Numerical Analysis.
Professor R. Usha.
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Indian Institute of Technology, Madras.
Lecture-48.
Practical Problems.

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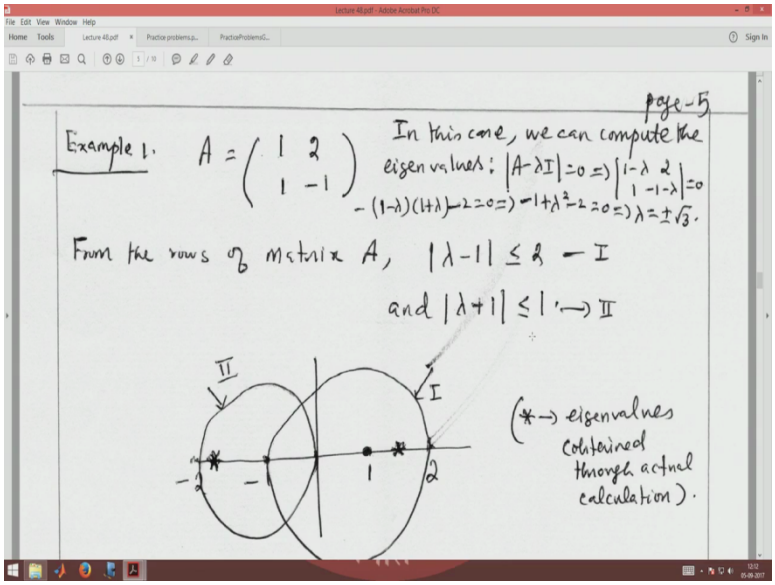

Good morning everyone, in the last class we learned how to compute Gerschgorin bounds which help us to specify the location of the eigenvalues of a given matrix A. So now we shall consider some examples and see how we can obtain the Gerschgorin bounds. Let us start with a simple example of a 2 by 2 matrix A. So here the matrix A has entries 1, 2, 1, -1 and we can actually compute the eigenvalues in this case. The eigenvalues are given by determinant of A - lambda I equal to 0. So if you compute the determinant of this matrix and equate it to 0 and solve the corresponding characteristic polynomial for lambda, then you get lambda to be + or - root 3.

So you know that one of the eigenvalues of matrix A is + root 3 and the other eigenvalue is - root 3. Now let us apply the Gerschgorin's theorem results and see what information that we get about the location of these 2 eigenvalues. So the 2 eigenvalues + - root 3 have actually been marked on the real line, they are real eigenvalues, I have marked them here by means of stars. This is lambda equal to - root 3 and this is lambda equal to + root 3. Now let us use Gerschgorin's theorem. So from the rows of matrix A I consider the 1st row, this is A11 and Gerschgorin's theorem tells us that modulus of lambda - A1 1 which is 1 is less than or equal to sum of the absolute values of the other entries in that row.

So here we have only one element which is 2, so the 1st result else that absolute value of lambda -1 is less than or equal to 2, this gives us one circle and some information about an eigenvalue of the matrix A can be obtained from here. Then taking the 2nd row we have lambda - of -1 which is A22 that this lambda +1 in absolute value must be less than or equal to the sum of the asset values of the other entries, hair in this case it is 1, so modulus of lambda +1 N equal to 1 is going to be the 2nd Gerschgorin circle that we have.

So we observe that for the matrix A we have considered, there are now 2 discs in the complex plane and the Centre of the discs are A 11 and A22 which appear in the matrix A. So each centred on one of the diagonal entries of the matrix A.

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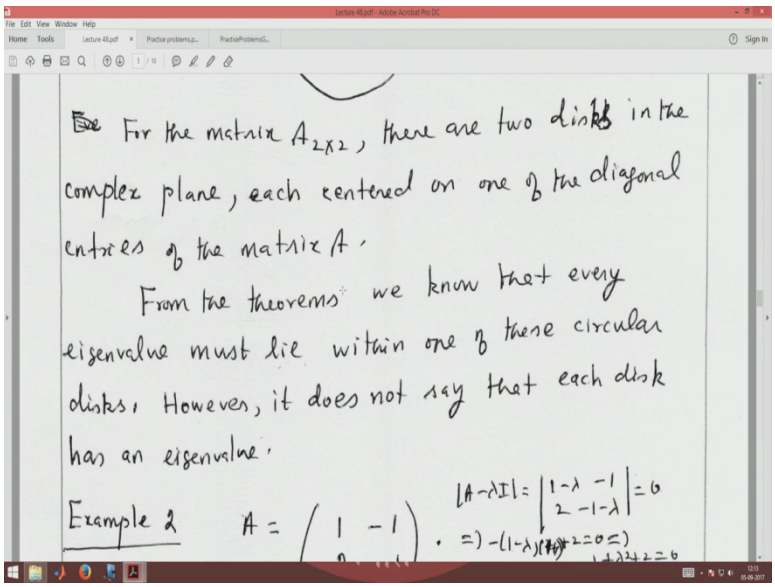

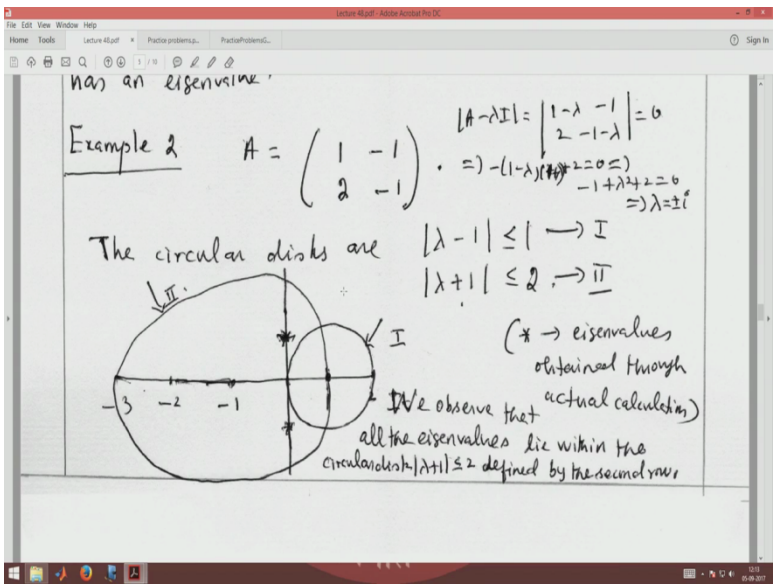

And from the theorems we know that every eigenvalue must lie within one of these circles and however we do not have any information about the fact that the each disc contains an eigenvalue. So this information is not obtained from Gerschgorin theorem. So let us see another example. So in the 2nd example, again I consider a 2 by 2 matrix, as before we can consider the eigenvalues of this matrix A, they can be actually computed by solving the characteristic polynomial even by determinant of A - lambda I equal to 0 and we observe that the eigenvalues are given by lambda equal to + - I.

So those eigenvalues are actually plotted here, this is lambda equal to i had this is lambda equal to - i, they are marked by star here in the complex plane. Now let us see what the Gerschgorin's theorems tell us. The Gerschgorin's theorems tell us that the circles within which the eigenvalues + or - i given by modulus of lambda - A 11 where A 11 is 1 is less than or equal to modulus of -1 that is 1. And modulus of lambda - of -1, that is lambda of modulus of lambda +1 is less than or equal to 2 and that gives us the 2nd circle. So 1 and 2 are Gerschgorin's circles and the eigenvalues i and - i lie in the union of these 2 circles is what is given by Gerschgorin's theorem.

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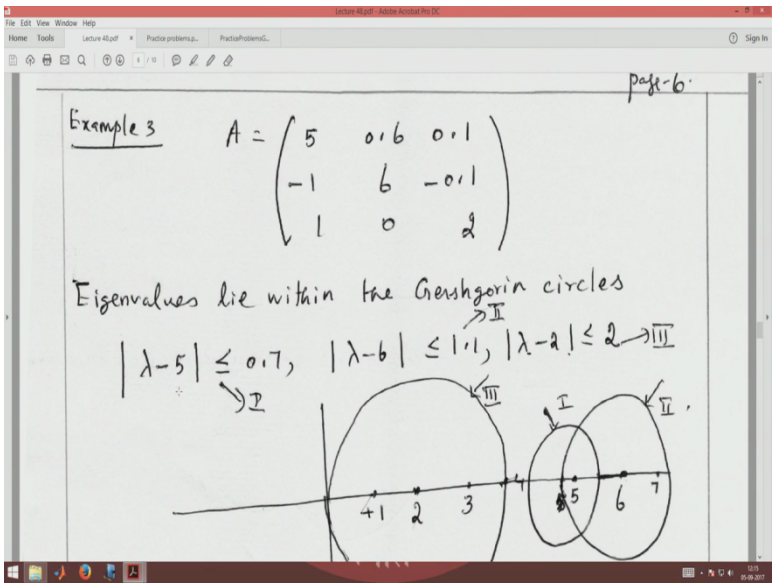

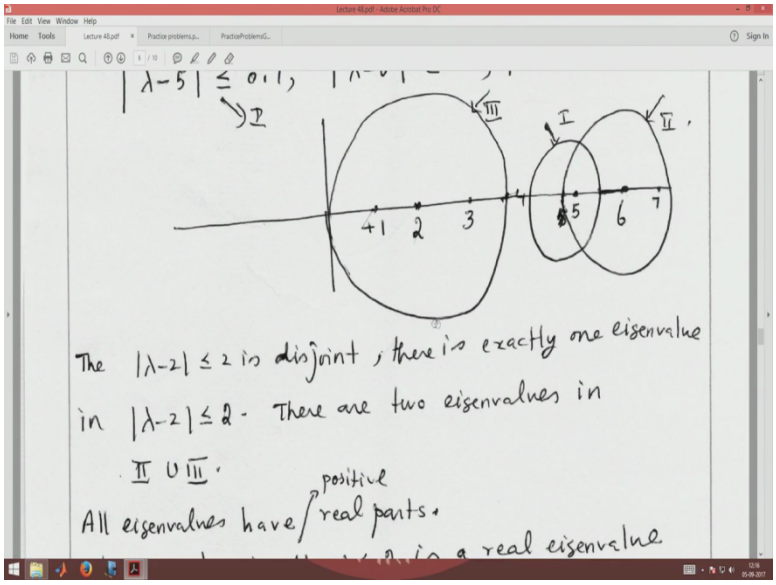

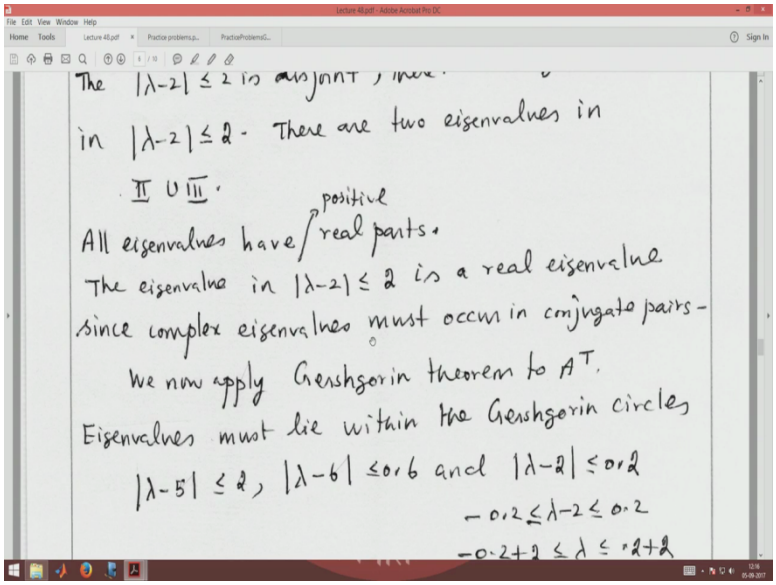

And we observe in this case that all the Eigen values, namely + i and - i lie within the circular disk given by modulus of lambda +1 less than or equal to 2 and that is defined by the 2nd row. Right, let us now consider another example. Namely consider a 3 by 3 matrix A and here Gerschgorin's Circle theorems tell us that all the eigenvalues lie within the Gerschgorin circles given by mod lambda -5 is less than equal to the sum of the absolute values of the other 2 entries in the row and that is 0.7. Then modulus of lambda -6 which is this is less than or equal to the sum of the absolute values of the other 2 entries that is 1.1 and finally modulus of lambda -2 is less than equal to 1 +0 and therefore we have the Gerschgorin's circles are given by these.

So finally we see that the circle which has its centre at 2 and radius 2 is disjoint from the other 2 circles which clearly tells us that there is exactly one eigenvalue in this circle which is

disjoint from the others. And there are 2 more eigenvalues and these 2 Eigen values lie in the union of other 2 circles which we have given. And we also observe that all the Eigen values are real because they have a positive real part. And the 2nd thing is therefore the circle, modulus of lambda -2 less than equal to 2 is such that it has an eigenvalue which is a real eigenvalue since complex eigenvalues, if they occur, they must after in conjugate pairs.

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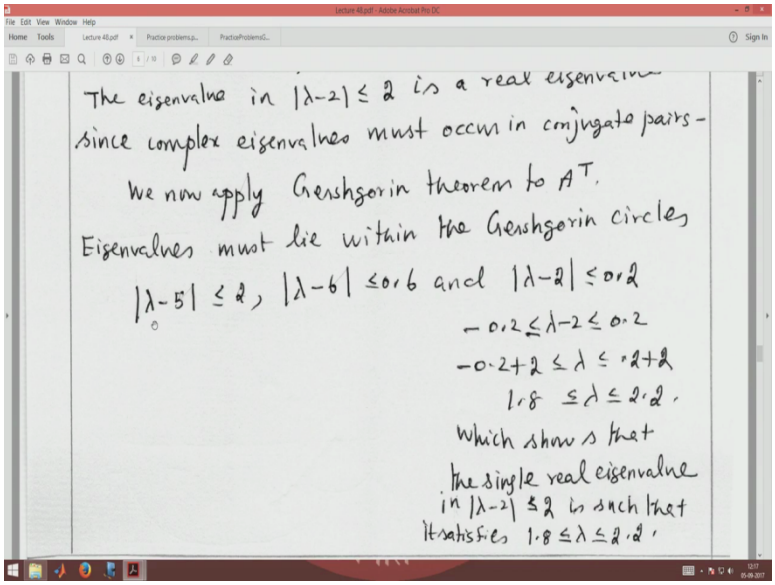

And so this circle, the disjoint circle encloses one and only one eigenvalue which is the real eigenvalue. Now to get more information, we apply Gerschgorin's theorems to the transpose of the matrix A, that is A transpose. Then we observe that eigenvalues must lie within the Gerschgorin circles which are given by modulus of lambda -5 less than or equal to the sum of the absolute values in the 1st column, that is 2 and modulus of lambda -6 is less than or equal to the sum of the absolute values in the 2nd column which is 0.6 and finally if you use the 3rd column of matrix A, that gives you mod lambda -2 is less than or equal to 0.2.

