applied to several vector spaces.

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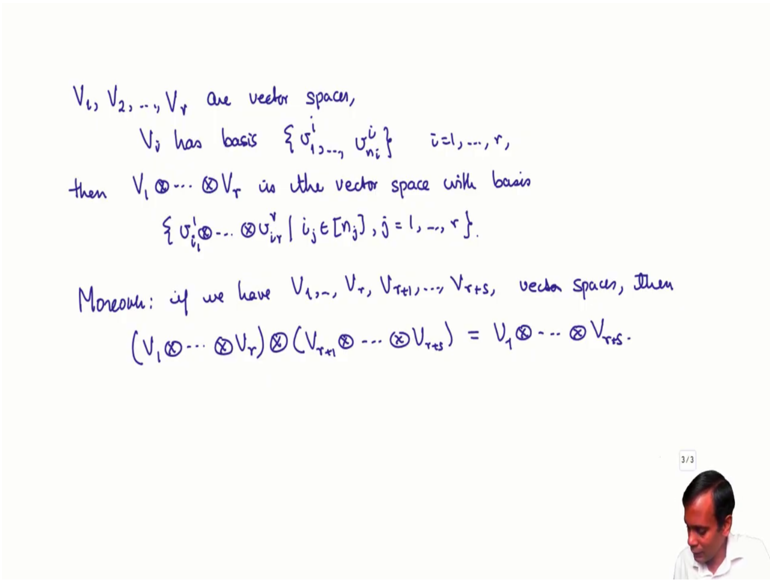

So, suppose you have V 1, V 2 up to V r are vector spaces. And let us say V i has basis v 1 1 let us say v i 1 up to v i n i. So, the ith vector space has dimension n i for i goes from 1 to r. Then V 1 tensor V r can be regarded as the vector space with basis v 1 i 1 tensor v r i r, where i j this lies between 1 and n j for j equals 1 to r. This is a vector spaces dimension is the product of the dimensions of the vector spaces V 1, V 2, V r.

And it is not difficult to see then with this definition that, if we have vector spaces V 1 up to V r and then a few more vector spaces V r plus 1 up to V r plus s. Then V 1 tensor V r, we take this tensor and then tensor it with the other tensor V r plus 1 tensor V r plus s. This is the same as the vector space V 1 tensor all the way down to V r plus s.

So, this is just you know, because these things essentially have the same basis think about it a little it is quite clear, ok. So, now let us apply this to a single vector space V, whose tensor product we take with itself repeatedly.

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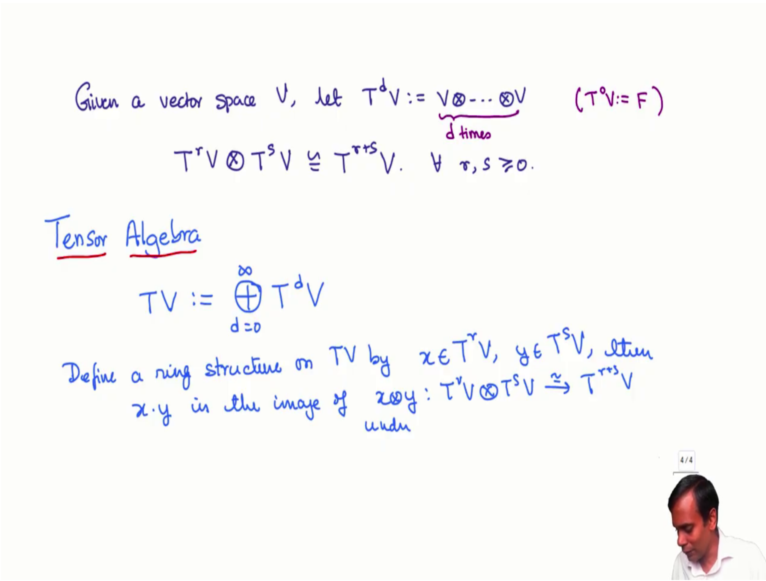

So, now given a vector space V, let T V let us say T d V we defined to be V tensor V tensor V tensor V, this taken d times ok. And what we have is that T r V tensor T s V is isomorphic to T r plus s V let for all r s greater than or equal to 0 at this point I should say something about, what is T 0 V.

So, T 0 V we will take it to be by definition just F. The one dimensional vector space F over F and now we are ready to define the tensor algebra. So, this is the algebra T V it is defined to be an infinite direct sum d goes from 0 to infinity T d V. So, this is the tensor algebra as an additive Abelian group and just an infinite direct sum of vector spaces. And how is product defined?

So, product is defined by linearly extending the map on graded pieces. So, what we do is define a ring structure on T V by, if you have x belongs to T r V, y belongs to T s V, then x dot y is the image of x tensor y from T r V tensor T s V to T r plus s V, this remember was an isomorphism.

So, the image of x tensor y under this isomorphism; of course, this is only defines multiplication of the elements which are in these some ends, but you can define it for those are called homogeneous elements. But you can define it for any element just by requiring this multiplication to be a bilinear map; we look at it very concretely using some examples.

But, firstly, what is the unit? So, we need to check that this is an algebra that it is associative additive and so on. I will not go into those steps you can try to check it yourself.

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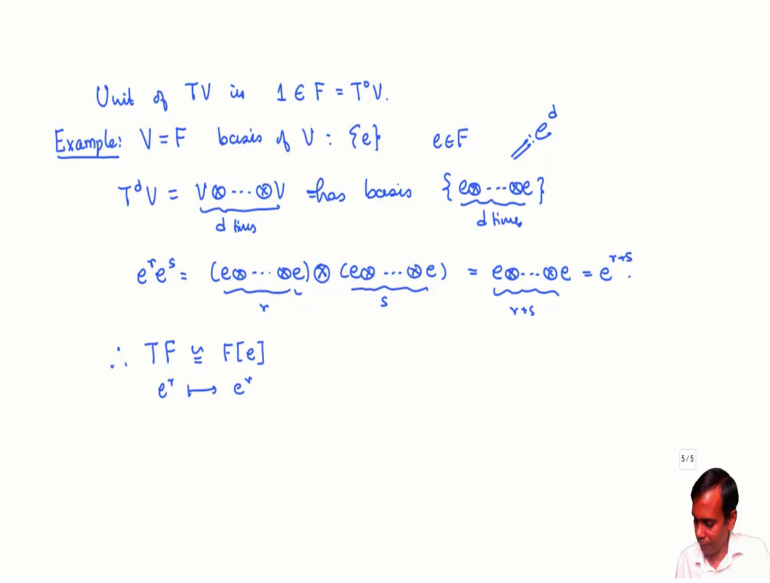

But, let me just point out that the unit of T V is the element 1 belongs to F which is T 0 V. So, the unit lives in the degree 0 part T 0 V. Let us look at the simplest example of a tensor algebra. Let us take V to be just the 1 dimensional vector space F over F and the other basis of V is let us say just pick an element. You could take the element 1 but, let us call it e. So, some element e of F if you want you can take the unit of F.

