Numerical Analysis
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Lecture No 42
Solution of Linear Systems of Equations -5 Gauss Jordan Method

Good morning everyone, in the last class we discussed Gauss elimination method with partial pivoting and obtained solution of systems of equation of the form A x = b. We shall now discuss another direct method known as Gauss Jordan method, in Gauss elimination method the given system A x = b is reduced to a system of the form U x = b hat where U is an upper triangular matrix so the coefficient matrix A is reduced to an upper triangular matrix by using elementary row operations. In Gauss Jordan method which is another direct method, given the system of equation in the form A x = b the coefficient matrix A is reduced to an identity matrix I so that the solution of the system can be directly obtained by solving I into x is equal to some b cap so that x will be given by the vector b cap which appears on the right-hand side.

For the procedure is similar to Gauss elimination method where we had used elementary row operations, in Gauss elimination method at any step that we will have a pivotal equation and when we make all the entries below the pivot at that step to be equal to 0, then we convert the coefficient matrix A to an upper triangular matrix. On the other hand in Gauss Jordan method at any step of our computation we not only make the elements which lie below the pivot in its column to be 0 by using elementary row operations but also the elements which lie above the pivot in its column to 0 and thus we try to reduce the coefficient matrix A to an identity matrix. So we will see how this is done so that we would have understood Gauss Jordan method, but it is a very expensive method from the point of view of computations and therefore this method is not usually used to solve a system of equations A x = b.

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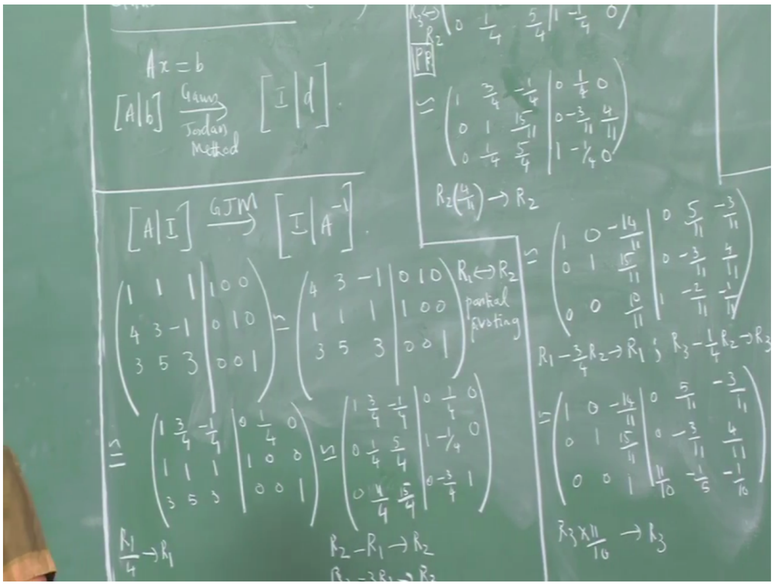

However, this method is used to compute the inverse of a matrix A which is a non-singular matrix so we demonstrate this by means of the following example and thereby you will understand how Gauss Jordan method can be used to solve a system of equations. So let us see how we can use effectively Gauss Jordan method to obtain an inverse of a given matrix A. So as I explained earlier, given a system of equation A x = b, if I write down the augmented matrix A along with right-hand side vector if I apply elementary row operations which describe the Gauss Jordan method then I end up with and equivalent system in which the matrix A is reduced to an identity matrix and the vector b is reduced to a vector D so that solving the system I x = d will give immediately the unknown vector x that is how you solve a system of equation by Gauss Jordan method.

But we would now like to use Gauss Jordan method in computing the inverse of a matrix A. So we are given a matrix A so what should I do? I should argument this matrix A by means of an identity matrix of the same order. So if A is the 3 by 3 matrix then take an identity matrix of order 3 by 3 and then apply Gauss Jordan method and try to convert it to the matrix I augmented by a new matrix and the resulting matrix that you get here is going to be the inverse of the given matrix A. So let us just illustrate this by taking A to be this and then I have augmented this by means of an identity matrix of order 3 by 3 because the given matrix is a 3 cross 3 matrix. So what is my first step? If suppose I am also asked to make use of partial pivoting so what should I do?

I should scan the first column and see which is numerically the largest element and I observe that the numerically largest element appears here namely 4, so I must exchange the row in which it occurs with the first row so that I bring the second equation to the first equation position. So that is what I do so I exchange R 1 with R 2 because I want to use partial pivoting strategy, so my second row comes as the first row so my elements are 4, 3, – 1, 0, 1, 0 and the first row comes to the second equation position so it is 1, 1, 1, 1, 0, 0, the third equation is written as it is so that is going to be the first step. So at any step take care to introduce partial pivoting strategy if necessary so this step is complete.