Right. So you observe that this is the disjoint circle with Centre at the point 2 and this result which is obtained from the columns of P tell us that that single eigenvalue which lies in the disjoint circle must be such that lambda must be greater than or equal to 1.8 and less than or equal to 2.2 and so lambda lies in the interval 1.8 to 2.2. So it clearly gives you an interval within which this single eigenvalue lie within that disjoint circle. So we are able to get the location of the eigenvalues of a given matrix A by applying Gerschgorin circle theorem in terms of rows and in terms of the columns of the matrix A or the rows of the corresponding matrix which is A transpose.

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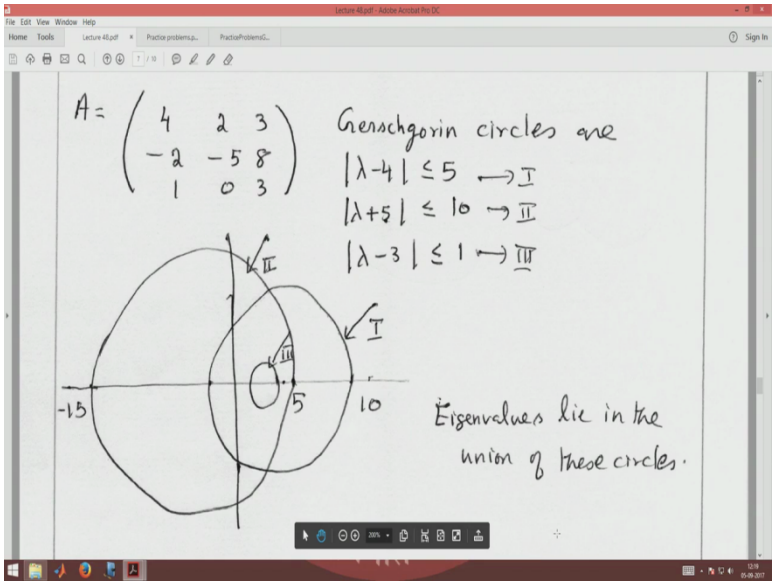

So we now consider another example in which say A is again a 3 by 3 matrix. Immediately we can write down the Gerschgorin circles, what are they, they are given by modulus of lambda -4 lead than or equal to 2+3 which is 5, modulus of lambda - of -5 that is lambda +5 is Y equal to modulus of - to +8 which is less 10 and modulus of lambda -3 is less than or equal to mod 1+0 that is one. So these are the 3 circles and the eigenvalues of this given matrix lie within the union of these 3 circles. So you just plot these circles because you know the Centre of the circles are going to be respectively 4, 5 and 3, I mean 4, -5 and 3.

And the radii of the circles are going to be 5, 10 and 1 respectively, so plot the circles in the complex plane and you know the eigenvalues of the given matrix lie within the union of these 3 circles in this case.

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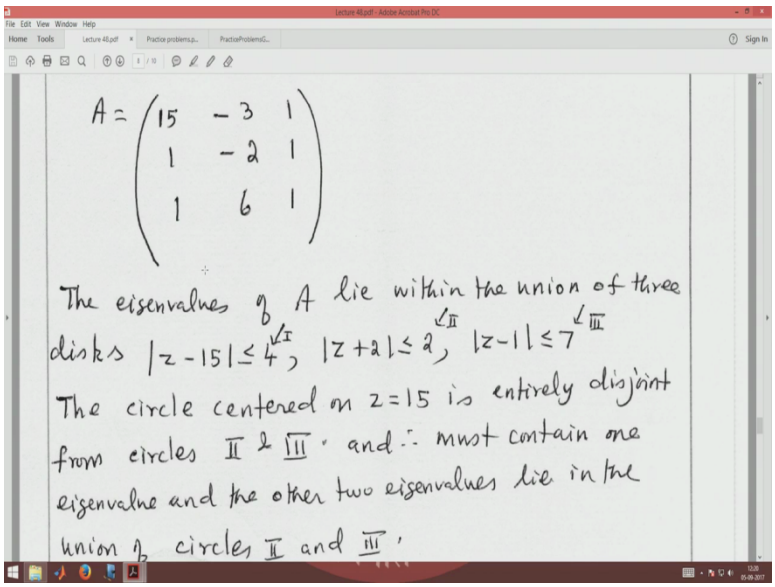

So let us now take another example. So again A is a 3 by 3 matrix and it is clear that the Eigen values of the matrix A lie within the union of the circles mod Z -15 less than or equal to 4, mod Z +2 less than or equal to 2 and mod Z -1 less than or equal to 7. And the one with Centre at 15 and radius is equal to 4 is going to be a circle which is the joint from the other 2 circles. So as before we argue and say and conclude that that circle which is disjoint from the other circles must contain one eigenvalue and the other 2 Eigen values lie in the union of the other 2 circles namely mod Z + 2 less than or equal to 2 and mod Z -1 less than or equal to 7.

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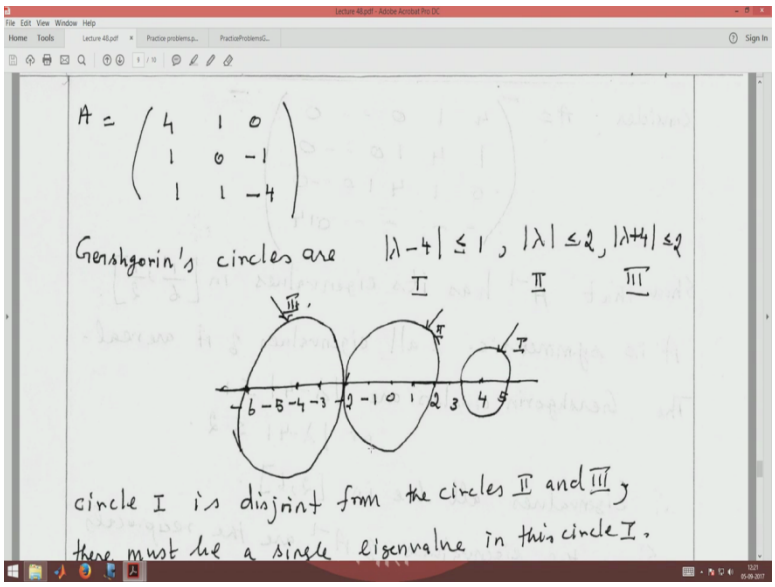

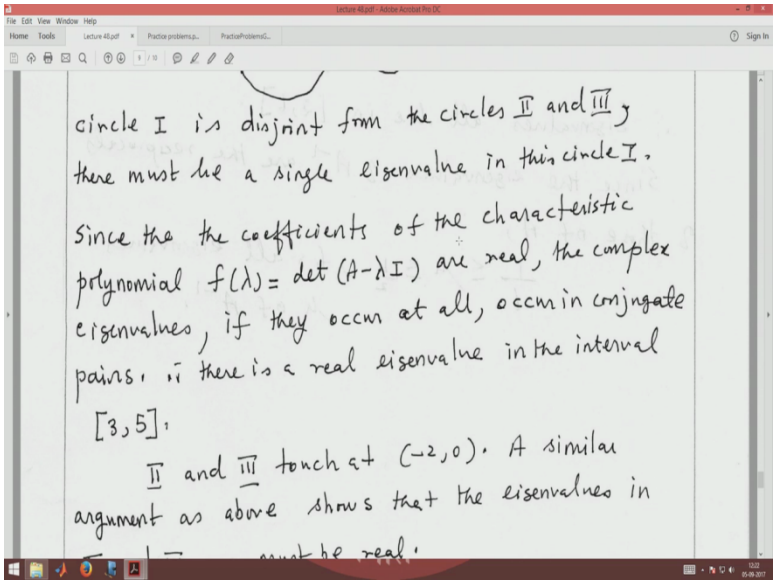

So the Gerschgorin's theorem is very very useful in understanding the location of the eigenvalues of a given matrix A. So let us look into another example, suppose A is given by these 3 by 3 matrix, we can write down the Gerschgorin circles, so mod lambda - 4 is less than equal to1, mod lambda -0, that is mod lambda less than equal to 2 and mod lambda + for is lesser equal to 2, so these are the rather circles which we have plotted here. And we observe that this circle 1 is disjoint from the other 2 circles and therefore it must contain a single eigenvalue within it.

And we also observe that if we write down the characteristic polynomial, the coefficients of the characteristic polynomial given by determinant of A - lambda I equal to 0 are real and hence the complex values which are eigenvalues, if they occur at all, then they can in conjugate pairs. Therefore there is a single real eigenvalue within this interval namely 3 to 5 because this is a disjoint circle, right, it single eigenvalue lies within it and radius of the circle is 1, so that single real, single eigenvalue is real and that real eigenvalue lies between 3 and 5. So we know the interval within which this real eigenvalue lies.

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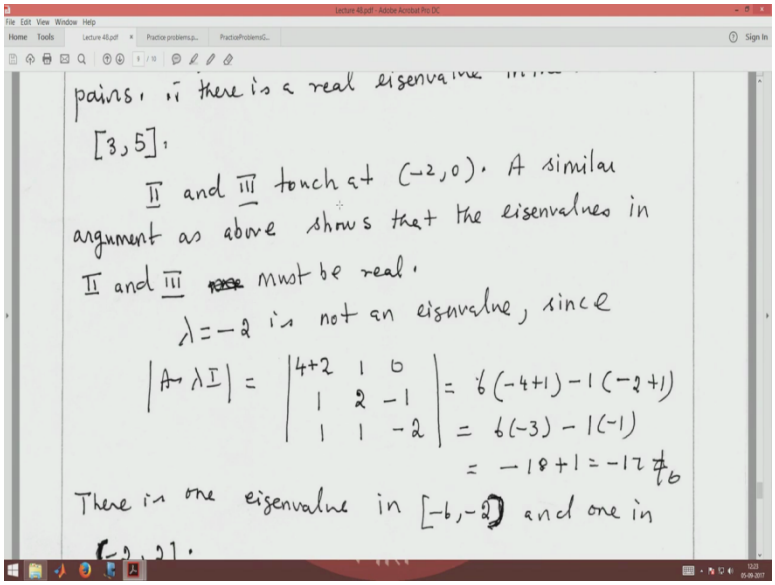

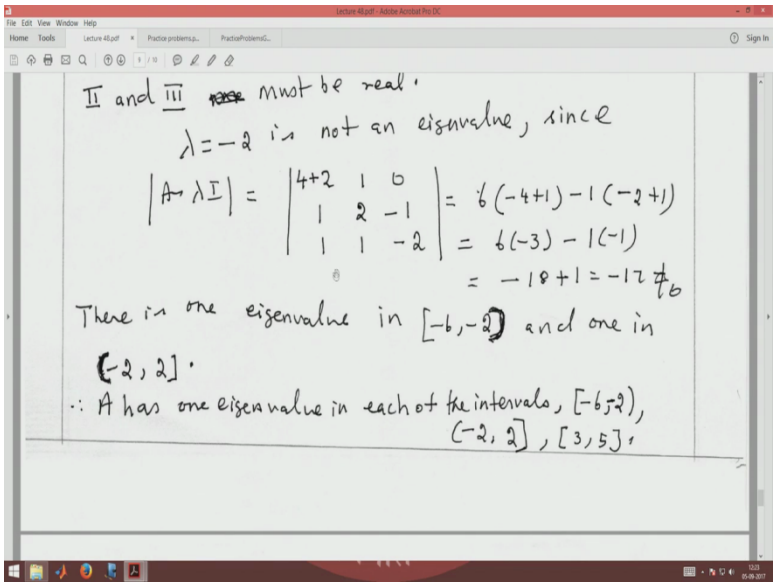

And then the other 2 circles, so the other 2 Eigen values lie in the union of these 2 circles is what we know. But we observe that these 2 circles meet each other or touch each other at the point -2. So we would like to see whether -2 is an eigenvalue of the matrix A. How do you check whether lambda equal to lambda 0 is an eigenvalue of the matrix A? The just have to see whether it is the root of the characteristic polynomial, determinant of A - lambda I equal to 0, that is what you have to check. So let us compute determinant of lambda - A - lambda I taking lambda to be -2, so if you write down the determinant and expand and you observe that that turns out to be -17 which is different from 0 and therefore the characteristic polynomial does not have lambda equal to -2 as its root.

And therefore lambda equal to -2 is not an eigenvalue of the matrix A is what we conclude. What does it mean, we said earlier that the other 2 Eigen values lie in the union of those 2 circles and the point where they touch each other, namely -2 is not an Eigen values, all right. And therefore the other 2 Eigen values must be such that each one must lie in each of these circles and that is the conclusion that we get from our discussion. So there is one eigenvalue in the interval -6 to -2 and another eigenvalue which is in -2 to -2. Why does that happen, so let us look at the circles, so this is a circle with Centre at -4 and radius 2.

And therefore it passes through the points -6 in -2, so one of the eigenvalues lies between -6 and -2. And then you have another circle whose centre is the origin and radius is 2 units, and there is an eigenvalue within this circle and it is a real eigenvalue and therefore it lies between -2 and 2. So the matrix A is such that it has one real Eigen value between -6 and -2, another real eigenvalue between -2 and 2 and 3rd real eigenvalue between 3 and 5. And so we conclude that A has one eigenvalue in each of these intervals -6 to -2, -2 to 2 and 3 to 5. So information about the location of the eigenvalues can be obtained by writing out the Gerschgorin's circles within which the eigenvalues are located and arguing out depending upon whether these circles are disjoint or the Eigen values lie within the union of these circles and so on.