Then what is T d V? T d V is V tensor V tensor V taken d times and it is basis is just singleton 2. It is just the singleton set e tensor e tensor e tensor e taken d times. Let us call this e to the d, just define it to be e to the d. Then, e to the r times e to the s is the image of e tensor e tensor e r times and e tensor e tensor e s times in t r plus s.

So, that image is just obtained by doing this and so this is e tensor e tensor e r plus s times. So, that is e to the r plus s. So, what we have seen here is that, T V or let us say T of F is isomorphic to the polynomial algebra, in one variable which we can call e. This isomorphism is simply defined on basis elements by taking e to the r or rather e tensor e tensor e r times to e to the r.

So, the tensor algebra of a one dimensional vector space is a commutative algebra. It is just the algebra of polynomials in one variable.

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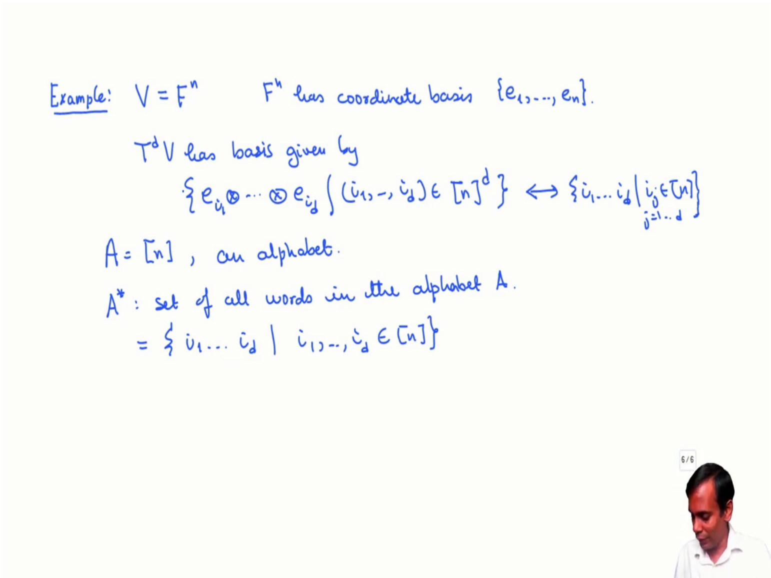

Now, let us look at the more general example, which is basically also the most general example for finite dimensional vector spaces. Every vector space is isomorphic to F to the n and every finite dimensional vector space is isomorphic to F to the n for some x.

So, let us just take V to be F to the n. So, then this has coordinate basis let us call it e 1, e 2, e n. So, e i is the ith coordinate vector, it has 1 in the eighth place and 0 everywhere else. And T d V as basis given by e i 1 tensor e i d, where i 1 up to i d this belongs to each of these lies between 1 to n and their d of them, ok.

So, this basis is in bijection with words. So, consider an alphabet A, so we will just regard this letter 1 to n as an alphabet. Like in English you have 26 letters in the alphabet, let us take a language where you have n letters in the alphabet. And a word in the alphabet is the set of all words in the alphabet A.

So, a word is just something of the form, i 1 i 2 i d where i 1 i d belongs to n. So, this basis is in bijection with i 1 i d, words of length d in the alphabet 1 to n for j equals 1 to d.

So, the basis of T d V is in bijection with words of length d in the alphabet 1 to n, you have seen these words in an alphabet before when you study free groups. So, before you constructed free groups you constructed this object called the free monoid.

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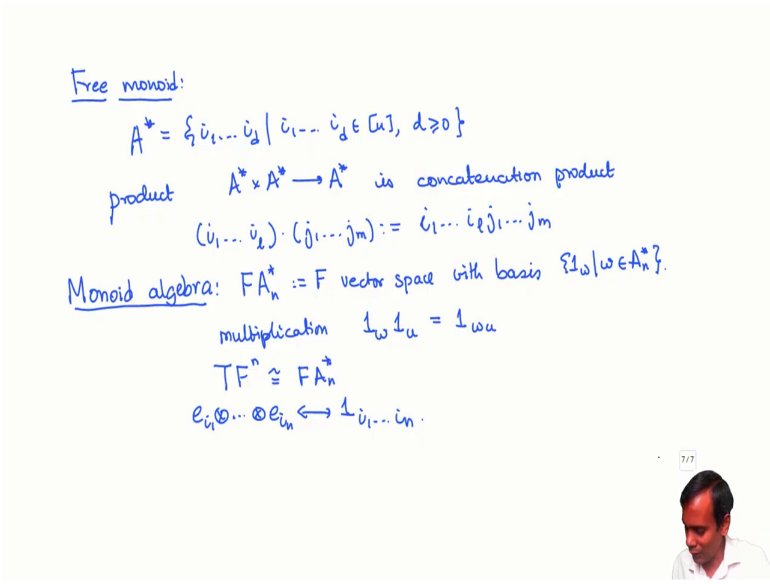

Let me just recall for you what that is. So, the free monoid. Well, a monoid is basically water down version of a group. It is a set with an associative binary operation and that binary operation must have a unit.

But the difference between a monoid and a group is that in a monoid, we do not require each element to have an inverse and while trying to construct the free group the first approximation that you saw was that of the free monoid and this is exactly what we are talking about here.

So, A n star as I said, this is going to be so yeah may be just a star is going to be all words of the form i d i 1, i d belongs to n and d can be greater than or equal to 0. So, if d is 0 then there is only 1 word namely the empty word of length 0 and the product operation on A star. So, it is a function from A star cross A star to A star is the concatenation product.

So, concatenation of words means, if you have two words you just write one word and then write the next word after it, ok. So, the concatenation of free and monoid is the word free monoid. So, it is just i 1 i l if you want to multiply it with j 1 j m it is just the word i 1 i l continue on j 1 j m. So, this is the word of length this is a word of length l plus m this is clearly associative it has a unit namely the empty word, but it lacks an inverse.

When you have a monoid like this, you can define an algebra. So, we will define the monoid algebra. I will call it a ring actually. Well, so I am using the word algebra to denote a ring whose additive rebellion group is actually an F vector space and whose multiplication is F by linear.

