Numerical Analysis
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Lecture -3, Part - 2
Polynomial Interpolation-2

The concept of finite differences is important for interpolation of a tabulated function. We introduce forward difference operator as follows.

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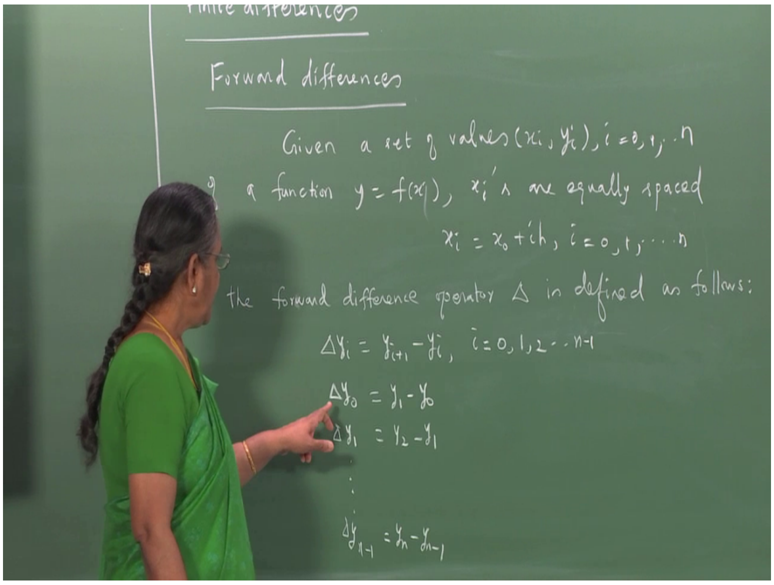

So given a set of values x i y i of a function y is equal to f(x), where x i are equally spaced and x i is x 0 plus i h, i is equal to 0 1 2 3 upto n. The forward difference operator delta is defined as follows. So delta on y i is y i plus 1 minus y i. For i is equal to 0, 1, 2, 3 up to n minus 1. So when i is say 0 delta on y 0 will be y 1 minus y 0. Delta on y 1 is y 2 minus y 1 and so on delta on y n minus 1 will be y n minus y n minus 1.

So what does delta do when delta operates on y 0 it is equal to the value that y takes at the next point namely y 1 which is y (x 1) namely f(x 1) minus y 0 which is f (x 0).

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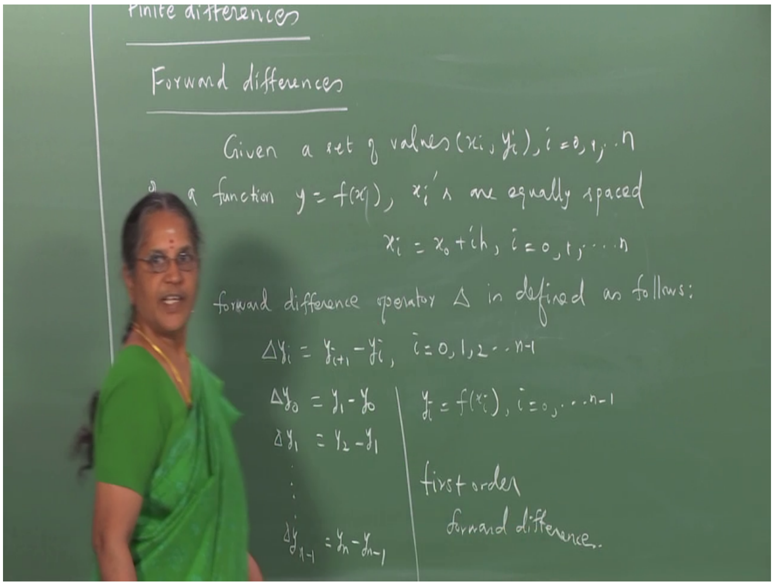

So here by y i I mean f(x i) for i is equal to 0 1 2 3 upto n minus 1. So these are the first order forward differences. The higher order forward differences can be similarly obtained.