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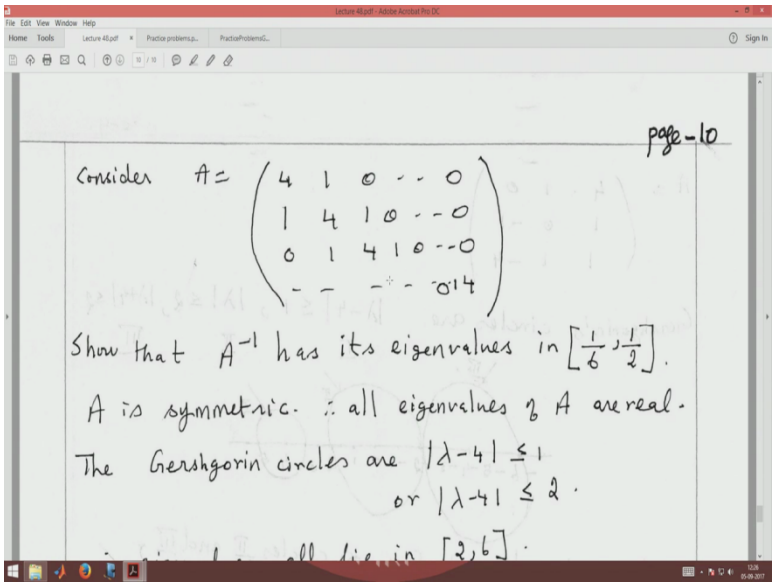

So now we have another problem where A is a matrix say it is it is an n Cross n matrix having entries given by these. Namely along the diagonals you have 4 appearing throughout and just above the diagonal, entries have values 1 and just below the angle, the entries are again 1, the rest of the entries are all 0. So you have an N cross N matrix where a I J are given by the

entries which are specified here. You are asked to show that the inverse of this matrix A has its eigenvalues in the interval 1 by 6 to 1 by 2. So you need some information about the location of the eigenvalues of the inverse of this matrix A.

We have already seen while doing the Gerschgorin theorems and other details, we have remarked that if lambda is an eigenvalue of A, then 1 by lambda is an eigenvalue of A inverse, we have actually shown this result. So now we want to show that A inverse has its eigenvalues interval 1 by 6 to 1 by 2. So let us 1st data mean the interval within which eigenvalues of A lie and then we can use this result and make a conclusion about the location of eigenvalues of the matrix A inverse. 1st of all we observe that A is a symmetric matrix, a IJ is a JI, so what do you immediately conclude, all its eigenvalues are, right, are going to be real.

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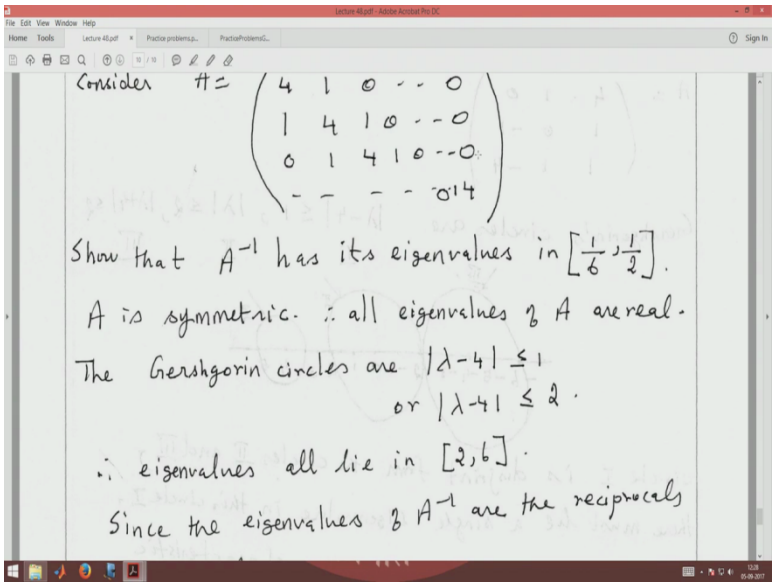

Namely all the Eigen values of medicine A are going to be real. That is the 1st information that we get by looking at what the matrix A and matrix A is symmetric. Now let us apply Gerschgorin's circles theorem to the rows of this matrix A. So every row has 4 along its diagonal and therefore the Gerschgorin circles will be given by modulus of lambda -4 is less than or equal to the sum of absolute values of the entries in that row. And we observe that the 1st row has entry to the right of 4, so the 1st row will give us the Gerschgorin circle to be modern lambda -4 less than or equal to 1.

Similarly the last row will give us mod lambda -4 is less than or equal to1 because the rest of the entries are all zeros. Now apply Gerschgorin's theorem to all the other rows, starting from

the 2nd row to the N -1th row. We observe that the diagonal entries are 4 and you have on the either side of the diagonal entry, value 1 appears to the right of it and value 1 appears to the left of it. And hence we get modulus of lambda -4 less than or equal to 2 to be the Gerschgorin circles. So what does this mean, if you write out this, this tells you -2 less than equal to lambda -4 less than or equal to 2 and therefore that immediately tells you that all the eigenvalues lie in the interval 2 to 6.

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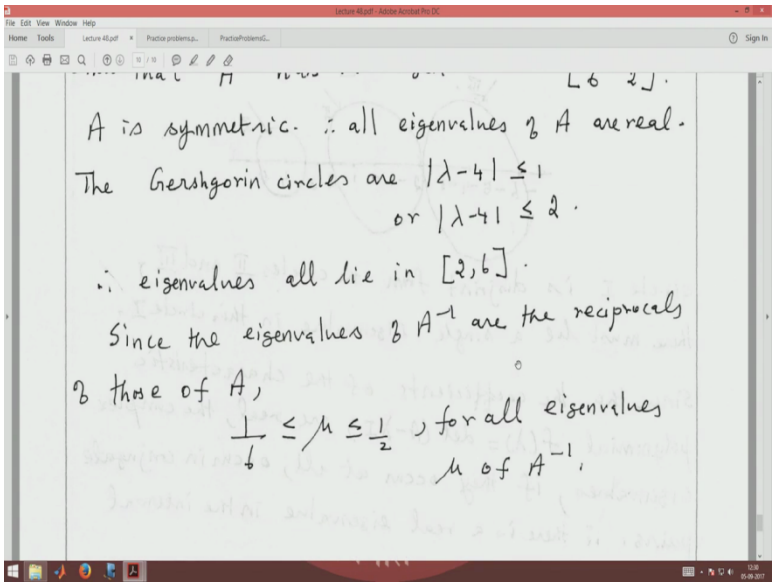

So what are these eigenvalues? They are the eigenvalues of matrix A. So Gerschgorin circle theorems tell us that all the eigenvalues of A are real and all the eigenvalues of the matrix A lie within the interval 2 to 6. So now we make use of the result that we have shown. What is it, eigenvalues of A inverse a reciprocal of eigenvalues of A. So A has eigenvalues lying in the interval 2 to 6 and therefore if mu denotes the Eigen values of A inverse, then mu has to lie between the reciprocal of 2 and reciprocal of 6, namely mu has to lie between 1 by 6 and 1 by2.

So new lies between 1 by 6 and 1 by2 and therefore the conclusion is that all the eigenvalues mu of A inverse lie in the interval 1 by 6 to 1 by2 and that is what we are asked to show. Right, so we have made use of the 2 results, what are the 2 results, we said that if A is a symmetric matrix, all eigenvalues are real and that secondly we use the fact that eigenvalues of A inverse are given by the reciprocal of eigenvalues of A, so we 1st computed the eigenvalues of A and then we used the result and got some information about the location of the eigenvalues of the matrix A inverse.

So these are some examples in Gerschgorin theorems and location of eigenvalues of a given matrix. Now we move onto some examples where we discuss the computation of norm of a matrix and then some problems from the error analysis by the direct methods of solving a system of equation AX is equal to B.

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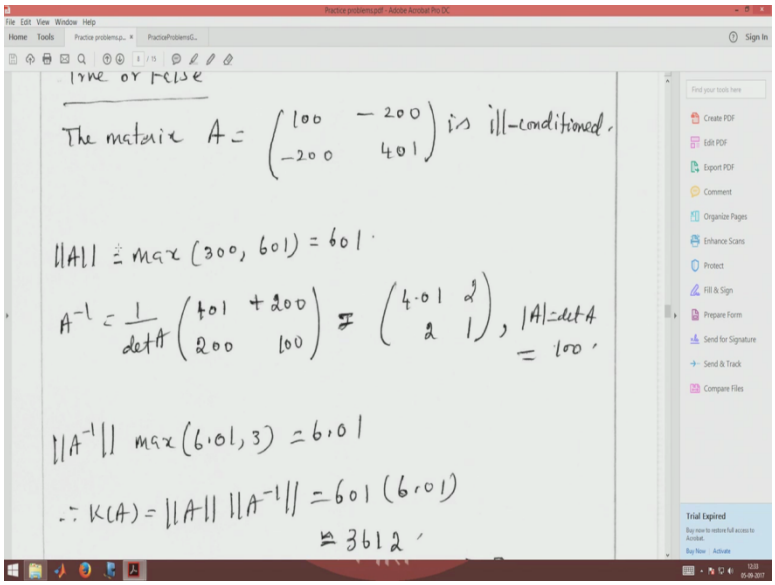

So suppose say I give you a matrix A which is this and I make a statement, I make the statement that the matrix A given by this is ill conditioned and I ask you to tell me whether this statement is true or false. It is just not enough if you would say that the statement is true or the statement is false, you have to justify your statement whether it is a true statement or a false statement, then only your answer will be correct and you have justified your conclusion. So how are we going to conclude whether the statement is true or false, how do I say or how do I find out whether the matrix is an ill conditioned matrix or not, I compute what is its condition number.

If the condition number is very large, then we conclude that the matrix A is ill conditioned. So I compute the condition number. And what is the condition number, condition number is norm A into norm A inverse. So given a matrix A, I compute its inverse and then I compute norm of A, then norm of A inverse and then find what K of A, the condition number of A is. So I compute norm A using maximum column sum norm. I have maximum of 100+ modulus of -200, so 300, then modulus of -200+401, that is 601, so norm A is 601.

So with respect to the maximum columns sum Norm, Norm A is 601. Now I compute norm A inverse with respect to the maximum column sum, so norm A inverse is going to be equal to

maximum of the some of the absolute values of the entries in the 1st column which is 6.01 and the sum of the entries in the 2nd column after taking the absolute value, right and that gives you 3 and so it is 6.01. And therefore norm A inverse is 6.01. So now compute the condition number of A, it is norm A into norm A inverse, so that is 601 into 6.01 and it turns out to be 3612 which is very much greater than the value 1, right.

For a well conditioned system, the condition number must be close to value 1 greater than 1 and here it turns out to be 3612 and therefore A is an ill conditioned matrix, so the statement that has been given, namely the matrix A is ill conditioned is a true statement. So whenever you are given some statement and we are asked to check whether the statement is true or false, right, in order to conclude what the answer is, you must work out the details and based on the results that you get, you must conclude whether the statement is true or false, in which case you have justified your conclusion.

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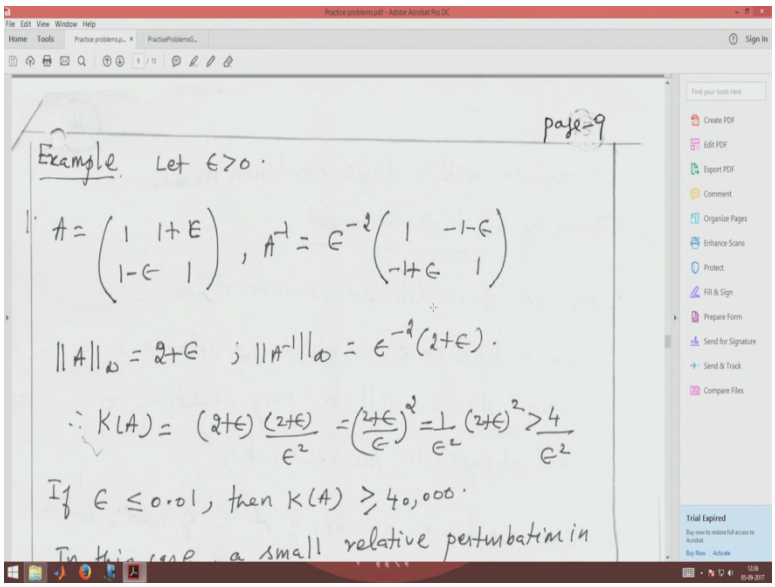

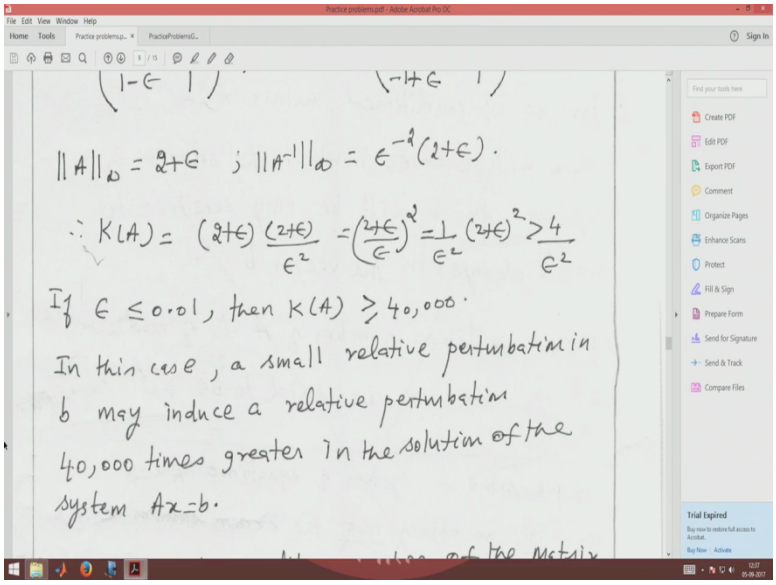

So let us now consider an example where we want to perform some error analysis. Suppose you are given Epsilon greater than 0 and a matrix A having entries 1, 1 + Epsilon in the 1st row and 1 - Epsilon and 1 in the 2nd row. Immediately I compute A inverse and that is given there. And this time I would like to obtain norm of A using the infinity norm. So what is, the condition number is norm A infinity into norm into A inverse infinity and that turns out to be 2 + Epsilon the whole square by Epsilon square. So this is surely greater than 4 by Epsilon square.