So, you have this monoid algebra F A n star. So, this is an F vector space with basis A n. Well, maybe I will give names to those things 1 sub w where w is a word in A n star. So, essentially the basis of this vector space is indexed by elements of the monoid A n star and multiplication.

So, addition is it is just a vector space. So, you use the vector space addition and multiplication is defined by bilinearly extending, if you have 1 sub w and you want to multiply it by 1 sub u, then it is just going to be 1 sub w. And then you this is defines bilinear map on basis elements and so you can extend it to bilinear map on vector spaces, right.

The it defines a function on basis elements and so you can extend it to a bilinear map from F A n star cross F A n star to F A n star. And this tensor algebra of V of F n, that we saw on the previous page is isomorphic to F of A n star via the isomorphism, e i 1 tensor e i n goes to 1 sub i 1 i. This is easy to see from the definitions.

So, the tensor algebra is the same as well the tensor algebra of the vector space F to the n is the same as the monoid algebra of the free monoid on n letters, having constructed the tensor algebra of a vector space a little more work can lead us to the construction of a very beautiful algebra called the exterior algebra. This algebra is the quotient of the tensor algebra by a certain two sided ideal.

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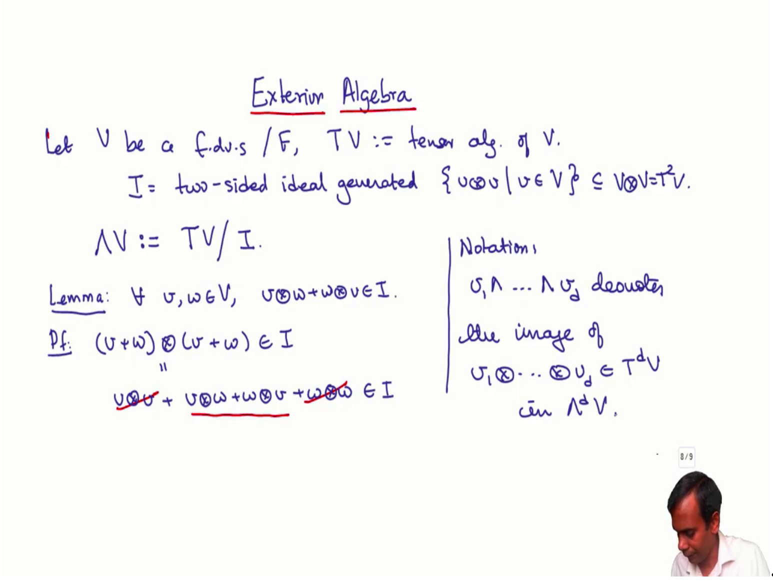

So, you start with a vector space V and you take it is tensor algebra. Let V be a finite dimensional vector space over F and T V be the tensor algebra of V. Now, inside this algebra I will take i to be the two sided ideal, generated by vectors of the form v tensor v where v is in V well tensors of the form v tensor. So, these are all inside V tensor V which is T 2 V, ok.

So, what do I mean by two sided ideal generated by a set? It simply the smallest two sided ideal that contains that set, to see that it exists you just take the intersection of all two sided ideals that contain itself. In intersection of two sided ideals is again a two sided ideal.

So, now I can define the exterior algebra wedge V is defined to be T V modulo I, ok. Now, before we start studying wedge V let us look at this ideal i a little more closely. The most fundamental fact about I is that I basically has vectors of this form for all v and w in V the tensor v tensor w plus w tensor v belongs to I. This is very easy it just follows from the fact that I is an additive Abelian subgroup of T V.

So, if you take v plus w tensor v plus w, sorry v plus w tensor v plus w that is; obviously, in I by definition of I it is some vector tensor with itself. But then, this is equal to v tensor v plus v tensor w plus w tensor v plus w tensor w. So, this is an I.

Now, among these four terms this term is in I. So, if I remove it also what remains will be in I and this term is in I. So, what is remains in the middle this is an I, which is exactly what I started off to prove. And we will use this observation when we are trying to understand the exterior algebra, ok.

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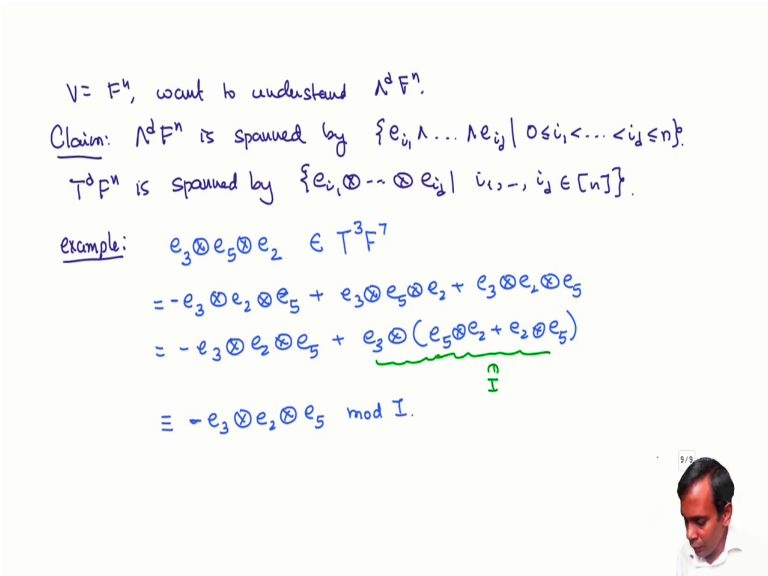

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So, firstly I will we want to understand a basis of wedge d F of a vector space. So, we are going to look at V equals F n now and we want to understand the basis of wedge d F n, ok.

So, firstly a first approximation of this is that, wedge d F n is spanned by, just to distinguish between vectors in T d n and wedge d n I will use a certain notation. I will say v i 1 v 1 wedge v d denotes the image of v 1 tensor v d belongs to T d V in wedge d V. So, what is wedge d V? It is just the image of T d V modulo I, ok.

I claim that wedge d F n is spanned by vectors of the form e i 1 wedge e i d, where i 1 we can take these things to actually be in increasing order. So, why is this? Basically, we are going to do certain moves. So, we are going to start with the general vector in. So, clearly you know T d F n is spanned by, e i 1 tensor e i d where we do not have any order on i 1 i d.

So, they are just elements between 1 to n. And then what we will say is that by modifying this element T d F n by elements of I we will be able to arrive at an element, where these indices are in strictly increasing order.

