

**NPTEL**  
**NPTEL ONLINE COURSE**  
**Introduction to Abstract**  
**Group Theory**  
**MODULE – 01**  
**Lecture – 03- “More examples of groups”**  
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So it's important to keep in mind that when we specify a group, we specify, we must specify a group operation, okay. So let's look at couple of more examples of groups, before we study properties of this, so I want to generalize the definition of  $S_3$  that we considered, so recall so recall that  $S_3$  was the set of bijections of the set  $\{1, 2, 3\}$ , and  $S_3$  under composition is a group, okay. We can generalize this, there is nothing special about 3. So now I define  $S_N$  for any positive integer  $N$ , so  $N$  is the positive integer,  $S_N$  is the, by definition the set of bijections of the set  $1, 2, 3$  up to  $N$ . And recall the exercise I gave in the first video, if you compose two bijections you get another bijection, so that exercise as that  $S_N$  is closed under composition, just like  $S_3$  was closed under composition, so  $S_N$  is closed under composition, it has an identity element namely, the identity function, remember this is the function which sends 1 to 1, 1 to 1, 2 to 2, 3 to 3, and so on  $N$  to  $N$ , it is the identity function because when you compose any function with this function you get that function back.

Similarly every element, okay this requires a little bit of thinking, but we can prove this as every element has an inverse, meaning if you give me any bijection of  $S_N$ , there is another, sorry, any bijection of  $1, 2, 3$  up to  $N$ , you can just construct another bijection such that composition gives you identity. For example if one bijection sends 1 to 2, you can simply send 2 to 1 under a new bijection, so the composition will send 1 to 1, so every element has an inverse and composition as always is associative, so in other words  $S_N$  under composition is a group, okay,  $S_N$  is a group under composition. We are going to spend in one of the future weeks, a lot of time on understanding the group  $S_N$ , it is called, it has a name it's one of the important groups in group theory, it is called the “symmetric group on  $N$  letters”, it is called the symmetric groups on  $N$  letters, letters being  $1, 2, 3$ , up to  $N$ , so it is a group on those letters, so it is called the symmetric

groups on  $N$  letters and as I said a very important group in the theory of abstract groups, and we will study this later in more detail, okay.

So one more example that I want to discuss, okay, so just to recap a little bit we have several examples of groups now, we have groups of numbers like  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ ,  $\mathbb{C}$ , positive rationals, nonzero rationals, positive reals, nonzero reals, nonzero complexes, under either multiplication or addition

groups, similarly we have symmetric groups, we studied  $S_3$  in the previous video in detail, now I define  $S_N$  in general that's a group, we also talked about group of rotations of an equilateral triangle.

And another group that I want to discuss this is also an important group for us, we will refer to this again in future, let's look at, I'm going to give you a set of complex numbers, so fix a positive integer, positive integer  $N$ , okay, so  $N$  is a positive integer, and I'm going to define a complex number  $\theta_N$  to be  $\cosine + I \text{ sine } 2\pi/n$ , okay, so complex numbers are of the form  $A+IB$ , where  $A$  and  $B$  are real numbers, so here I'm taking  $\cosine 2\pi/n + I \text{ times sine } 2\pi/n$ , so if you know little bit about complex numbers, it's not important in general for the course but in this example you need to know this, if you have a complex number of this type when you take the  $n$ th power of this, in other words I'm taking  $\cosine 2\pi/n + I \text{ sine } 2\pi/n$ , so I'm taking the whole power  $N$ , this by the properties of complex numbers simply happens to be  $\cosine 2\pi + I \text{ sine } 2\pi$ , so what happens is, you multiply  $N$  with this  $2\pi/n$ , with this  $2\pi/n$ , but  $\cosine 2\pi$  is 1,  $\text{sine } 2\pi$  is 0, so this is 1, okay. So  $\theta_N$  power  $N$  is 1, so we say that  $\theta_N$  is a "primitive  $n$ th root of unity", okay, so it's a  $n$ th root of unity because  $\theta_N$  power  $N$  is 1, and it's primitive because  $\theta_N$  power  $M$  is not equal to 1, if  $M$  is positive number less than  $N$ , so it's not an  $M$ th root of 1 for  $M$  less than  $N$ , it's the smallest positive integer such that  $\theta_N$  power of  $\theta_N$  is 1 is  $N$ , okay, so this is what makes it primitive. Again this is, if you do not know complex numbers you can disregard this example, but now let me define  $G$  to be  $1 \theta_N, \theta_N^2, \theta_N^3, \dots, \theta_N^{N-1}$ , okay, here powers are, so I'm here the operation is, is simply multiplication of complex numbers, okay, so I'm only, when I write  $\theta_N^2$ , I mean  $\theta_N$  times and  $\theta_N$ ,  $\theta_N^{N-1}$  is  $\theta_N$  multiplied with itself  $N-1$  times, and remember when I do  $\theta_N^N$  I get 1 back, so this is what I get, so this I claim under multiplication is a group.