If suppose in the matrix that is given, I take my Epsilon which is positive to be such that it is less than or equal to 0.01, then I observe that in that case K of A0 than 4 by 0.01 the whole square and that will turn out to be 40,000. So K, condition of condition number of A will turn out to be greater than or equal to 40,000 which is a very very large quantity.

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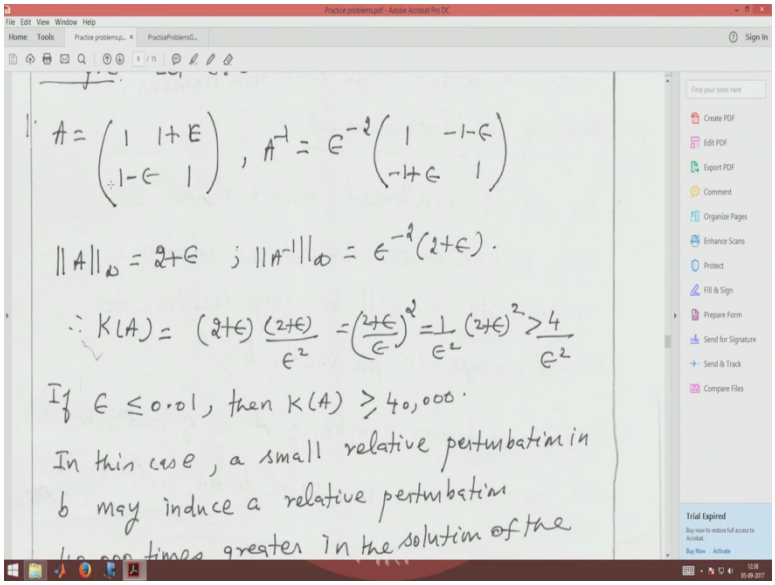

Suppose I give a small, suppose I give a small change in one of the entries in the matrix A, namely I add Epsilon to this entry and subtract Epsilon from this entry, so if I perturbed entries in the given matrix A, then what happens to the condition number. In that case the condition number becomes very very large and so this says that a relative perturbation in the right-hand side may induce a relative perturbation 40,000 times greater in the solution of the given system AX is equal to B. So change in some entries in A will give the corresponding changes in the solution and that is going to be such that it is going to be 40,000 times greater in the solution of the given system AX is equal to B.

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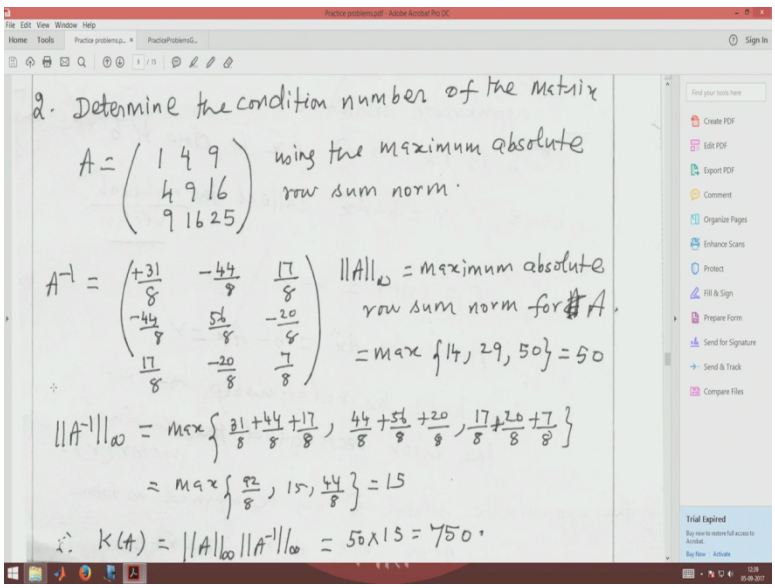

So let us now consider what happens if you want to compute the condition number of a matrix A which is a 3 by 3 matrix using the maximum absolute by row sum Norm. So A is given, so you can compute A inverse using Gauss Jordan technique, now you have learnt that. So you know how to compute the inverse of a given matrix using Gauss Jordan technique and now you compute norm A infinity. What is it, it is maximum absolute row sum Norm for A. So it is a maximum of the sum of these entries in each of these rows after taking the absolute value. And that turns out to be maximum of 14, 29 and 50 and that is 50.

Now compute norm A inverse infinity, here again I want to use the maximum row sum and so compute the sum of the entries in absolute values in each of the rows and from among them choose that which is the maximum that turns out to be 15. So norm A inverse infinity is 15, so the condition, condition number of given matrix A is K of A given Norm A infinity into A inverse infinity which is 50 from here and 15 from norm A inverse infinity, that gives you 750. So we can say that the matrix A is an ill conditioned matrix.

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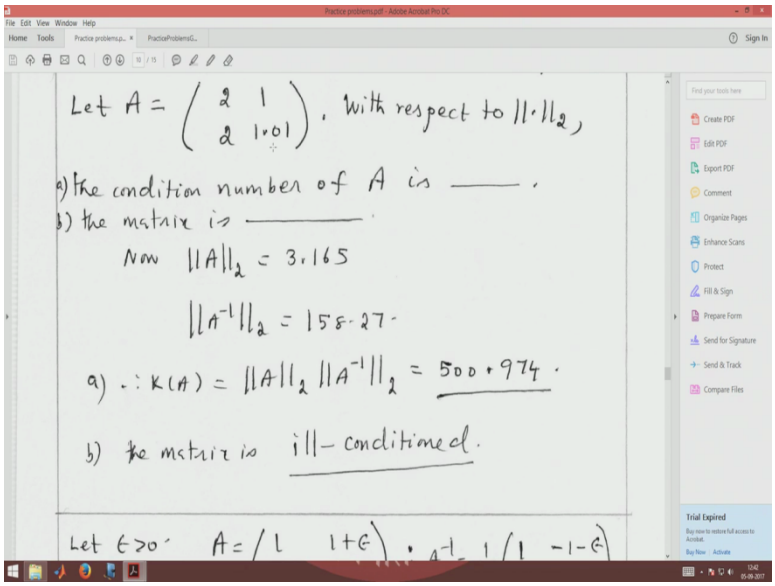

Now I consider another example. Taking A to be a 2 by 2 matrix and I want you to suit the condition number of A with respect to the 2 norms. So when I work out the 2 norms for the matrix A, that turns out to be 3.165. I also require the 2 norms for A inverse because I am asked to compute the condition number of A. So norm A inverse 2 turns out to be 158.27. Now that you know the definition of 2 norms for a matrix A, you can compute A inverse and work out the details and show that norm A inverse 2 is 158.27. And therefore the condition number is norm A into norm A inverse with respect to the 2 norms and so it is the product of these 2 values and that turns out to be this.

So you have actually computed it and you are asked to fill in the blanks here, so write out the results here that it is 500.974. And you observe that this is a large quantity and therefore the, you are asked to make some conclusion about what the matrix is. And you know since the condition number is large, the matrix is ill conditioned, so you fill in the blanks here by saying that the matrix is an ill conditioned matrix.

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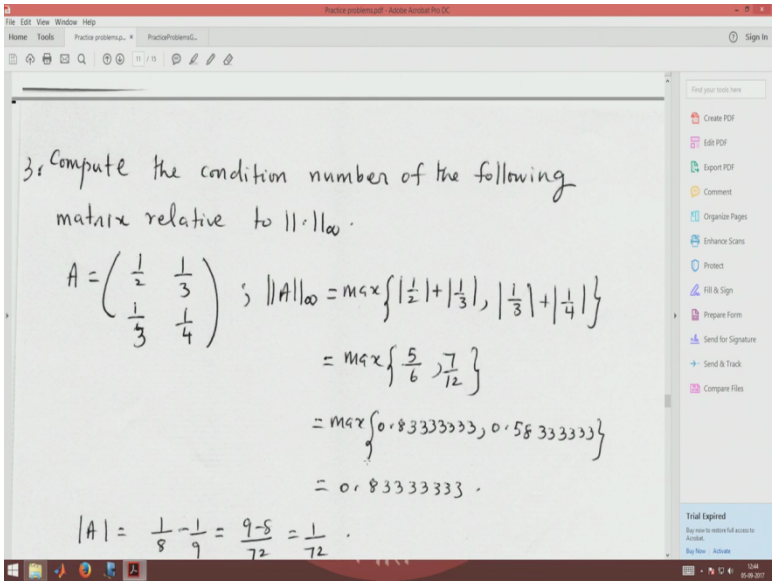

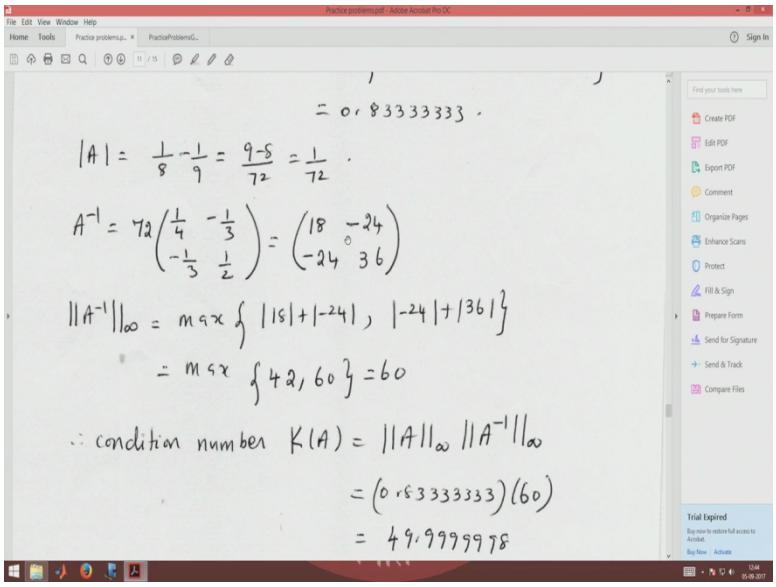

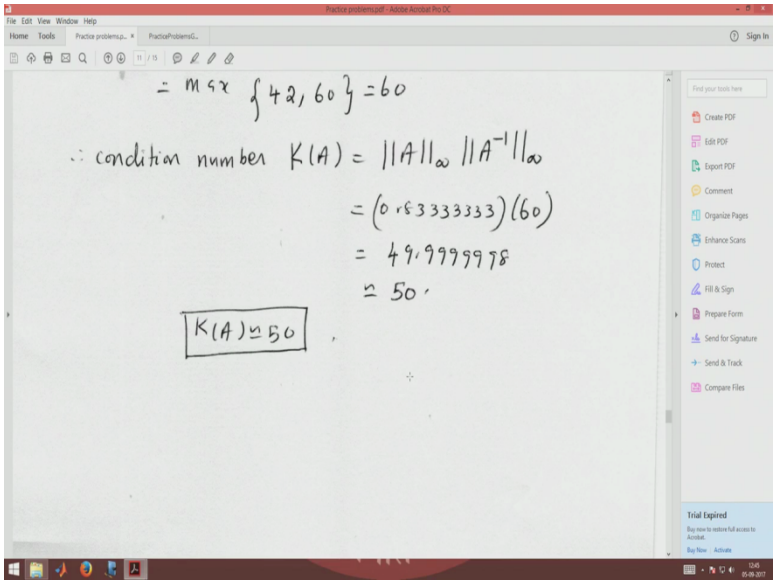

Compute the condition number of the following matrix relative to infinity norm. So A is a 2 by 2 matrix, norm A infinity is maximum row sum, so compute the row sum, namely the 1st row sum in absolute value is and the 2nd row sum and you conclude that norm A infinity turns out to be this. Now what is the determinant of A, that turns out to be 1 by 72. Get A inverse and workout what norm A inverse infinity is. Again maximum row sum Norm and so we have maximum of 42 and 60 to give you norm A inverse infinity and that turns out to be 60. So what is the condition number? The condition number is norm A infinity into norm A inverse infinity.

We have computed each of these, so the product of this gives you norm A A infinity into norm A inverse infinity and that is 50 and you are asked to get what the condition number is. So the condition number of A is 50. So now we know how to compute norm of a vector or norm of a matrix associated with the norm of a vector which is given. So let us work out some details about the error analysis for direct method.

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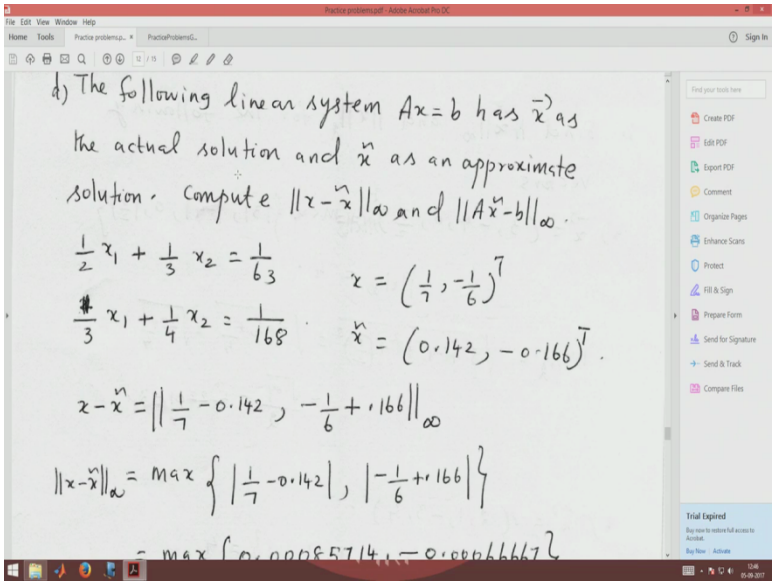

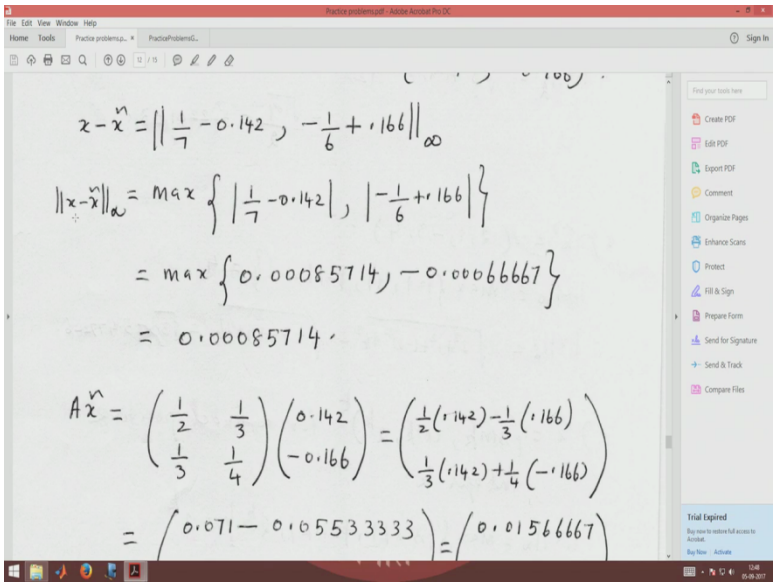

So the problem is, the following linear system AX is equal to B has X as the actual solution and X tilde is an approximate solution. So you are asked to compute the magnitude of the absolute error and also compute what is the residual a namely norm of A X tilde - B with respect to the infinity norm. So you are given a system of 2 equations in 2 unknowns. And you observe that these coefficients have 1 by 3, 1 by 3, 1 by 4 are the same as the examples which we have considered earlier. And X is equal to 1 by 7 - 1 by 6 transpose is the actual solution and X tilde which is this is an approximate solution.