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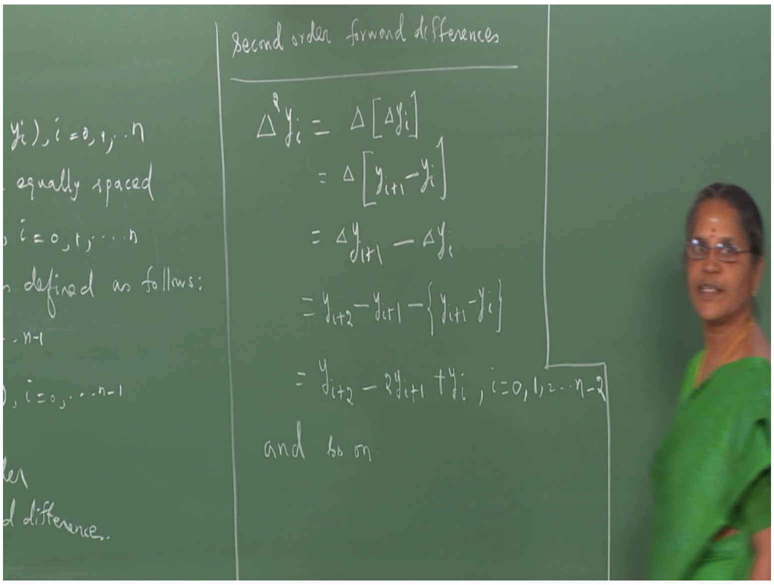

Say if I want the second order forward differences then I would like to get what is del square on y i so this will be delta on [delta y i], that is delta on we know delta y i is [y i plus 1minus y i]. So this will be delta on y i plus 1 minus delta on y i. So these are first order forward differences so I use the definition and write delta y i plus 1 as delta y i plus 2 minus y i plus 1 minus y i plus 1 minus y i. So that will give me y i plus 2 minus twice y i plus 1 plus y i. So I can compute the second order forward differences using this definition for values of i starting from 0 1 2 3 etc up to say n minus 2.

And we can follow this definition and compute higher order differences using this definition. So now that we know how to obtain the forward differences given a set of say n plus 1 values x i y i for equally spaced points.

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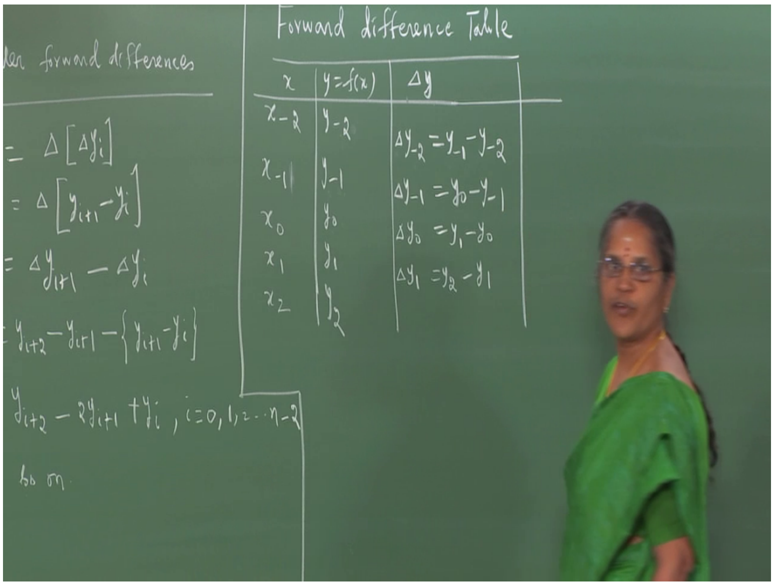

We present the forward difference table here. Why is it needed with the help of this table when we form this table we will be using this table in writing down an interpolation polynomial of degree at most n when we are given a set of n plus 1 points. It is easier to write down the interpolation polynomial for a set of given discrete values.

So we would like to see how we can form a forward difference table. So let us be given the values of x and the corresponding values of y at a set of points say x minus 2 x minus 1 x 0, x 1, x 2. So the corresponding values of y are denoted by y minus e, y minus 1, y 0, y 1, y 2. So we form the various order forward differences. So let us compute delta y the first order forward difference.

So delta y minus 2 will be y minus 1 minus y minus 2. So the value at the next point minus the value at this point. So delta on y minus 2 is y minus 1 minus y minus 2. What is delta on y minus 1 it is the next value y 0 minus y minus 1 delta on y 0 will be y 1 minus y 0 will be y 1 minus y 0. Delta on y 1 is y 2 minus y 1. So these are the first order forward differences.