So, the basic idea is the following. So, let us look at an example of such a reduction. So, suppose you have e 3 wedge e 5 wedge e 2. This belongs to T 3 let us say F 7 sorry wedge 3 F 7. But let us not do that, let us write it as a tensor product ok, e 3 tensor e 5 tensor e 2 this belongs to T 3 F 7 fine.

And now, I can write this as e 3 tensor e 2 tensor e 5 plus e 3 tensor e 5 tensor e 2 plus e 3 tensor e 2 tensor e 5. I have not really done anything here; I have just added and subtracted e 3 tensor e 2 tensor e 5. But let us club these two things together; this is minus e 3 tensor e 2 tensor e 5 plus e 3 tensor e 5 tensor e 2 plus e 2 tensor e 5.

Now, in this previous lemma, we saw that v tensor w plus w tensor v is an I for any vector v vectors v and w in v. So, this thing belongs to I this thing belongs to I and but then you are left multiplying with something in e 3. So, this whole thing belongs to I. So, what we are saying is that this is congruent to minus e 3 tensor e 2 tensor e 5 modulo I.

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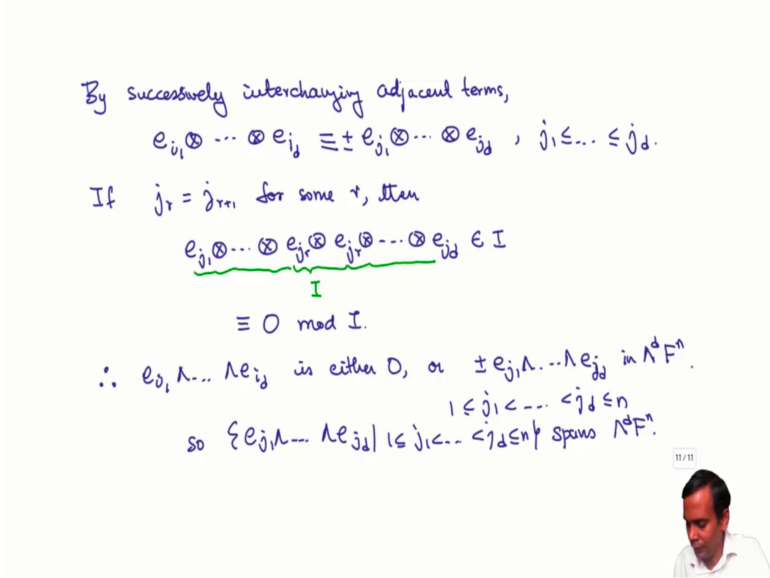

So, by successively interchanging adjacent terms, by successively interchanging adjacent terms. We can start with e 1 tensor e i d, we can show that using some sort of something similar to the bubble sort algorithm which you may have seen. By successively interchanging consecutive terms in a list, you can take the list and turn it into a sorted list in increasing order.

So, this is the same as e j 1 tensor e j d where these j 1 j 2 j d are the same as these indices e 1 e 2 i 1 i 2 i d, but they are written in weekly increasing order. But now suppose two of these indices are equal, if j r is equal to j r plus 1 for some r.

Then we have this thing e j 1 tensor and then we have e j r tensor e j r plus 1, which is also e j r and then some other stuff. But this stuff is in I just by definition of I and therefore, since I is a two sided ideal even if you multiply something in I on the left and right by things in I you get a two sided ideal an element of I.

So, this whole thing is in I, so this belongs to I. And so this is congruent to 0 mod I. So, the only terms that survive are where j 1 is strictly less than j 2 is strictly less than j d. Therefore, the images therefore, e i 1 then e i d is either 0 or plus or minus e j 1 wedge e j d, where j 1 is strictly less than j d in wedge d F n.

Since, the image of a basis is a basis this is of a basis of a vector space module or subspace is a generating set, this is a generating set. So, e j 1 spans wedge d f n here, this should be a plus or minus e j 1 e j d. Because each time you change interchange to consecutive tensors sign flips, ok. Now, we are ready to prove the somewhat more difficult result that the image.

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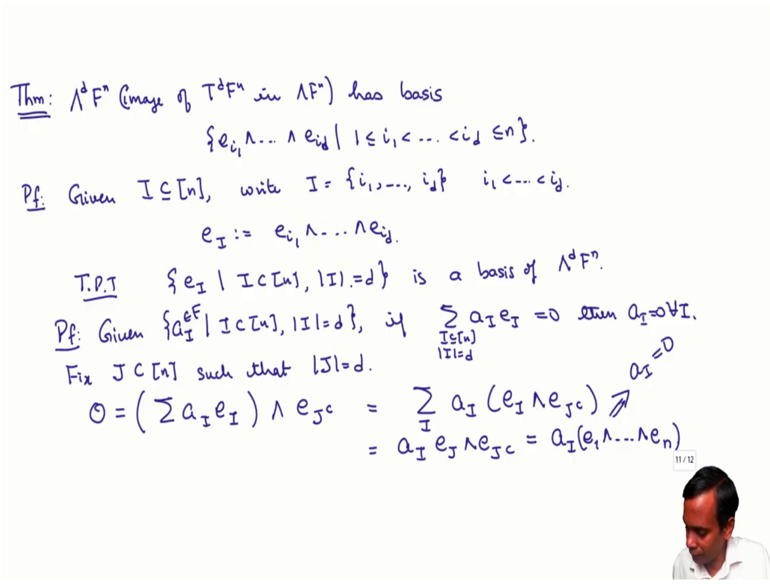

So, let us this wedge d F n we call is the image of T d F n in wedge F n is has a basis, e i 1 wedge e i n e i d where 1 less than or equal to i 1 less than i d less than or equal to n. This we can call a theorem and we have already seen that this set a spans wedge d F n and we want to show that these elements are actually linearly independent, in order to do that it is convenient to use somewhat set theoretic notation.

So, given a subset I of n write I in increasing order. Let us say I has d elements, then you write e I to be defined to be e i 1 wedge e i d. So, what I am basically saying is that collections of strictly increasing indices between 1 to n are the same as subsets of the set 1 to n.

And so what we want to show is that, e I I subset of n cardinality of I equals d is a basis of wedge d F n. Well, the proof goes as follows. Well what do we need to show? So, given scalars a I I in n I d and these a I should be in F, if summation aI e I equals 0, then a I equals 0 for all I, this is what we have to show. So, what we will do is, we will multiply this by another element.

So, let us say fix one of these ''s let us call it J, and now you take this thing summation a I e I and then you so this is I will leave out the index of summation here just gets cumbersome, which e j complement. So, let us look at this element. So, this is equal to summation over I using distributivity e I a I, then e I wedge e J complement.