So solve this system say by applying Gauss elimination method with partial pivoting and get the solution correct to 3 decimal places and that is what your X tilde. So now you want to compute the error, what is the error, it is X - X tilde. So X is 1 by 7, X tilde, 1st component is

0.142. 2nd component is -1 by 6 and the 2nd component of X tilde is -0.166, so it is this quantity. And you want norm X - X tilde infinity. So it is the maximum of the 1st entry and the 2nd entry. So maximum of absolute value of the 1st entry and the absolute value of the 2nd entry, so compute this.

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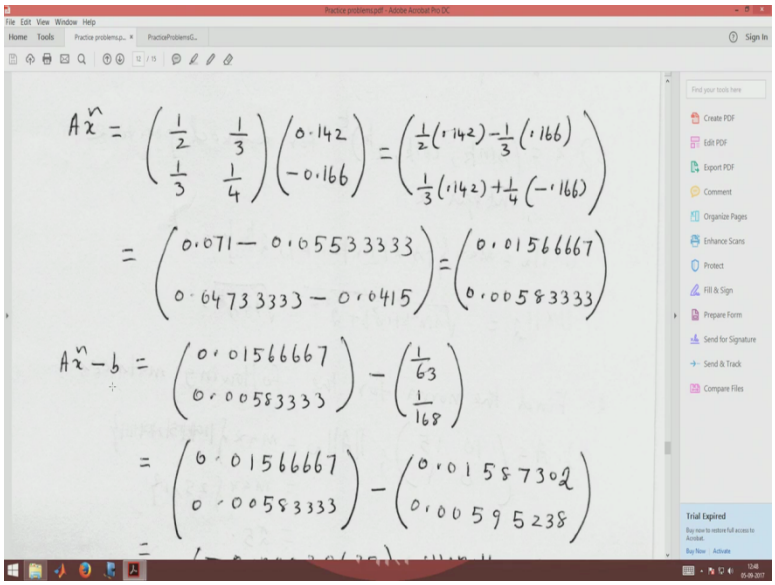

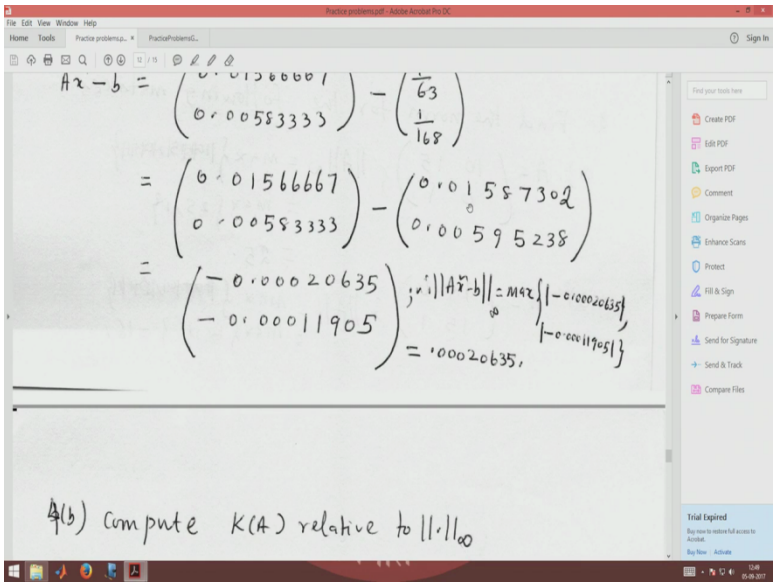

So you get this to be maximum of this and this which turns out to be the value 0.00085714, so that is the absolute error in the solution X of the system AX is equal to B. So you would like to find what is the residual, so compute A into X tilde. So A matrix is given, X tilde is given, so work out what is A into X tilde which turns out to be this vector. So what is the residual, residual is AX tilde - B. Why is it a residual, you want X to satisfy the equation AX

is equal to B, so AX - B must be 0 but X tilde is an approximate solution, so AX tilde - B will not be equal to 0, it gives you residual, and so you call r as AX tilde - B.

Are you computed AX tilde as this, so that vector - you know what the right-hand side vector is, so substitute that and then evaluate what is the difference and compute norm AX tilde - B, that is the maximum of the absolute values of the entries that you have written here and that turns out to be 0.00020635. So you will be in a position to compute the absolute error which is norm X - X tilde as well as the norm of the residual r which is norm AX tilde - B with respect to any norm that will be asked. In this example you had been asked to get these values using the infinity norm and you have computed them with respect to infinity norm.

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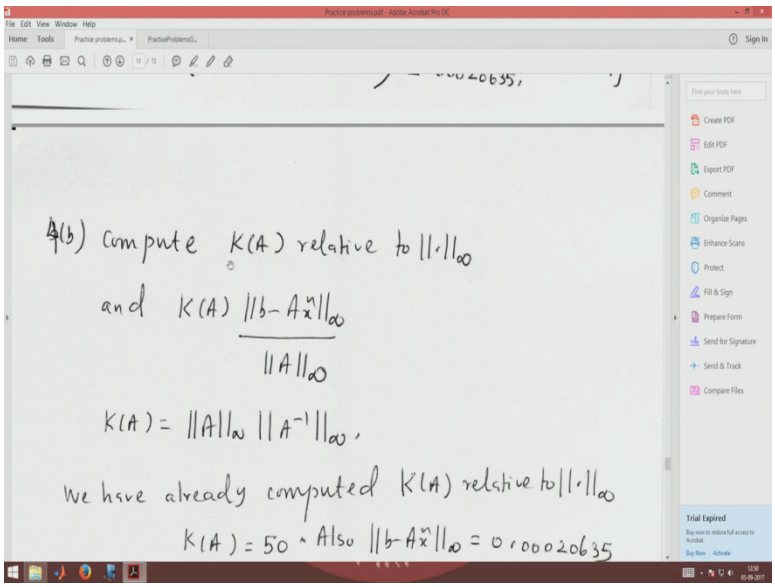

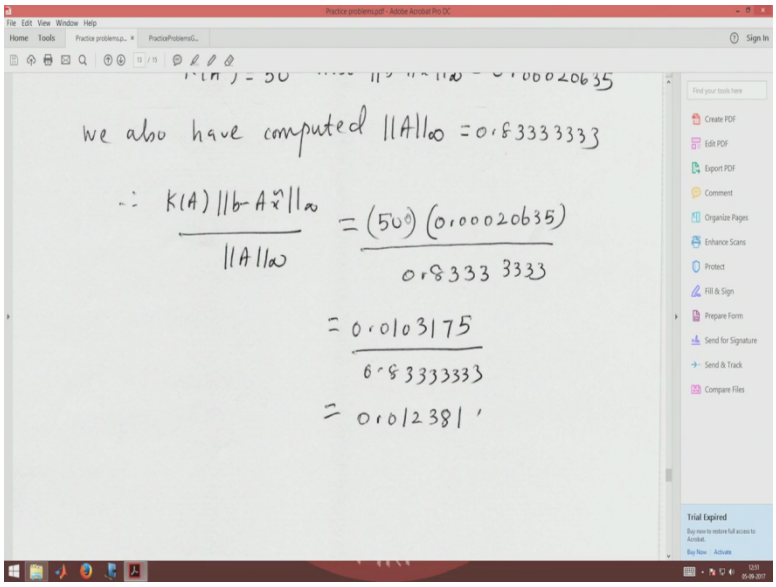

And now you are asked to compute the condition number relative to infinity norm and also the condition number times the residual by norm A with respect to infinity norm. So you compute what is condition number, it is norm A Infinity into norm A inverse infinity. We already have computed the condition number earlier when we worked out the example, it turned out to be 50. And now we have computed what is the residual, namely norm B - AX tilde with respect to infinity norm and that is 0.00020635. And we also have computed norm A with respect to infinity norm which is this so we simply have to substitute these values which appear on the left-hand side, namely these and then simplify and that gives you the value of this namely condition number multiplied by norm of the residual by norm A infinity to be equal to this.

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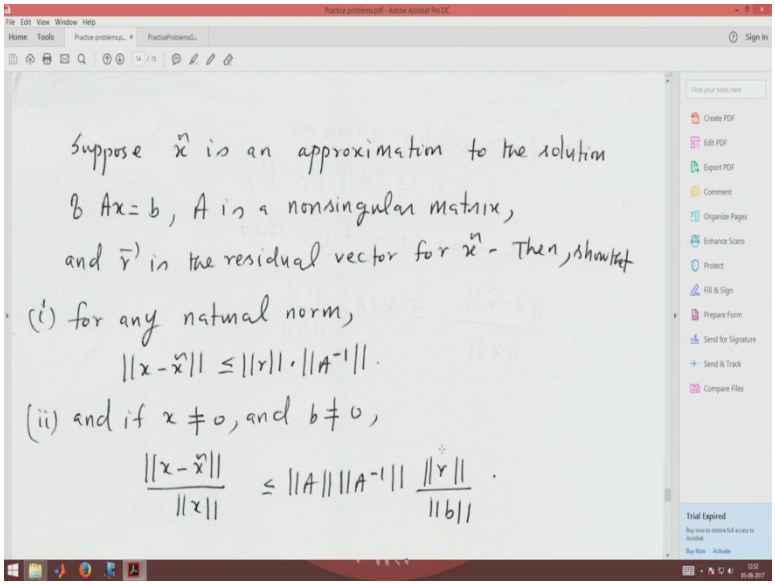

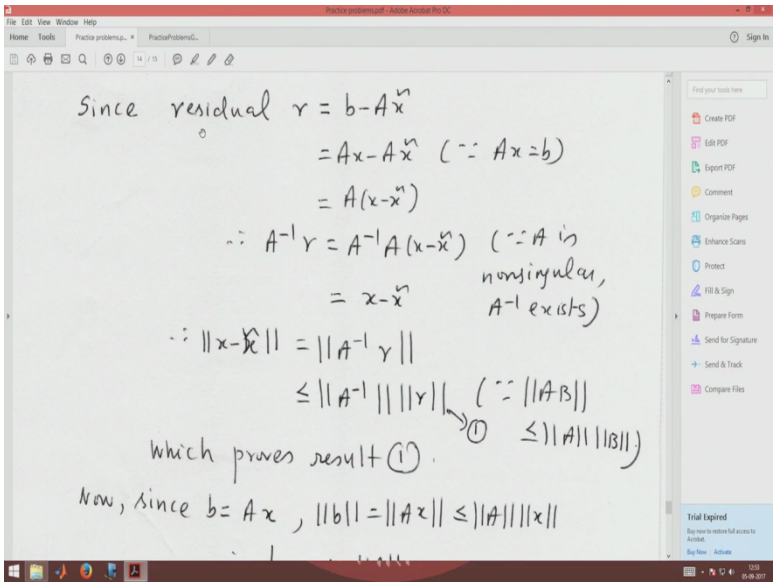

If we know how to compute the matrix norm given the vector norm, then we will be in a position to compute the absolute error and also the relative error. So that is what is given in the next example. Suppose that X tilde is an approximation to the solution of AX is equal to B, A is a non-singular matrix and r is the residual vector for X tilde, that is r is AX tilde - B. Then you are asked to show that for any natural norm, namely infinity norm or 2 norm, you should be able to show norm X - X tilde is less than equal to norm r into norm A inverse.

And in addition, if X is different from 0, is a nonzero vector, and B is also a nonzero vector, then show that the relative error in our computation, what is the relative error, it is change in X with respect to X, so it is norm X - X tilde by norm X, that must be less than or equal to the condition number which is norm A into norm A inverse multiplied by the norm of the residual by the norm of the right-hand side vector, this is what we have to show. So these 2 results clearly tell us the bound on the absolute error, namely this is the right-hand side, your absolute error cannot exceed the value on the right-hand side.

Your relative error in the computation of solution of a system AX equal to B cannot exceed the value on the right-hand side, what is it, it is the condition number of the matrix A times the norm of the residual by norm of the right-hand side vector B. So let us work out the details. Now residual is r, which is B - AX tilde. But what do you know about B, B is AX, so A X - AX tilde, that is A into X - tilde. So I premultiply both sides by A inverse because it is given A is a nonsingular matrix, so A inverse exists and therefore I multiply by A inverse.