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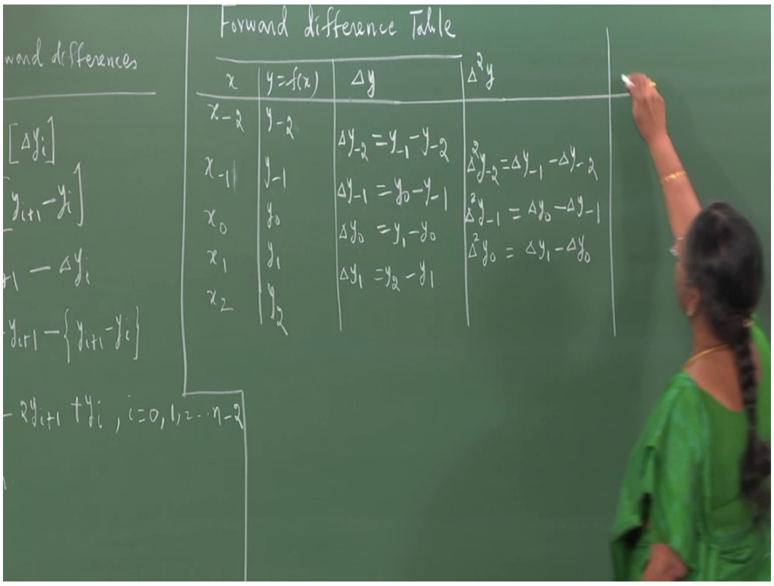

Let us compute the second order forward differences. So they are denoted by Del square of y minus 2 and that is equal to it is going to be now delta on y minus 1 minus delta on y minus 2. Then Del square minus 1 is delta on y 0 minus delta on y minus 1. Then Del square on y 0 will be delta y 1 minus delta y 0. So these are the second order forward differences.

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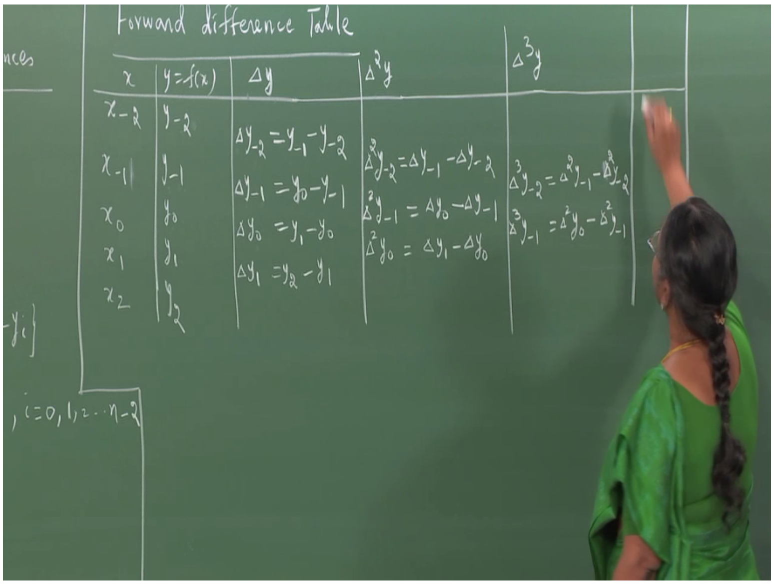

Let us go to the third order differences. Del cube y so we have Del cube on y minus 2 will be Del square on y minus 1 minus Del square on y minus 2. Then Del cube on y minus 1 will be Del square on y 0 minus del square on y minus 1.