Now, if I is not equal to J, then I being on subset of size d and J being a subset of size d as well. So, J complement is a subset of size n minus d, I will have at least one element in common with J complement, right. So, if I is not equal to J then I has an element in common with J complement and so e I wedge e J complement will be 0 except when I is equal to J.

So, this just becomes summation no summation, all the terms die out except a J e J wedge e J complement and by sorting out these elements we can say that this is the same as a I e 1 wedge e n. Firstly, one thing I should have said before is that, if n is greater than d then of course, all these if d is greater than n then of course, none of these elements e i can be non 0. So, we are restricting ourselves to the case where a d is less than or equal to n, ok.

So, we get this e i times e 1 wedge e 2 wedge e n and I want to show that this vector e 1 wedge e 2 wedge e n is non-zero. Because if I can show that then that means that well we know that summation a I e I is 0. So, then this would imply that a i is 0.

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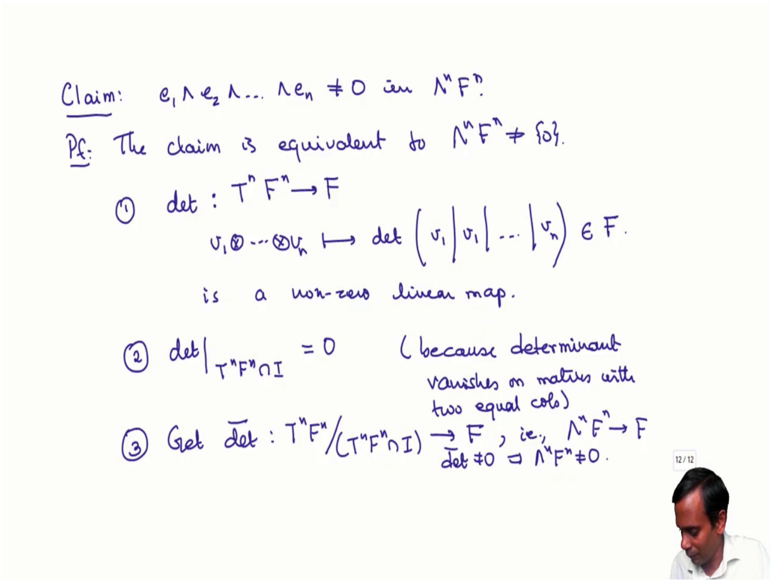

So, I want to show that a 1 sorry e 1 wedge e 2 wedge e n is non-zero in wedge n F n. So, how do you prove this? So, this is a very interesting proof, it actually uses the notion of determinant. So, what I am going to do is I am going to show that so, this is equivalent to saying that, the claim is equivalent to saying that wedge n F n is not a 0 dimensional vector space.

Why is that? Because we know that wedge n F n is spanned by this vector e 1e 2 wedge e n, we saw that in the previous lemma. So, if this vector is 0, then wedge n F n will be 0. So, this claim is equivalent to showing that wedge n F n is not 0, but I will prove this that wedge n F n is not 0 using determinant.

So, we have this map determinant and it is a map from matrices n by n matrices to F, but I will think of it as a map from T d F n to F. How does it work? If you have a vector v 1 tensor v d, then it goes to determinant of the matrix whose columns are the column vectors v 1 v 2 sorry, I want this n n, v n because then this is a square matrix and we can take its determinant.

So, this gives rise well so firstly, you know that the determinant is multilinear in the column. So, this gives rise to a well defined linear map from the tensor product the nth tensor power of F n to F. And this is a non zero map, because there are non-zero linear map. Because well there are matrices with non-zero determinants such as the identity matrix and the other fact, that we know is that determinant restricted to T n F n intersection I is 0.

Why is that? Well, this just corresponds to the fact that even matrix two of whose columns are equal, then its determinant is 0. So, this determinant will vanish on any matrix which is in T n F n intersect I; because determinant vanishes on matrices with two equal columns. So, then what we know is that, so determinant induces determinant bar from T n F n mod T n F n intersect I to F and this is non-zero.

So, determinant gives rise to a non trivial linear functional from T n F n mod T n F n intersect I to F, but this is the same as wedge n F n to F. So, determinant gives rise to a non trivial non zero linear functional on the vector space wedge n F n. But if the vector space wedge n F n were 0, it could not possibly have a non zero linear function. Therefore, we conclude that wedge n F n is not 0 and therefore, e 1 wedge e 2 wedge e n is a non zero vector in wedge n F n.

So, what we get is that 0 is a I times a non zero vector therefore, a i is equal to 0. And hence these so we can do this for every I and hence we see that this e i as I runs over subsets of size d in n forms a basis of wedge d F n.

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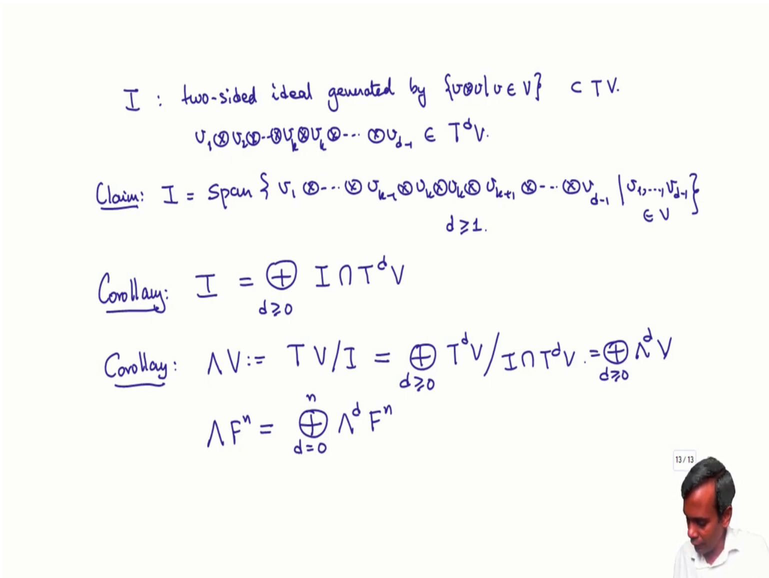

Let us take a closer look at the ideal I. So, the ideal I is the two sided ideal generated by v tensor v v n v. This is a two sided ideal in T V. And of course, so this ideal contains all vectors of the form v tensor v, but it also contains tensors of the form v 1 tensor v 2 tensor and then at some place you have v k tensor v k and then you have tensor v d minus 1, this would be a tensor in T d V.