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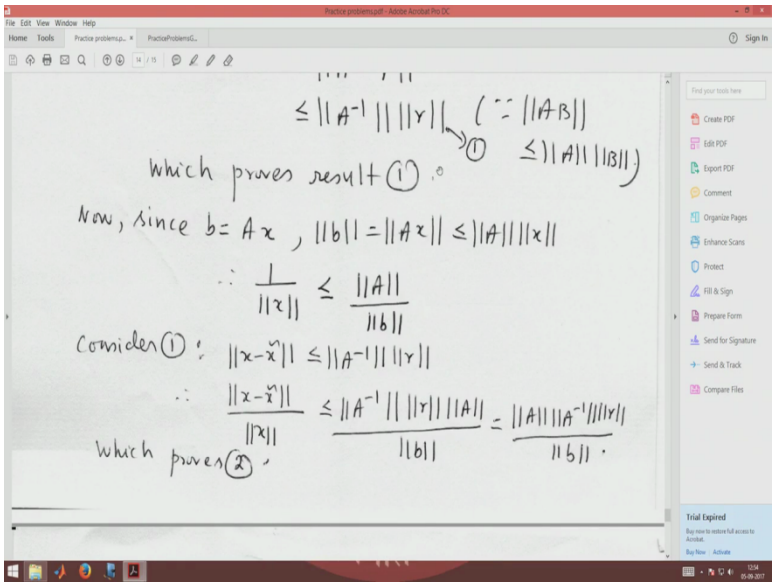

So we have A inverse into r to be A inverse into A into X - X tilde, so that will be X - X tilde. And therefore norm X - X tilde will be norm of A inverse into r but we have shown already norm of A B, where A and B are matrices will be less than or equal to norm A into B, so let us make use of that result. So norm A inverse r is less than or equal to norm A inverse into norm r. So this proves our 1st result, that is what we have to show, namely norm of X - X tilde is less than or equal to norm r into norm A inverse and we have shown that result 1.

Now, we are, we know that B is A X, so norm B is norm AX, that is less than or equal to norm A into norm X and therefore 1 by norm X is that an equal to norm A divided by norm B. Now consider norm X - X tilde less than or equal to norm A inverse into norm r. And therefore divided by norm X Throughout, so you get norm X - X tilde by norm X is norm A inverse into norm r and what about norm X, we know AX is B, so I also know norm B is less than or equal to norm A into norm X and so I can replace 1 by norm X by a quantity which is a miracle to norm that able norm B. So I replace this 1 by norm X by this and this give me norm A inverse into norm A into norm r by norm B and that is what we have to show here, namely norm A, norm A inverse, norm r.

We can immediately get the bound on the absolute error and the relative error for a given problem when we get an approximate solution using the direct methods, namely Gauss elimination method or any of the direct methods that we have studied. So given a system AX is equal to B and if you are if you apply direct methods to solve the system, then you can immediately specify the amount of absolute error and relative error that are involved in your computations by using the bounds which are given by these examples. What is that we have done all along in this course?

We have focused our attention on 5 important topics which are very useful to us in obtaining approximation of a function whose values are specified at discrete points. Namely, we discussed polynomial approximation and then when the information is given at a set of equally spaced points, we used Newton's forward interpolation polynomial or backward interpolation polynomial and obtained approximate polynomial, namely the interpolating polynomial that interpolates the function at a certain of given point. If the information is given at a set of arbitrarily located points, then we had Lagrange interpolation method and divided difference method to compute the interpolating polynomial that interpolates the given data and we have discussed error in interpolation and also the bound on the error in interpolation.

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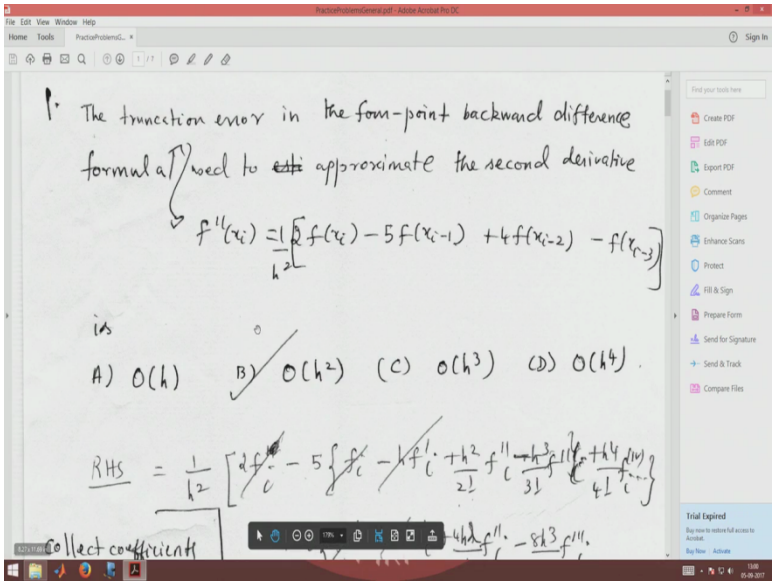

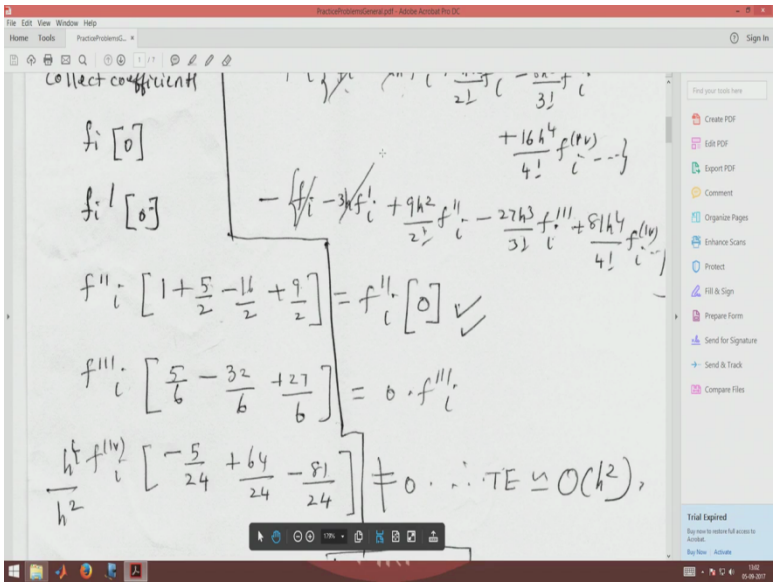

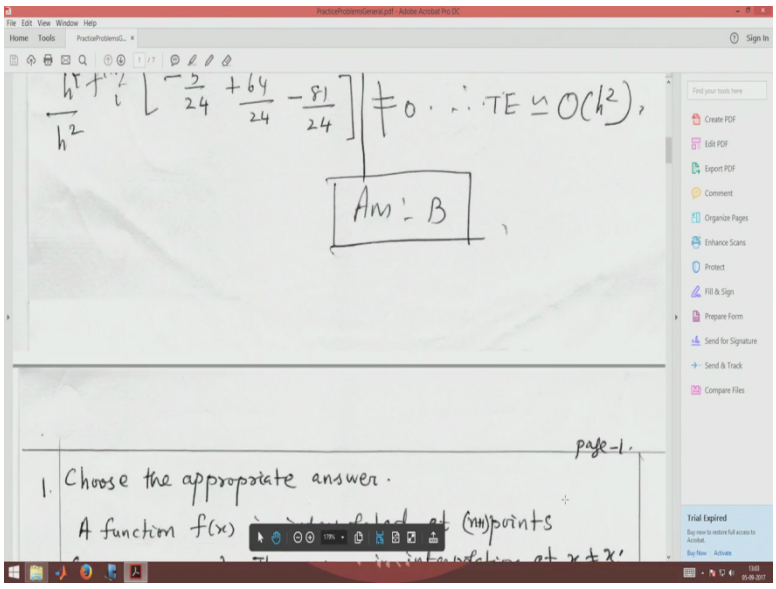

Then we considered the topic on numerical differentiation. So then we derived finite difference approximations to various ordered derivatives which can be given in terms of the values of the function at a certain set of points. So let us consider an example here. Suppose say I give you a formula which approximates the 2^{nd} derivative, F double dash of X I is given by this and I asked you the following. Show that the truncation error in the 4 point backward difference formula which is given by this, that is used to approximate the 2^{nd} derivative is, is it order of H, order of H square, order of H cube or order of H to the power of 4.

I ask you to choose the correct answer from the 4 choices which I have given here. Unless you work out the details, you will not be able to give the results. So you start with the right-hand side, expand using Taylor's theorem and get the coefficient of FI, F dashed I, F double dashed I, F triple dashed I and the 4^{th} derivative of I and so on. And then that is equal to F double dash I which appears on the left-hand side. So collect the coefficients of FI, you will see it is 0, F dashed I, that is also 0, F double dashed I also turns out to be 0, F triple dashed I also turns out to be 0.

On the other hand the 4^{th} derivative of I coefficient turns out to be different from 0 and so you have right-hand side has H power 4 into 4^{th} derivative at i divided by H square because the formula had 1 by H square in it. So we observe that you have the 4^{th} derivative term appearing, multiplied by H power 4 by H square and that is a nonzero term, so it is H square times the 4^{th} derivative. And so that is the 1^{st} term which is a nonzero term that appears in this formula when you approximate the 2^{nd} derivative by means of the formula on the right-hand side. So you conclude that the truncation error is of order of H square.

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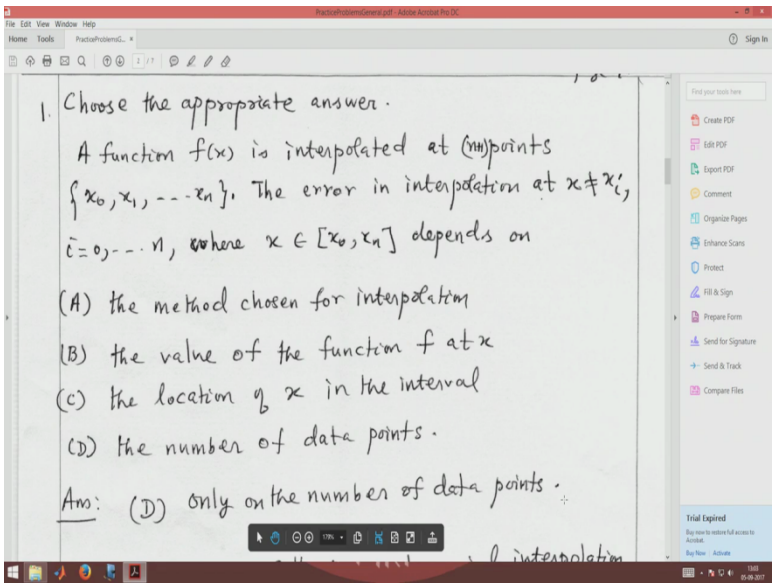

And we have also discussed after numerical differentiation some results on numerical integration when we derived the closed type formulas, which are Newton Cotes type from last, namely the trapezoidal rule and Simpson's rule and we derived certain integration methods which are exact of polynomial is of degree N when we use N nodes in the interval say A, B which is a closed interval.

Then we said that if we can sacrifice the requirement, that these N nodes are equally spaced and look for methods which are quadrature methods, such that they are exact for polynomials of degree greater than N, when N nodes are used within an interval of the form A, B in particular an interval of the form -1 to 1, then we said that we obtain open type integration methods where we have used properties of the Legendre polynomials and the integration methods turn out to be methods which are exact for polynomial is of degree less than or equal to 2N -1 where the function values at only N nodes are required.

And these N nodes turn out to be the zeros of the Legendre polynomial of degree N and these formulas were of the open type. The notes were all within the interval -1 to1. And then we moved on to the solution of ordinary differential equations where we derived some single step methods like Taylor's series method, Euler smothered, modified Euler's smothered and Runge Kutta methods of order 2 and 4 and solved initial value problem using any of these methods. Then we said that it is possible to consider iterative type of methods where we have a predictor to predict the solution and then a corrector which would recorrect the solution and so that we can obtain solution correct to the desired degree of accuracy.

And these methods involved a number of points close to the point at which the initial condition is specified. So that information at 4 points were used to compute the solution at the 5th point. And these were multistep methods which we used as predictor corrector methods and we had solved an initial value problem using Milne's predictor character and Adam Bashforth's predictor character methods. Then we moved on to solution of nonlinear equations of the form F of X is equal to 0.

So we discussed 2 types of methods, namely the enclosure methods and then the fixed point iteration methods and discussed the order of convergence of these methods and we learnt some techniques of solving equations of the form F of X is equal to 0. And finally be moved on to the solution of a system of algebraic equations and then discussed the direct methods and iterative methods. And the direct methods included the decomposition methods of Doolittle, Crouch and Cholesky and Gauss elimination method incorporating partial pivoting and then we discussed the iterative methods such as Gauss Seidel method and Gauss Jacobi method.

But then we wanted to see how in the, in the case of direct methods, we can have some information about the error that has been committed by us. So we performed error analysis and so we introduced norm of a vector, norm of a matrix associated with the given norm for a vector and we have shown how to get a bound on the absolute error and the relative error when we know the condition number of a given matrix A which appears as the coefficient matrix in the system AX is equal to B. And when we know what the norm of the residual which is norm AX tilde - B where X tilde is an approximate solution B is the right-hand side vector that appears in the system AX is equal to B.

And finally we moved to the matrix eigenvalue problems and computed the most dominant eigenvalue by power method and discussed the Gerschgorin bounds using Gerschgorin circle theorem and Brauer's theorem. And we have illustrated the all these methods which we have learnt in this course by means of some examples, problems and you have solved number of assignment problems, you also have looked into a number of problems which we have given to you for practice and finally you are ready to appear for the end semester examinations.

The type of problem that you will get in your end semester will be such that you will have questions where you may have to choose the appropriate answer with proper justification, you may have to choose whether a given statement is true or false, you may have to fill in certain blanks where you may have to work out the details 1st and then fill in the results in the

blank and there will be also be problems where you will have to completely work out the details and present the solution. Some questions similar to what may come in your examination, I have included a few examples here. So let us quickly go through them.