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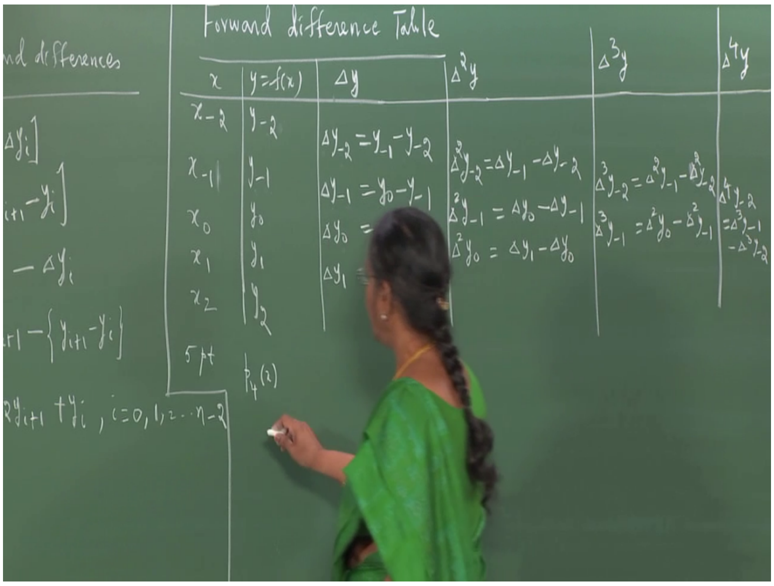

So let us now move on to compute the fourth order forward difference. Del power 4 on y so that will be del power 4 on y minus 2 and that is equal to del cube y minus 1 minus del cube y minus 2. So we observe that we are given a set of 5 points and the corresponding values at these points in this table.

So we know that there exist a polynomial an interpolation polynomial of degree at most 4 that interpolates the function f(x) at the set of these 5 points x i y i. And we observe that we have computed the various order forward differences of this polynomial namely the first degree polynomial that interpolates the function at a set of 5 points.

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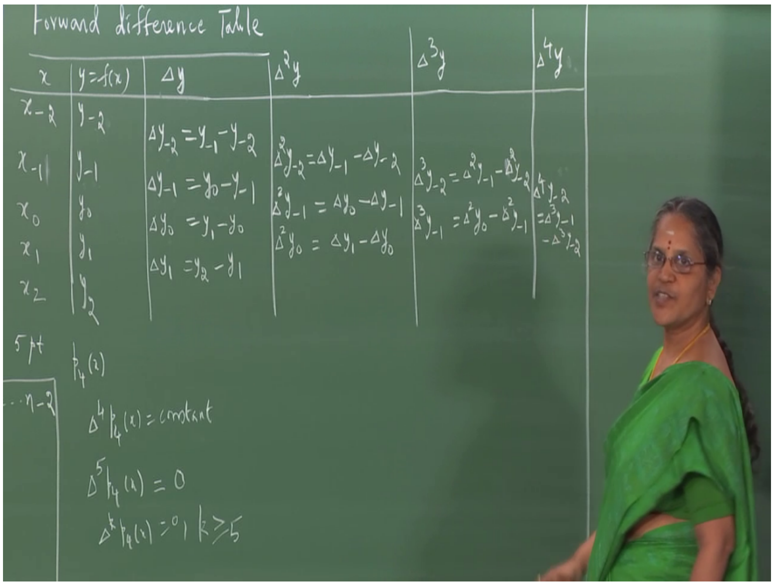

We will show that when we take forward differences of this polynomial of degree 4 which I call as p 4(x) the fourth order difference of p 4(x) will be constant and the fifth and the higher order differences will all be equal to 0. So the k th order difference of p 4(x) will be 0 for k greater than or equal to 5. I will just repeat again we have a set of 5 discrete points at which the function values are given.

So there exists an interpolation polynomial of degree 4 p 4(x) that interpolates this function at a set of these discrete points we will show later that the fourth order difference of a polynomial of degree 4 is constant and higher order forward differences of this polynomial of degree 4 will be 0. This result will be used when we find out the interpolation polynomial that interpolates this function at a set of discrete points.

And you also observe from this table that you have fourth order difference of y computed and this is constant and the higher order differences will all be 0.

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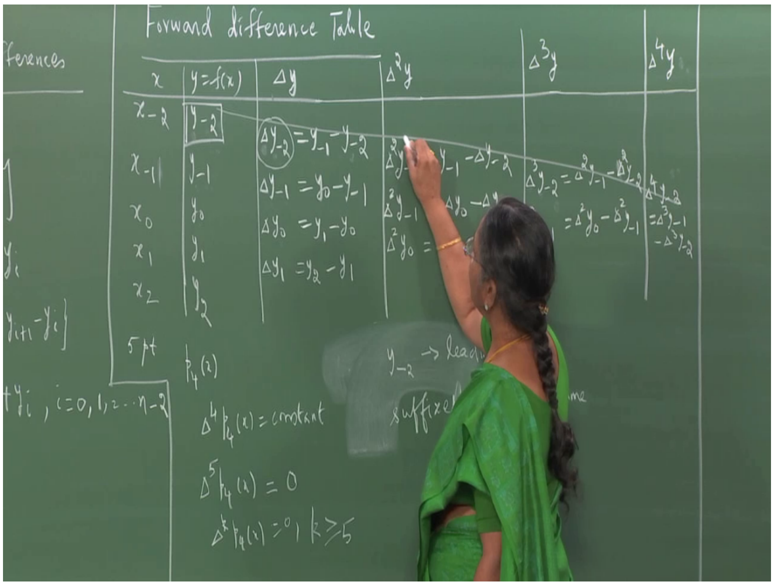