I claim that the ideal I is actually the span of v 1 tensor v k minus 1 tensor v k tensor v k v d minus 1 where, v 1 v d minus 1 belongs to v. As d runs over all yeah let us just say d greater than or equal to 1. And this is easy to see because firstly, clearly I would contain all these vectors because it contain v k tensor v k and it is closed under left and right multiplication by elements of T V and secondly, you just see that this span is actually an ideal, so a two sided ideal.

And so, I is this span, a consequence of this is that I is the direct sum over d greater than or equal to 0 I intersect T d V, that means I itself is a sum of its intersections with the different degree decomponents of tensors and a corollary of that, is that wedge V which is T d V not T d V, T V mod I is equal to direct sum d greater than or equal to 0 T d V mod I intersect T d V.

And this we have seen is wedge d. In fact, wedge d V is 0 if d is greater than or equal to dimension V and so what we have is, wedge F n is equal to this that sum d equals 0 to n, wedge d F n and this wedge d F n is the span of subsets of size d of F n.

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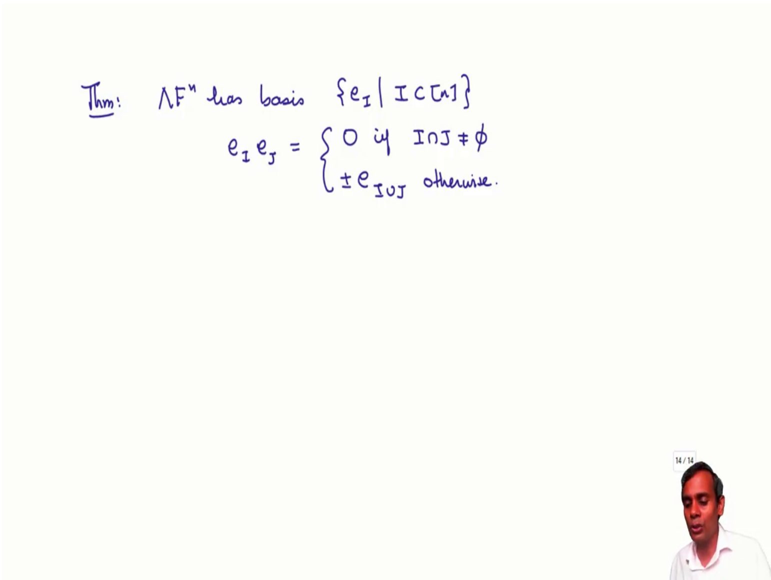

So, what we have is that finally, wedge F n has basis e I I subset of n. Now, these subsets could have size anything from 0 to n and you can write down the product of two such basis elements by taking the words. So, if you have e I wedge e J, then you can write down the words the elements of.

So, this is going to be 0 if I intersect J is non empty and if I intersect J is empty, then this will be plus or minus e of I union J otherwise. The sign has to be worked out when you concatenate the words corresponding to I and J and then you try to sort them back into increasing order, you have to see how many times you have to switch successive terms. So, that is the algebra the exterior algebra also known as the Grasman algebra.

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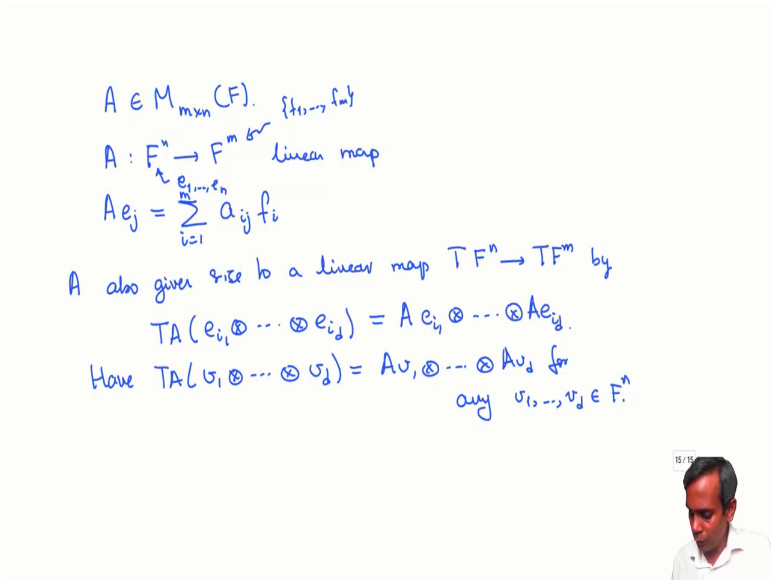

Now, suppose A is a matrix with entries in F, let us say it is an m by n matrix. Then A defines a linear map which I will also denote by A from F n to F m and what this linear map does is A e j. So, let us take for F m the basis e 1 e n and let us take for F m the basis F 1 up to F m and a e j is summation i goes from 1 to m a i j f i. So, this is the usual way in which we think of matrices as linear maps.

Now, this A also gives rise to a linear map, T F m, T F n to T F m as follows. I will define it on basis vectors, T A of e i 1 tensor e i d is T e i 1 tensor not T A e i 1 tensor a e i d. This linear map has the property that T A of v 1 tensor v d is A v 1 tensor A v d for any v 1 v d in F m, ok.

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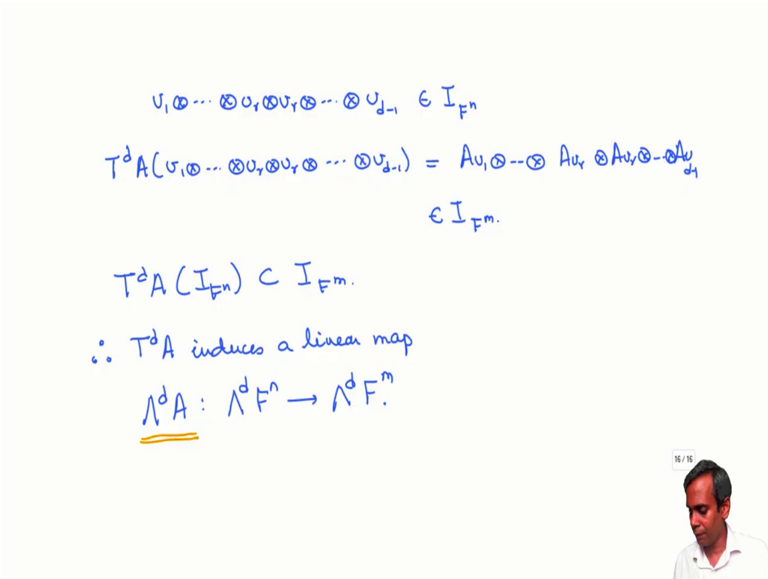

So, one interesting feature of this map is that, if you take something like v 1 tensor v r tensor. So, you have this thing repeated, then this is these kinds of vectors span the two sided ideal whose by which we take the quotient to get the exterior algebra.