Let's spend two minutes on why this is the case? Why is it a group? Because is it closed under multiplication, if I multiply  $\theta_N^i$ , what are the elements of this?  $\theta_N^i$  is an element times,  $\theta_N^j$  that's another element, what is  $\theta_N^i$  times,  $\theta_N^j$ , it is

$\theta^N$  power  $i+j$ , but because  $\theta^N$  power  $N$  is 1,  $\theta^N$  time power  $i+j$  will be equal to one of these, because I mean it's easiest if I say by example, if  $\theta^N$  power  $N-1$  times  $\theta^N$  squared is  $\theta^N$  times  $2N$ , sorry,  $N+1$  but this is  $\theta^N$  times  $N$ , power  $N$  times  $\theta^N$  which is  $\theta^N$ , okay, so this is again in the group, in the set  $G$ , so it is closed? Yes, identity? Yes, 1 is there, right that's identity, inverse? So I'm asking again, remember what is a group? It's closed, binary operation is closed that we checked here, quickly checked, identity is there, yes, multiplication is certainly associative so again that we don't need to check separately, is there an inverse? Yes again, because what is the inverse of  $\theta^N$  power  $i$ ? That is simply  $\theta^N$  power  $N-i$  because if I do  $\theta^N$  power  $i$ , times  $\theta^N$  power  $N-i$  I get  $\theta^N$  power  $N$  which is 1, okay, so inverse also exists, so this is a group.

Okay, so this group  $G$ , the group  $G$  being what I define earlier in the previous page, and  $X$  is the multiplication is a group, this cross is a multiplication is a group, is a group it is called, it's an important group, it is called the group of  $N$ th roots of unity, it consist of  $n$ th roots of unity, remember that every element of this group is an  $N$ th root of unity, because if you do  $\theta^N$  power  $i$  power  $N$ , then this is certainly equal to  $\theta^N$  power  $N$  power  $i$ , because this is just  $iN$  and I can pull out  $i$ , and this is  $1$  power  $i$ , and this is 1, so everything here is an  $N$ th root of unity, however everything here is not primitive  $N$ th root of unity, so it's called the group of  $N$ th roots of unity.

A familiar example that you all know is, if you take  $N = 4$ , remember  $\theta^4$  is  $\cosine\ 2\pi/4 + i\ sine\ 2\pi/4$ , okay, so my notation I realize now is a bit confusing, here  $I$  refers to the square root of  $-1$ , here  $i$  is just an index, so what is  $\cosine\ 2\pi/4$  that is just 0, that's  $\cosine\ \pi/2$  so this is simply  $I$ , so  $\theta^4$  is a primitive,  $4^{th}$  root of unity which is  $I$ , so what is  $G$ ? So I should really call these groups  $G_N$ s, so I should maybe call this  $G_N$ , so  $G_4$  is 1,  $\theta^4$  which is  $I$ ,  $\theta^4$  squared, what is  $I$  squared?  $I$  squared is  $-1$ ,  $\theta^4$  cubed which is  $-I$ , and that's all, so this is simply  $\{1, I - 1, -I\}$ , so this is the group of  $4^{th}$  roots of unity.

And one more point that I'll make which is useful sometimes to keep in mind is all the roots of unity for any  $N$  are on the unit circle, so if you look at the unit circle this is 1, that's a first root of unity, this is  $-1$  you have  $I$  here and  $-I$  here, in more general you have  $\cosine\ 2\pi/N + i\ sine\ 2\pi/N$ , that's this name, okay, and this angle is  $2\pi/N$ , okay, so these are elements on the unit circle, so this is another example, so again in this video also I'm basically trying to give you examples of the groups, we have various groups now, we have groups of numbers, we have groups of bijections of a given set, and now I have a group of  $N$ th roots of unity.