Y minus 2 is called a leading term and you also observe that along this diagonal the suffix for y minus 2 delta y minus 2 del square y minus 2 del cube y minus 2 del power 4 y minus 2 so the suffixes are all the same. And when you compute say first order difference of y you write that entry between any two elements in the previous column.

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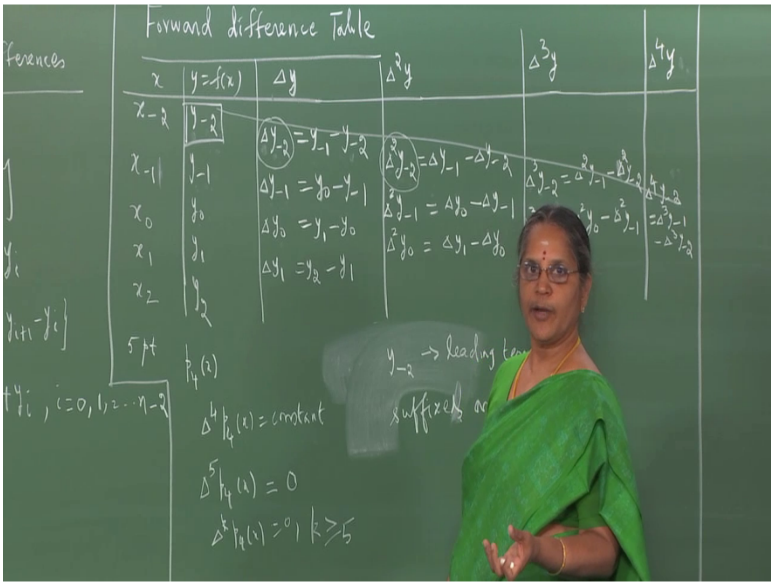

That is how you have formed this forward difference table, for example how did you write del square y minus 2. You placed it in the column for second order differences and this is located in between the 2 entries which you used to compute the del square y minus 2 in the previous column.

What are those entries they are delta y minus 1 and delta y minus 2. So you place this second order difference del square minus 2 between the two entries which appeared in the previous column. So this is how a forward difference table is formed for a set of values of a function which is given in the form of a table of values. So as we said we would like to make use of this forward difference table and find an interpolation polynomial for a set of given discrete points.

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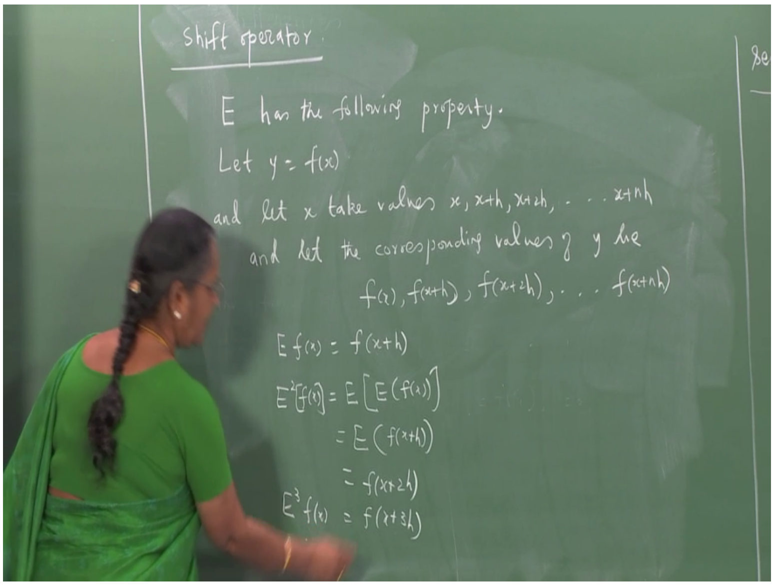

At this stage I would like to introduce another operator which is called the shift operator. The shift operator is denoted by E and it has the following property. So let y be equal to f(x) and let x take values x , x plus h, x plus 2h, and so on x plus n h. And let the corresponding values of y be f(x), f(x plus h), f(x plus 2h) and so on f(x plus nh).