So, this is an ideal in this is a vector in I just to be specific here I will say I F n. And if you apply T d of A to this, then this will be A v 1 tensor and again you will have A v r tensor A v r tensor A v d minus 1 which belongs to I F m.

So, what we have is that, T d A takes I F n to I F m and therefore, T d A induces a linear map which I will denote by wedge d A from T d A mod I F m to T d A mod I F m. So, this is a map from wedge d F m to wedge d F m. So, what we have seen is that, every matrix also induces a linear map on the exterior algebra. So, the question I want to ask now is what is the matrix of this linear transformation wedge D A?

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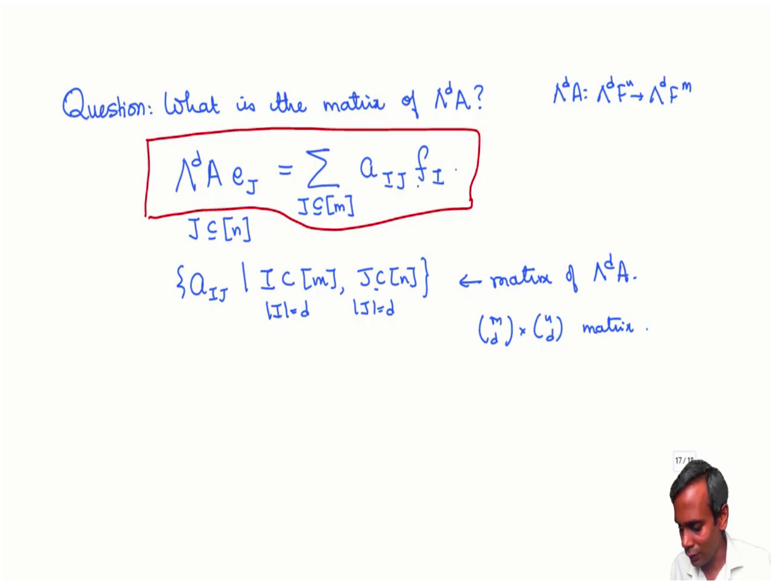

What do I mean exactly by that? What I mean is that, you take this wedge d A and so this wedge d A goes from wedge d F n to wedge d F m and so we take a basis vector of wedge d f n.

So, we take so we had taken for the basis of F n we are taking e 1 e 2 e n. So, we take a basis vector for this would be of the form e J for some J subset of n. And then this is a vector in wedge d F m and so we expand it in terms of the basis that we have for wedge d f m. So, this is going to be summation over I subset of m a I J f I.

So, this system of constants a I J indexed by I subset of m J subset of n, I will call refer to this as the matrix of wedge d A. If you somehow order the subsets of m and the subsets of n maybe here these are size of I is d size of J is d. So, if you take all the subsets of size d and order them somehow, then this becomes you can really write this as a matrix whose the number of rows will be m choose d and number of columns will be n choose d.

So, you can think of this is an m choose d times n choose d matrix, but let us just think of it as a system of coefficients defined by this equation here, for every subset g of n ok. So, to figure out this matrix is not very difficult, so let us just write out, unwind the definitions.

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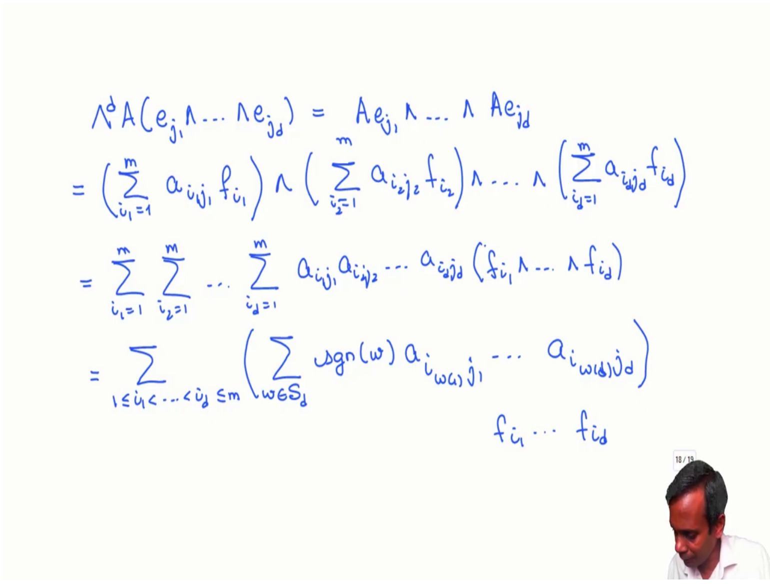

So, what happens if we start with wedge d A and then apply it to e j 1 wedge e j? So, we started with a j which is subset of n d, well by definition this is going to be A e j 1 wedge A e j d ok. Now, let us expand all these things.

So, this first thing is going to be summation i 1 equals 1 to m a i 1 j 1 f i 1 wedge i 2 equals 1 to m a i 2 j 2 f i 2 and so on, up to i d equals 1to m a i d j d f i d. And if we pull out all the constants we will get this multiple sum i 1 equals 1 to m i 2 equals 1 to m i d equals 1 to m.

And then we get this product of coefficients i 1 j 1 a i 2 j two a i d j d and then finally, we get our vectors which are f i 1 wedge f i d. But I want to these things are not linearly independent, if I take some of these vectors and if two of them if two of these I s are equal then it is going to be 0 in wedge d of f m and also if these are not in increasing order, when I reorder them they will there will be a sign change.

So, if I account for that, I can rewrite the sum as sum over 1 less than or equal to i 1 less than i 2 less than i d less than or equal to m and then all the possible reorderings of the vector. If two indices are the same then this thing is 0, so I do not worry about those cases. But all possible reorderings means I need to go over permutations in S d.

And every time I interchange two of these factors, there is a sign change and that corresponds to the sign of the permutation w and S d and then I get a i w 1 j 1 a i w d j d. But this and yeah this is the coefficient of the term f i 1 f i d. So, this is exactly what we were looking for.