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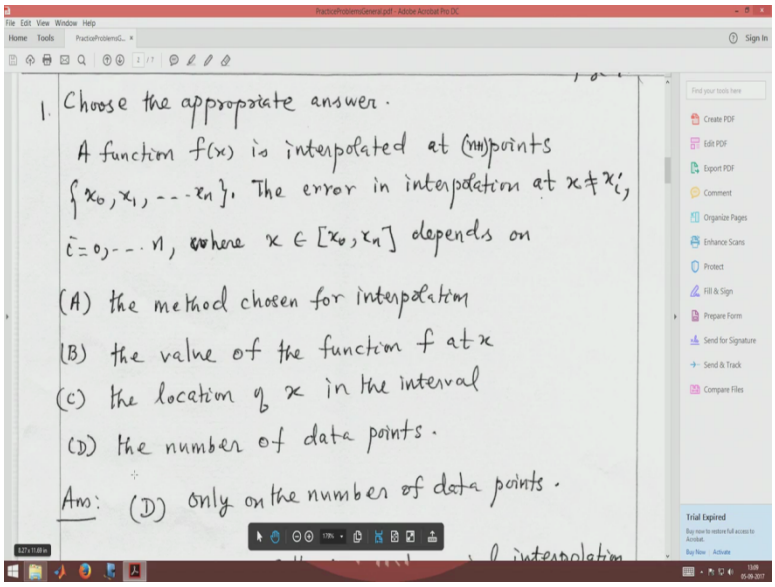

So you have the 1st question to be, choose the appropriate answer. So you are given a function F of X is interpolated at N +1 points, X0 to XN. The error in interpolation at any point X which does not, which does not coincide with any of the X I for I is equal to 1 to N, depends on what is the question. The answer is, is it a dependent on the method that is chosen for interpolation, surely not, it is clear. Does it depend on the value of the function F at X, no, you do not even know what the value of F at X. Thirdly is it dependent on the location of extremely interval, no, not at all, X can be anywhere in the interval X0 to XN.

Is it dependent on the number of data points, yes, of course. If you choose 3 points from the data, you can given an interpolating polynomial of degree 2. If you choose 10 points from the data, you can give and you can get an interpolating polynomial of degree N. So the error in interpolation depends on the number of data points and not on any of the other 3 statements which are given. So you conclude that the result is only on the number of data points and Mark D to be the correct result and giving reason for that.

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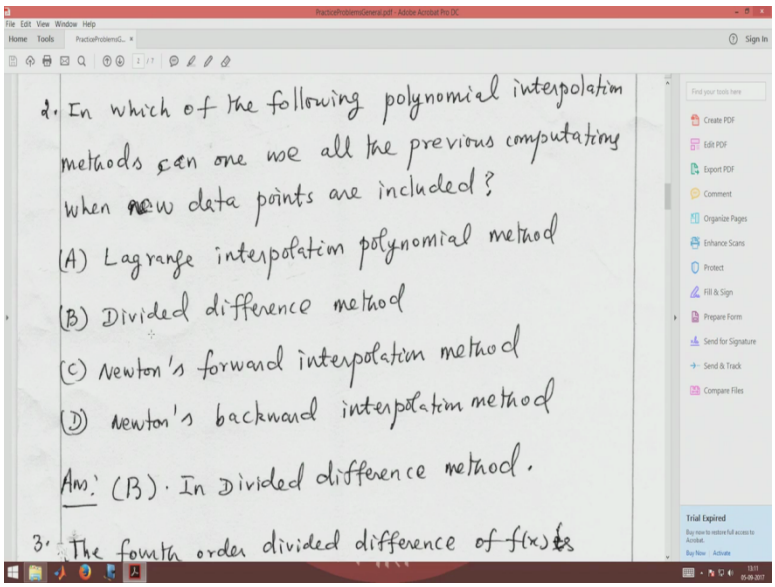

Then you have the another question which comes in interpolation again. In which of the following polynomial interpolation methods can one use all the previous all the previous computations where new data points are included. We have already discussed this in the class and therefore I straightaway go to the answer, we know that the use in Divided difference method, all the computations that have been used earlier because suppose say we are given 3 points and we are asked to get the Divided difference interpolation polynomial. Then we would be getting the interpolating polynomial of degree 2 that interpolates the setup given data which are 3 in number.

But now suddenly I have another information coming about this function, so another point and the value of that function at that point is given, now I have 4 points, so I can get an interpolating polynomial of degree 3 which I call as P3, then what is P3, P3 is simply P2 of X, namely the quadratic interpolation polynomial that we already have computed. So P 2 of X + an extra term, that extra term which arises due to the new information that is given to us. Right. And you know what the extra term should be, so if X0, X1, X2 are the previous point and X3 is the new point, then the extra term is going to be +A times X - X0, X - X1, X - X2.

How do you determine A so your P3 of X is P2 of X + A times X - X0, X - X1, X - X2. How do you determine A? This P3 must be such that it should interpolate at the new point that is given. So substitute the X value as the X coordinate of the point, new point that is given and the Y value as a function value that is specified and determine A and you now have a cubic polynomial which interpolate the given data where you have used the computation of P2 of X which you have already done. So it is in the divide and difference formula that you are able to

use the previous computations. So you write the result as B, that is so in Divided difference method.

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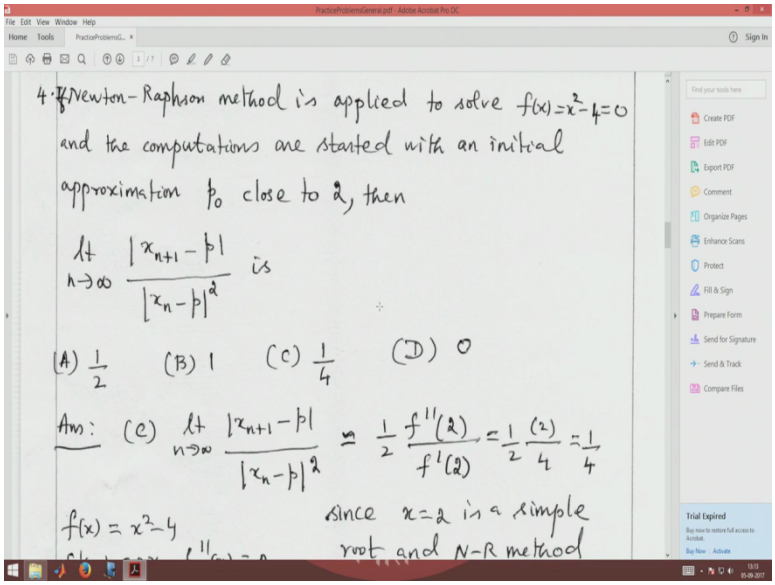

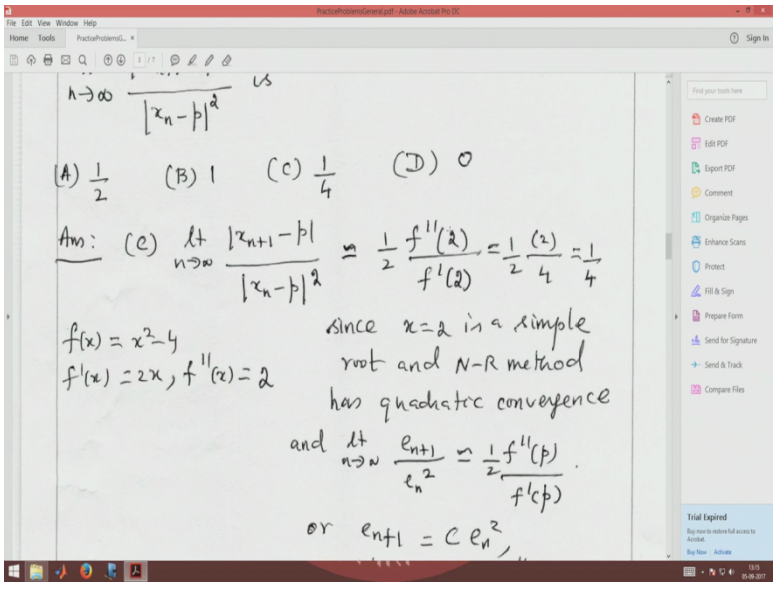

So you have another problem on Divided difference which you can try to work out. Now if I consider examples in say methods of solving a non-linear equation of the form F of X is equal to 0 and I say apply Newton Raphson method to solve this equation. And the computations are started with an initial approximation P0 close to 2, that is an initial approximation to a root of the equation. And you are asked to find what is limit N tending to infinity of modulus of XN +1 - P by modulus of XN - P the whole squared. Immediately you have recognised that this is your error at the N +1th step, this is your error at the Nth step.

So you are asked to get what is limit as N tending to infinity of EN +1 by EN square, is it any one of these values. Unless you work out, you cannot give the answer, so let us work out the details, what is it. I start with EN +1 by modulus of EN square. We have already done error analysis and we have shown, when we apply Newton Raphson method, EN +1 by EN to the power of Alpha where P is an approximation is nothing but half of the 2nd derivative of F at the root which is P divided by the 1st derivative of the function at the root. So P is given, because it says close to 2, that information is given to us.

So P is 2, and therefore we take this value, we know the function, how do we know the function, we are given F of X is equal to X square -4. Compute the 2nd derivative, compute the 1st derivative, substitute these derivatives, evaluate it at 2 and give the values here and that gives you 1 by 4. So immediately you know that the answer C is correct because you have been able to show that it is equal to 1 by 4. Because the Newton Raphson method is such that the error at the N +1th step is given by the C into the error at the Nth step squared whereas C, the asymptotic error constant is given by this information and that result has been used.

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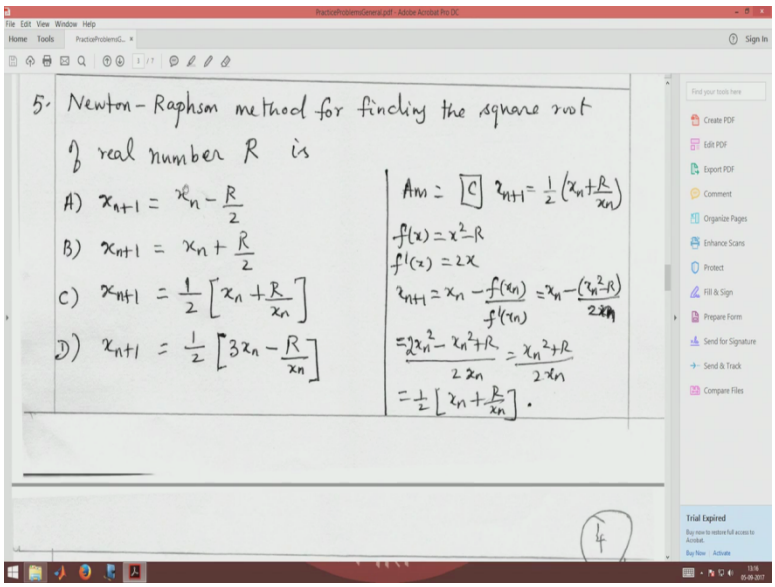

So all the results that we have shown in that there are very important in the sense that those results like what we have now shown, those results can be used to solve some examples and that has been illustrated above. Let us consider this example, Newton Raphson method for finding the square root of a real number R. What is it, is it any one of these, is the question. Yes, take the new trend of their method, take the function you want to get the square root of a real number R, so what is equation that we want to solve, X square - - R equal to 0 is equation, call that is F of X, so find F dash of X, substitute in XN +1 equal to XN - F of X N

by F dash to of XN, that is what appears here, substitute, simplify and you end up and you observe that that appears here wherein this half of XN + R by XN. So you Mark C as the correct answer.

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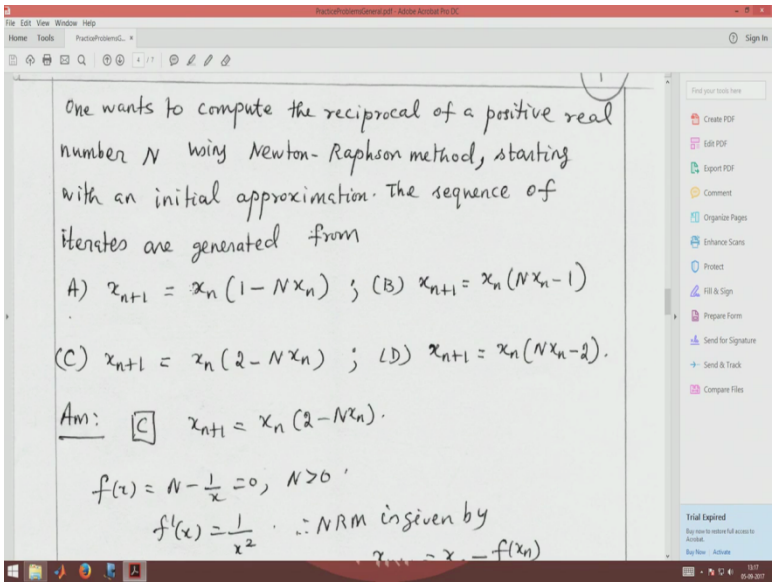

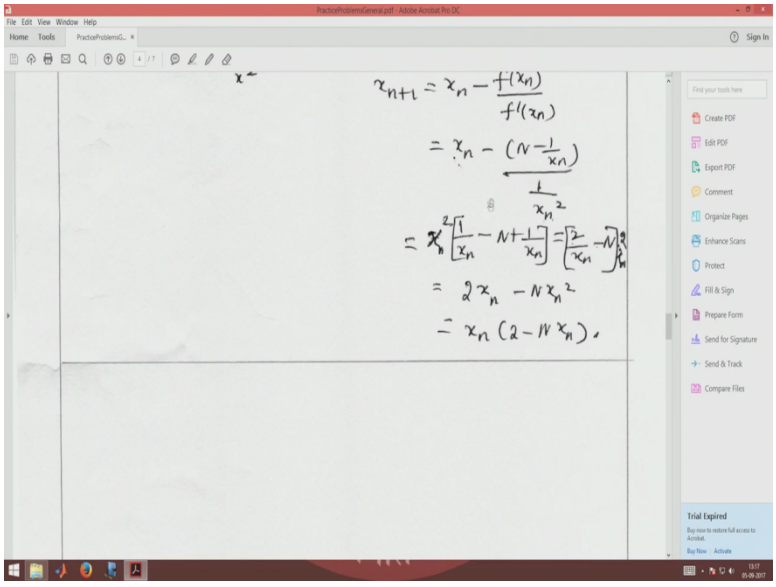

Similarly I have another example where we say the reciprocal a positive real number N using Newton Raphson method starting with an initial approximation. So the sequence of iterates are generated from which one of these? So we are asked to do by Newton Raphson method, you want to get the reciprocal of a number, so the equation that you want to solve is actually equal to 1 by N or the equation is N -1 by X is 0, compute F dash of X, substitute in Newton Raphson method, XN +1 equal to XN - F of XN by F dash of XN and that gives you this and that appears as one of the results, namely C and so you can fill in and say the answer is C.