So what is the property of E , E on f (x) is f (x plus h) and What is E square on f(x)? It is E on E of f(x) but E on f(x) is f(x plus h) and what is the property of E, E on f(x plus h) is f(x plus 2h). So you observe that E square on f(x) is f(x plus 2h). So now you should be able to tell me what is E cube on f(x) it is going to be f(x plus 3h) and so on.

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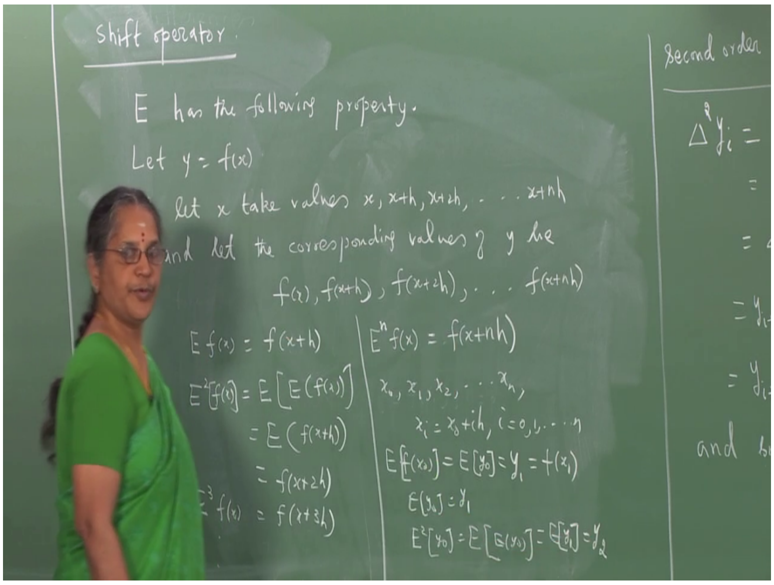

And we can write what is E to the n on f(x) that is going to be f(x plus nh). Suppose I take these values as x 0, x 1,x 2 etc x n with x i as x 0 plus x i h for i is equal to 0 1 2 3 upto n. Then I know that E on [f(x 0)] but what is f(x 0) it is y 0. What is E on [f(x 0)] E has the property that it produces the value f(x plus h) when it acts on f(x). So here E acts on f(x 0) and therefore it gives you the value y 1 but what is y 1 it is f(x 1). So what is E square on [y 0] it is E on [E (y 0)] which is E on [y 1] so that produces y 2. So E square on y 0 is y 2 So continuing this we have E to the n on[y 0] will be equal to y n.

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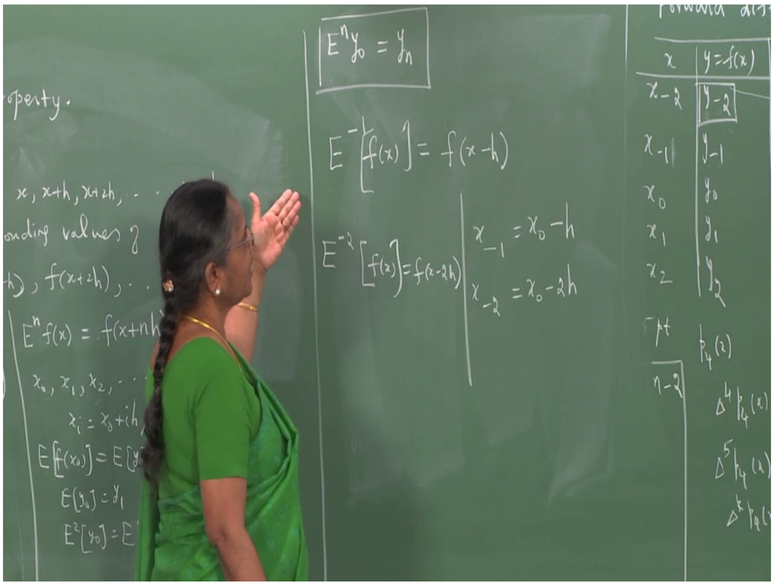