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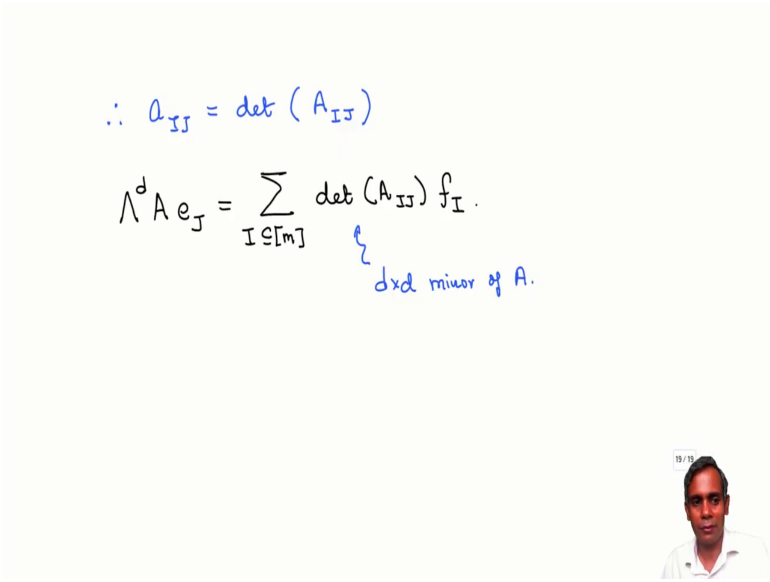

So, we are looking for the coefficient of such a term therefore, a i j is precisely the determinant. This thing here is a determinant it is a determinant of what? A sub matrix of the original matrix A, it is the determinant of the sub matrix of A obtained by choosing rows according to the subset I and choosing columns according to the subset J.

So, what we have is the formula, wedge d A applied to e J is summation I subset of m determinant of A I J f I. So, this is the formula which shows how wedge d A acts on wedge d F m. So, this determinant is what is often called a d by d minor of the matrix A. So, what I am saying is that, the matrix entries of the linear map wedge d A are the d by d minors of the linear map A itself.

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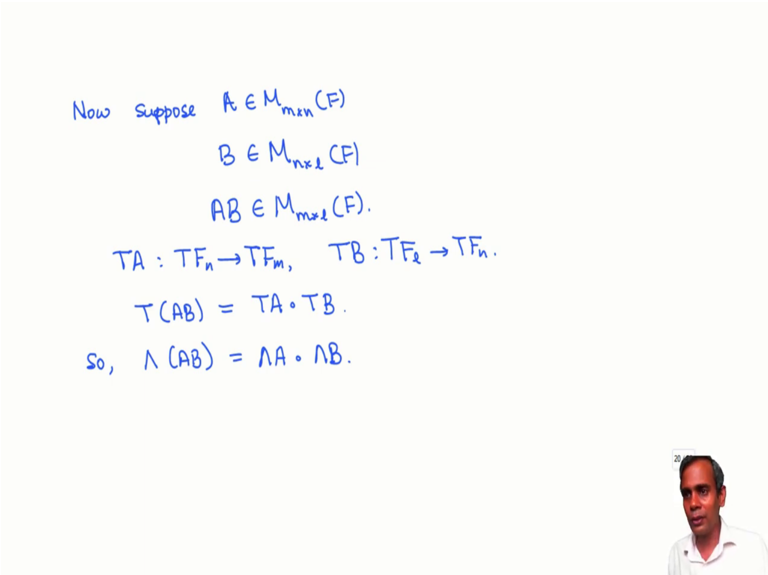

Now, suppose I have two matrices. A belongs to M m by m F, B belongs to let us say m by n F and b belongs to m n by l F, where m n and l are three possibly different positive integers. Then you can multiply these two matrices and A B will belong to A M by l F.

And so we have T A from T F n to T F m, we have T B from T F l to T F n and what is clear from the definition of T A and T B is that T of the product matrix A B is T A composed with T B, just see how it acts on basis vectors. And so, also wedge of A B is wedge of A composed with wedge of B, ok.

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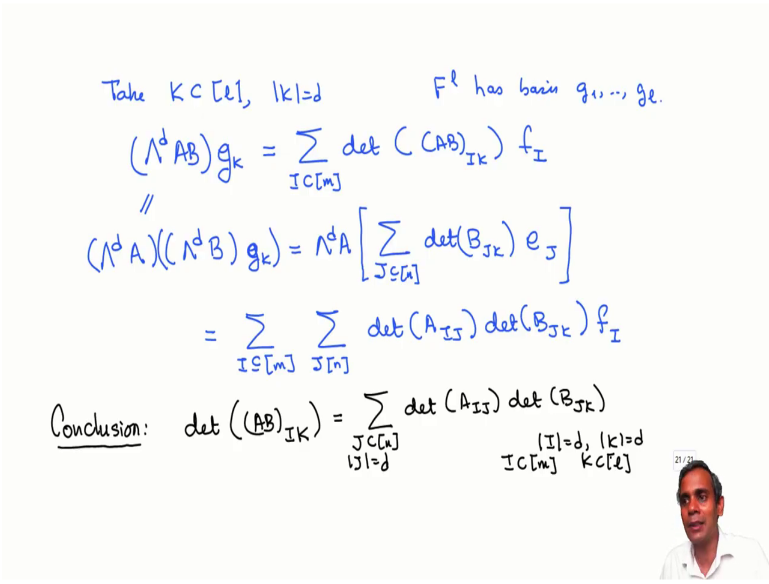

So, now let us see how these are related to the matrices. So, suppose you take some K subset of the set 1 to l and the size of K is d and we ask what is wedge d A B when it is applied to e K? This is on the one hand given by K by K sorry d by d minors of the matrix A B. So, this is summation I subset of m determinant of A B I K f I, on the other hand it is a composition of matrices of minors.

So, on the other hand this is wedge d A applied to wedge d B applied to e K, but this is just wedge d A applied to summation J subset of n determinant of B J K the minor of B corresponding to rows J and columns K e J just by what we did earlier. And then, we apply wedge d A to this and we get summation over I subset of m summation J subset of n and then we get determinant of A subscript I J determinant of B subscript J K F yeah maybe, I should not have called this e k let us call it g k.

So, let us take F to the l has basis g 1 g 2 g l then yeah F i. So, the conclusion that we draw from all this is that, the determinant of you take the product of a matrix and take its I Kth d by d minor where I and K are subsets of the rows and columns of you know I is a subset of K is a subset of l and I is a subset of 1 to m.

Then you get that, this is sum over all subsets J of n of size d, determinant of a I J determinant of B J K for all I of size d, K of size d I subset of m K subset of l. This beautiful identity involving minors of a product of expressing the minors of a product of two matrices in terms of the minors of the matrices themselves.