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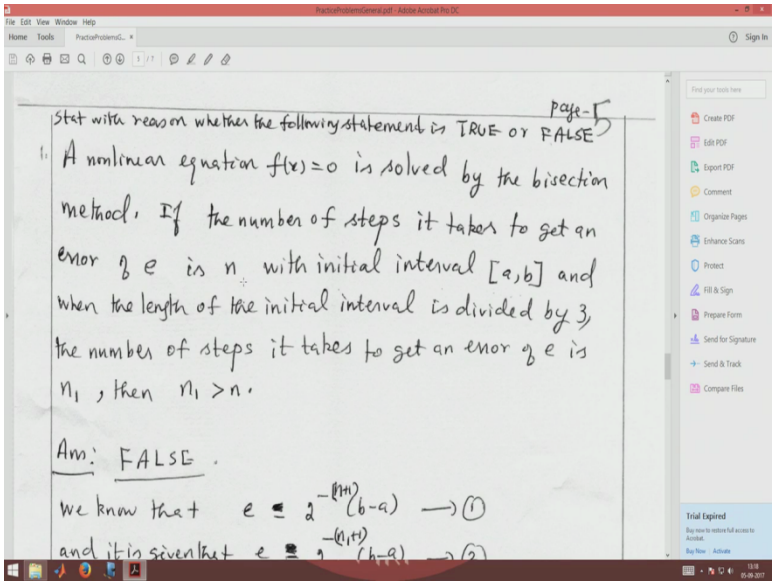

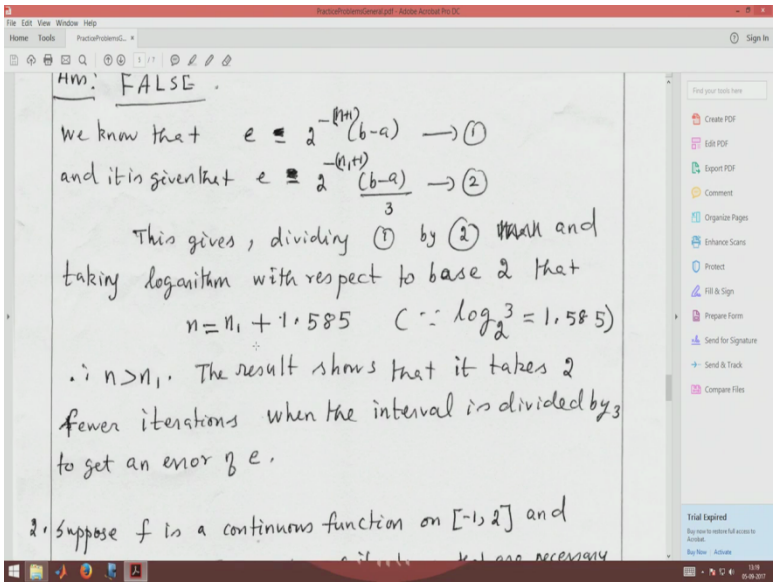

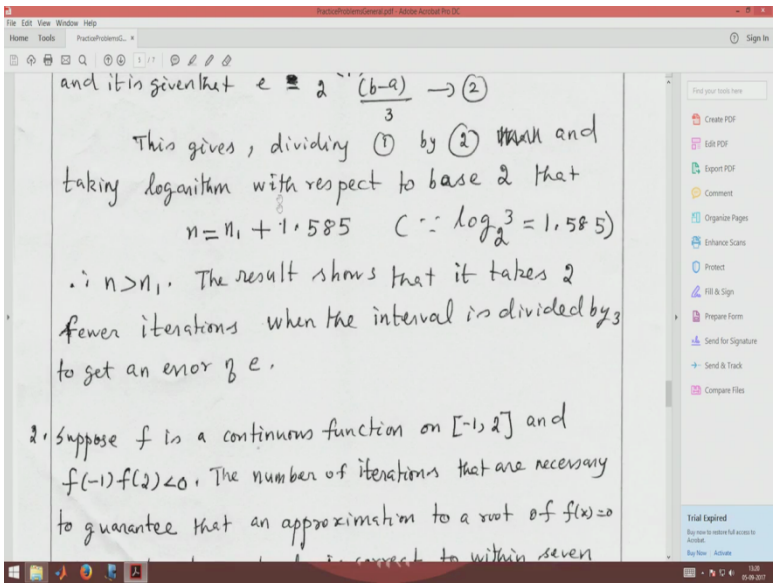

Then we have a desert which says states with reason whether the following statement is true or false. So the statement is, a non-linear equation F of X is equal to 0 is solved by bisection method. So it is a direct method for solving an equation of the form F of X is equal to 0, it belongs to the class of enclosure method. Now if the number of steps to be taken, so that you want to get an error of E and that is N steps, with initial interval A, B. And you divide the interval in such a way that the length of the initial interval is divided by 3. And the number of steps that you need to commit the same error E is say N1.

Then the statement says N1 has to be greater than N, is your statement correct is the question. So do not worry about what you should write, start writing down what is given. What is given, we know that the error is in bisection method given by the length of the interval by 2 to the power of N +1 at the end of N steps. So you start, then what do you do, you divide the interval by 3, so what is the length of the interval now, it is B - A by 3. So what is the error at step, it is B - A by 3 divided by 2 power, now the number of steps is N1, so 2 power N1 +1.

Now what is the error that you commit, it is the same e. So I have written whatever that is given in the problem, now I divide one by the other. And I see that when I take logarithm with respect to base 2, N turns out to be N1 + some quantity. So N is bigger than N1, so N is greater than N1. The question says N1 is greater than N, no, the statement is therefore false, why because of all this. So you have to justify by giving the details of why you are saying that the statement is false.

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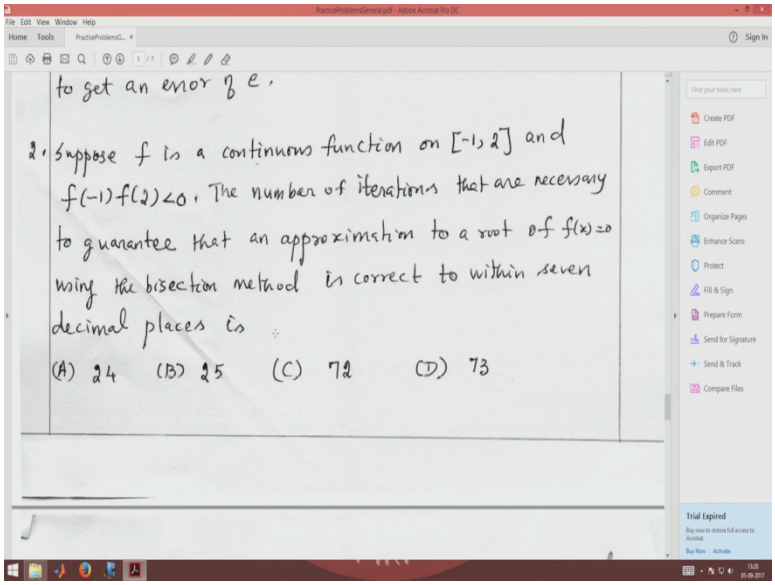

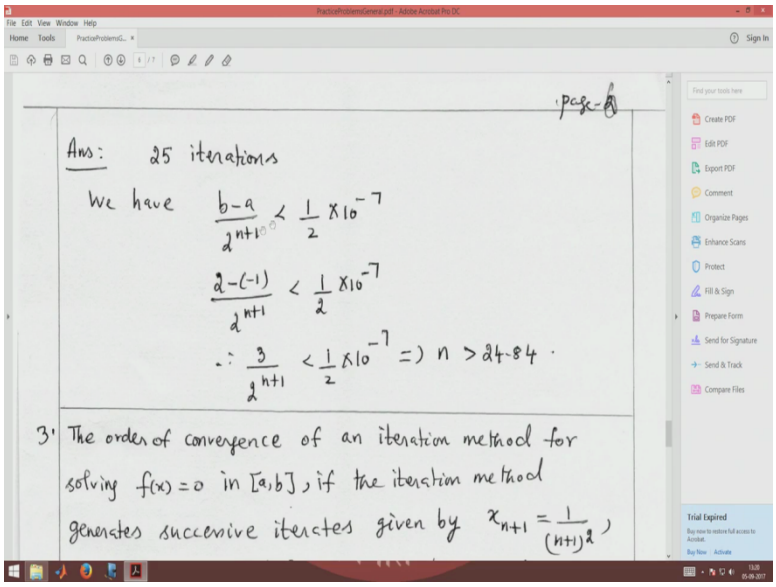

Then suppose say F is a continuous function on the interval -1 to 2. And F of -1 into F of -2 is less than 0, what does it mean, the root lies in -1 to 2. The number of iterations that are necessary to guarantee that an approximation to a root of F of X equal to 0 using bisection method is correct to within 7 decimal places is what, which one of them is correct. So you have a formula which says B - A by 2 power N +1, you want it to be less than half of 10 to the power of -7 because correct to within 7 decimal places is what your requirement is. Simplify and you get N to be bigger than 24.84 and you observe that N greater than N is equal to 25 is what appears here.

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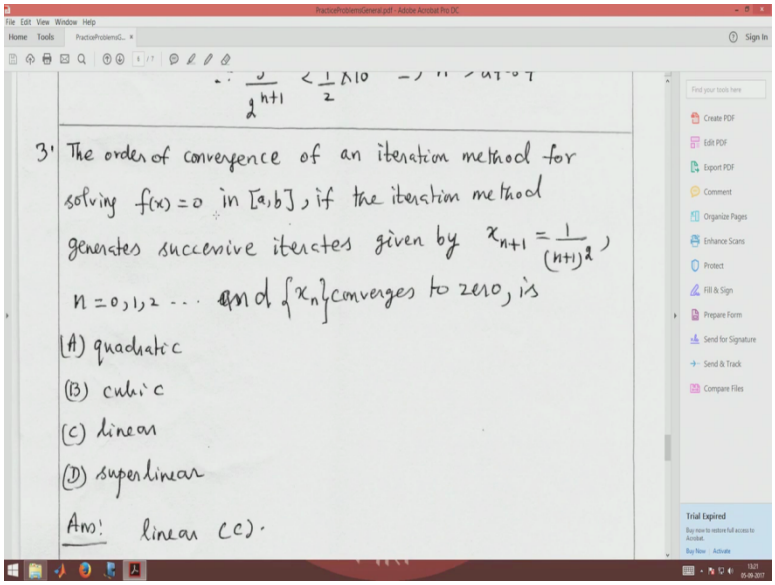

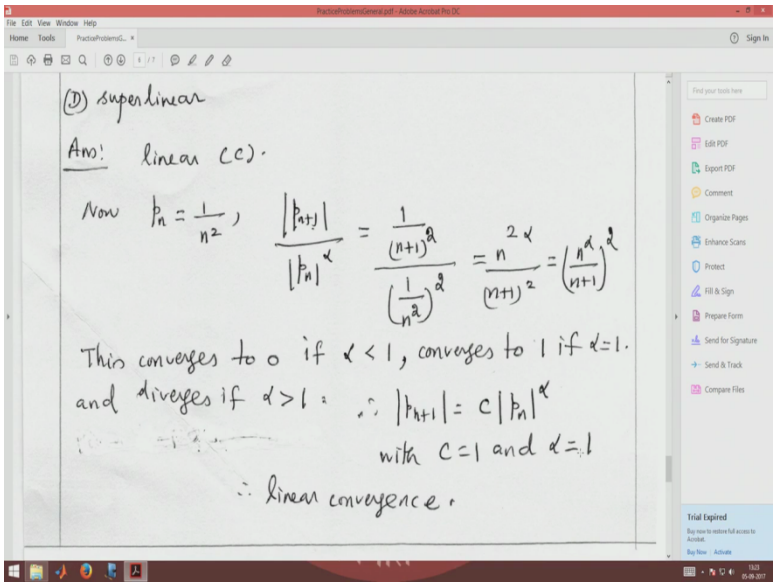

So now, the order of convergence of an iteration method for solving F of X is equal to 0 in A, B, if the iteration method generates successive iterates given by XN +1 equal to 1 by N +1 the whole square and XN converges to 0 is what. So you are asked to get the order of convergence of a method. You are given a sequence that generates, sequence which generates the successive iterates and which converges to 0. So what is the order? So Pn is given, how do you compute the order, you must get what is EN +1 divided by EN power Alpha, that must be an asymptotic error constant times, right, EN, asymptotic error constants, which is C.

So I repeat again, you have to show that EN +1 is equal to EN to the power of Alpha multiplied by C. Alpha gives you the order of convergence of the method and C give you the asymptotic error constant. So let us see what is this in this example. Now the sequence

converges to 0, so it is PN + 1 - 0 is EN +1, PN -0 is EN. So I consider mod EN +1 by mod EN power Alpha. So it is given 1 by N +1 the whole square by 1 by N square the whole square. So that is N to the 1 by N square to the power of PN power Alpha. So this must be Alpha.

So that gives you N power 2 Alpha by N +1 the whole square, this is Alpha, right. You have N power Alpha by N +1 the whole square. Now when will this quantity converge to 0? That will converge to 0 if Alpha is less than 1, it will converge to 1 if Alpha is 1 and it will diverge if Alpha is greater than 1. And you want the sequence to converge, so Alpha must be such that it converges to 1 if Alpha is equal to 1. So modulus of PN + 1 must be C into modulus of PN to the power of Alpha, with C is equal to 1 and Alpha equal to 1 and so you conclude that the method converges and its order of convergence is 1, so it has linear convergence.

I have illustrated some examples from the different topics that we have discussed in our course so that one can expect questions of this type also in your examination. So I hope you have enjoyed this course and you have had learnt some good portion of the topics from numerical analysis from this course and I wish you all the best in your career and in future. Thank you very much.