And let us see what is E minus 1 on f(x) it is going t be f(x minus h) that is x minus 1 is a point which is x 0 minus h. x minus 2 is a notation for the point x 0 minus 2h. So E to the minus 1 [f(x)] will give you the value of the function at a point which is previous to x namely it is x minus h. So that is what we have denoted by x minus 1. What is E to the minus 2 on f(x) it is going to be f(x minus 2h). So we will use the shift operator and the forward difference operator and see whether we can connect the two operators.

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Let us start with E on y 0 and that is equal to y 1. What about delta on y 0 delta on y 0 is y 1 minus y 0 but I know I can write y 1 as E on y 0. So delta on y 0 is E y 0 minus y 0. So it is E minus 1 on y 0. So the role of delta on y 0 is identically the same as the role of the operator E minus 1 on y 0. So delta is related to E minus 1 by delta is equal to E minus 1 or E is 1 minus delta. So the shift operator and the forward difference operator are related by E is equal to 1 plus delta.

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So our goal now is to make use of the forward difference table and the property of the shift operator and the relation between the shift operator and the forward difference operator in writing down an interpolation polynomial that interpolates the given function at a set of discrete points which are equally spaced.

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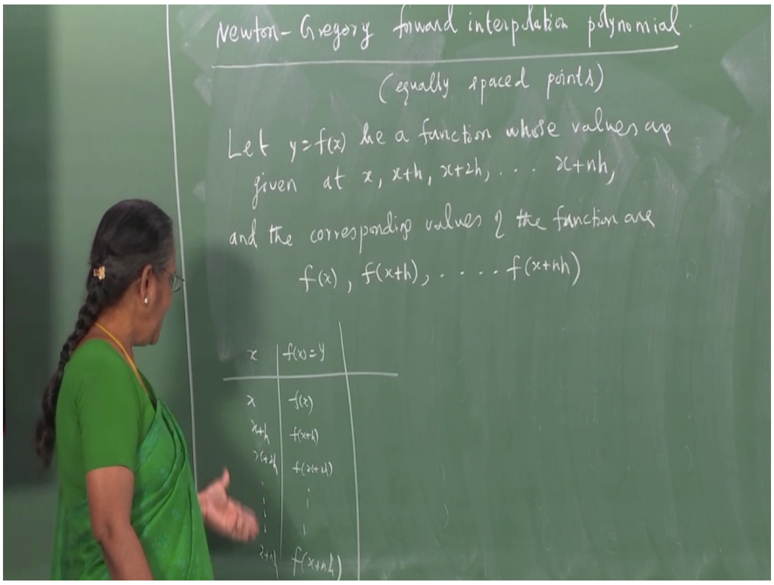

We now derive Newton Gregorys forward interpolation polynomial that interpolates a function which is given in the form of a table of values so our derivation is for a set of equally spaced at which the function values are specified in the form of a table. So let y is equal to f(x) be a function whose values are given at x , xplus h , x plus 2h etc x plus nh. The corresponding values of the function are f(x), f(x plus h) etc f(x plus nh).

So we have a table of values of the function at a set of points x plus h etc x plus nh and the corresponding values of this function namely f(x), f(x plus h), f(x plus 2h) and so on f(x plus nh). What is our problem given a set of n plus 1 points we would like to obtain a polynomial of degree at most n that interpolates at a set of points x , f(x), xplus h, f(x plus h) etc x plus nh, f(x plus nh). So we want to construct this polynomial of degree at most n that interpolates the set of n plus 1 points which are given to us.