One more important example is the group, groups of matrices, so groups of matrices are good examples of groups in future for us, and I'm going to spend just a couple of minutes describing them, and then as and when needed we will discuss more of these, so if you know what groups are, matrices are, let's say you fix two positive integers  $M$  and  $N$ , and we consider  $M \times N$  matrices, we have to specify where entries come from, so let's just take  $\mathbb{R}$ , so these are, matrices where entries are real numbers, okay, so the set I think I called them  $M, N$  and so let's just take the notation is  $M \times N(\mathbb{R})$  is the set of all  $m \times n$  real matrices, okay, so these are, for example this is  $M$  rows and  $N$  columns, so we can there is an addition here, right, so you can add matrices entry wise, so  $M \times N(\mathbb{R})$  is a group under addition, this is easy to check, for example if you take, let's say  $2 \times 3$  matrices, so you have  $A_{11}, A_{12}, A_{13}, A_{21}, A_{22}, A_{23}$ , you can always add two matrices  $B_{11}, B_{12}, B_{13}, B_{21}, B_{22}, B_{23}$ , there is no, the fact that it's a matrix is irrelevant here, it's just about arranging certain real numbers, so you just add them component wise, okay, so it's just the same. Okay, so because addition under reals is a group you can quickly check that, this is also a group the identity element for example will simply be  $0, 0, 0, 0, 0, 0$ , so there is no surprise, this is simply several copies of  $\mathbb{R}$ , so that's not surprising.

On the other hand you can also ask matrices form a group under multiplication? that also you can ask, because there is a way to multiply matrices, immediately you have to impose certain restrictions, because you can't multiply a  $2 \times 3$  matrix with a  $2 \times 3$  matrix you can't multiply, right. In order to multiply 2 matrices the number of columns in the first one must equal the number of rows in the second one, so restrict our attention to square matrices, remember that in terms of the language that we are developing, multiplication is not a binary operation on the set of matrices, if I don't specify that they are square matrices, because I cannot combine  $2 \times 3$  matrices under multiplication, but now I overcome that difficulty by considering square matrices, here at least we can multiply and we do get a binary operation, so we do have a, because any two square matrices can be multiplied to get another square matrix. However this is not a group because not every matrix has an inverse. If you have studied matrix theory earlier you will see that there are many matrices which do not admit inverses under multiplication, in order to do that the matrix has to be what we call a an invertible matrix, so if you look  $MN(\mathbb{R})$  to be all  $N \times N$  matrices, over  $\mathbb{R}$ , if you do that, we agree I hope, that  $MN(\mathbb{R})$  is not a group under multiplication of matrices, because even though it is a binary operation, in other words multiplication is closed, not every matrix has an inverse, but if you look at, so the reason is let me just write it here because inverses do not exist in general, okay, so that is the reason why it's not a group. But we will restrict our set of matrices in order for inverses to exist, so now define, it's usually denoted  $GLN(\mathbb{R})$ , to be the set of invertible  $N \times N$  matrices over  $\mathbb{R}$ , okay, so we have artificially restricted our attention to only those which admit inverses, then  $GLN(\mathbb{R})$  is a group, okay, so again this requires a small verification which I'll leave for you to do, but I'll just run through what you required to do. We need to check that multiplication is a binary operation, certainly it is, if you multiply 2 invertible matrices you do get another invertible matrix. Is there an identity element in

this? Yes, there is because the identity matrix is invertible. Are there inverses? Yes, because we have restricted our attention to only those matrices which admit inverses, so every element in  $GLN(\mathbb{R})$  has, this is  $GLN(\mathbb{R})$ , every element in  $GLN(\mathbb{R})$  has an inverse and matrix multiplication is associative that you have studied in some matrix theory course, so  $GLN(\mathbb{R})$  has all the required properties of a group, so it's a group, okay. This is an important group for us, so I'll end the video now, but to recall what we have done today in this video is first we started with the definition of a group, and then have seen a series of examples of groups which will be very important to understand in future when we study properties of groups. So in my next video I'm going to quickly recall all the groups that we have learned and then study some properties of groups.