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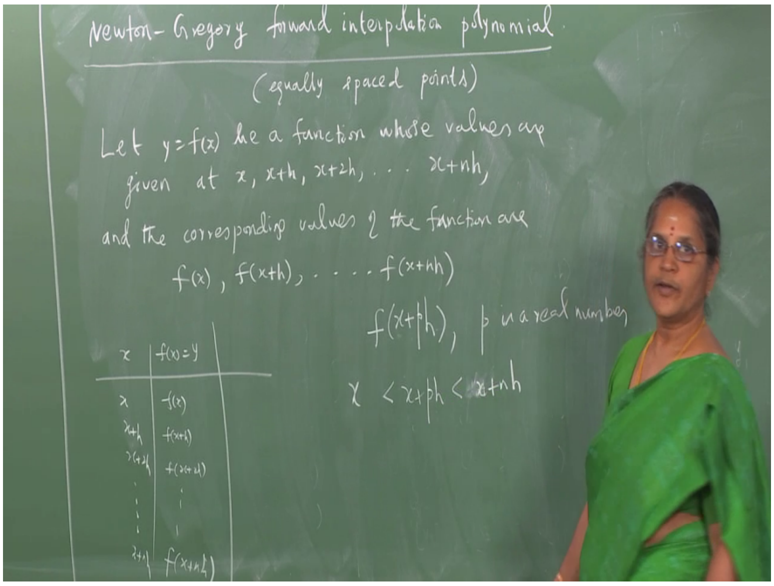

Suppose say we are able to do that and we would like to now estimate the value of the function at some point x plus ph where p is a real number and x plus ph is such that it lies between x and x plus nh. That is our problem.

Problem 1 is to obtain an interpolation polynomial that interpolates the function at a set of given points when we are done with that what is the estimate of the function f at x plus ph where p is a real number and x plus ph lies between x and x plus nh.

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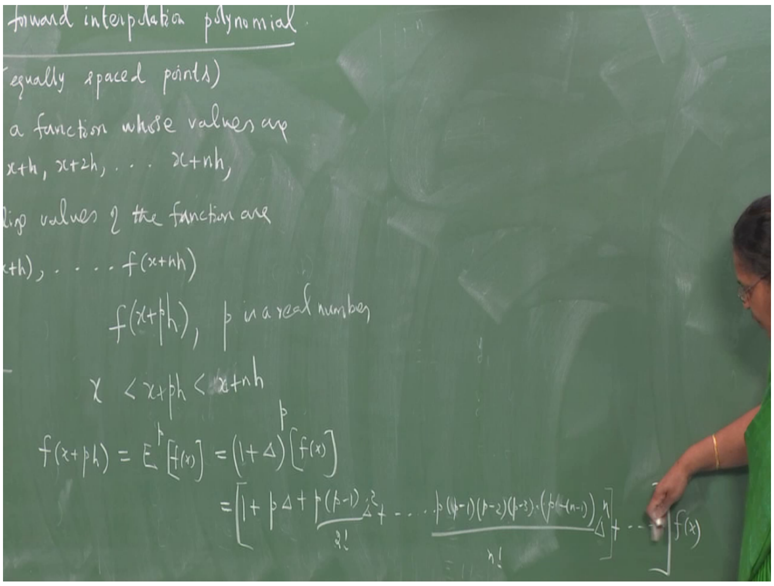

So I can write down f (x plus ph) as E to the power of p on f(x) that is the property of the shift operator which we have seen E to the p on f (x) gives you f(x plus ph). But we know that E power p is 1 plus delta to the power of p on f(x) because E and forward difference operator are related by E is equal to 1 plus delta. So at this stage I can expand 1 plus delta power p using binomial theorem.

So it is 1 plus p into delta plus p into p minus 1 by factorial 2 into del square plus etc p into p minus 1 into p minus 2 into p minus 3 etc. p minus n minus 1 by factorial n into when I have p into p minus 1 by factorial 2 I have del square when I have p p minus 1, p minus 2, p minus 3 etc p minus n minus 1 then it is del to the power of n plus etc operating on f(x).

So the question is how many terms will appear in this expansion we have already seen from our forward difference table which we formed for a set of 5 values we can fit a polynomial of degree 4. And the fourth order forward difference is constant and higher order differences are 0. So we shall now prove that result in general namely given a set of n plus 1 points one can fit a polynomial of degree at most n namely p n(x) and the n th order forward difference is constant and higher order forward difference is beyond n plus 1 are all 0.

We will make use of that result so that the terms here will be such that they will terminate here because the n the order difference is constant and the higher order difference will be 0 so the terms will such be those which appear within the brackets operating on f(x). So before continuing this step and finding what the polynomial is we shall prove the result that given a set of n plus 1 points where the x i equally spaced. The n th order forward difference of a polynomial of degree n is constant and n plus 1 th and higher order differences are 0.