Then we would like to apply Gauss Jordan method which is actually similar to Gauss elimination method. What do we do? We keep the first equation as the pivotal equation and then call this element as the pivot and try to make the entries below the pivot in its column to 0 that is what we should do. So before doing that I would like to make this entry as one because my goal is to reduce A matrix to an identity matrix so in that case this element should have a value which is one and therefore, I divide the throughout by 4 and take this entry to be 1, so 1, 3 by 4, – 1 by 4, 0, 1 by 4, 0. So I perform the row operation that the new first row will have the old R 1 by 4 so R 1 by 4 will replace the old R 1 namely the old first equation.

The second and third equations are there, so at this stage (I observe that this is my pivotal equation and this is my pivot so I must make these 2 entries into 0s. So what should I do? I should take R 2 and subtract R 1 from it because both the entries are 1, 1 so R 2 – R 1 will replace the second equation so I will have 0. Then I will have 1 – 3 by 4, 1 – of 1 by 4 then 1 – 0 then 0 – 1 by 4, 0 – 0 so that will give me the second row entries. Then I also have to make this entry 0 so what should I do, R 3 – 3 R 1 will give me entries in the new third row so I will have 3 – 3 which is 0 then 5 – 9 by 4 that will give me 11 by 4 then 3 – of 3 by 4 that will give me 15 by 4.

0 – 3 times 0 so 0, – 3 times 1 by 4 so – 3 by 4, 1 – 0 into 3 is 1 so that is going to be my third row so I observe that I have entries below the pivot in the first row in its column to be 0 at this stage. So I have completed the first step now I move over to the second step, so what should I do? I should take my second equation as the pivotal equation but before selecting that pivot equation I just scan the columns or I just scan the equations namely the second equation and all the equations which lie below the second equation and then scan the second column elements so that I can have numerically the largest element to occur in the pivot namely in the a 22 position in the second row.

So when I scan the column elements, I observe that 11 by 4 is greater than 1 by 4 so that suggests that I must use partial pivoting strategy here and bring the third equation to the second equation and the second equation should be moved to the third equation location. So I do that and that is what we have written, the elementary row operation is interchanged row 3 with row 2 and so the first equation remains as it is, the third equation comes as the second equation, the second equation is now the third equation so we are at this step here after applying partial pivoting. So now I at this stage I see what is the pivotal equation so this is going to be my pivotal equation and what is the pivot, 11 by 4 is the pivot.

But what do I want? I want the reduced matrix to be an identity matrix and therefore this pivot must be unity and therefore I divide the second row throughout by 11 by 4 or I multiply the second row throughout by 4 by 11 and rewrite the new R 2 equation namely the first row remains as it is, I multiply throughout by 4 by 11 so I get 0, 0, 15 by 11, 0, – 3 by 11 and 4 by 11, this is my new second equation, the third equation remains as it is. So I observe that at this stage this is my pivotal equation with this as my pivot clear so what should I do? In the Gauss elimination method we would look at the entries in the same column in which pivot appears and make those entries into zeros right.

But in Gauss Jordan method we not only look at those entries in that column where the pivot appears below the pivotal equation but also make the entry in its column above the pivot and make that also 0. So what should I do? I have my pivotal equation as it is here, I want to make this element into 0 so what should I do, I must take R 3 and subtract 1 by 4 of the pivotal equation namely R 2 and write that value in the new R 3 equation, so when I do that, I take R 3, – 1 by 4 of 1 that given by 4 and subtract and that will give me 0. Having made this entry 0 I should look at the entry above the pivotal equation in the column in which pivot appears and make that entry into 0.

So it concerned with row 1 and the new row 1 is obtained by taking the old row 1 – 3 by 4 times this 1 in the pivotal equation, so R 1 – 3 by 4 R 2 will give me the entries in the new R 1 so you observe that when I perform the elementary row operation I get the entries in the first row as this. Now my system at this stage is this and I observe that in the next step I must take as my pivotal equation and since I want to reduce A to an identity matrix, in the pivot I must have unity and therefore I divide the third equation throughout by 11. Multiply equation 3 by 11 by 10 and so I make this entry into unity and the other entries are multiplied by 11 by 10 and this is what we get. What should we do at this stage in Gauss Jordan method?

We must make all the entries in the column in which the pivot appears below it and above it to be 0 but there are no more equations below it, we only have equations above this pivotal equation at this stage so I must make this entry into 0 and this entry into 0. So what do we do, we take the first row and add to it 14 by 11 times the third row that will give you all the entries in the new (first row and so I get 1, 0, 0, 7 by 5, 1 by 5 and – 2 by 5, so I have reduced this to 0. Now I must reduce this to 0 what should I do? Take the old R 2 and subtract 15 by 11 R 3 this will give you the entries in the new R 2 so this entry is also reduced to 0 and the corresponding computations have been carried out in this row.

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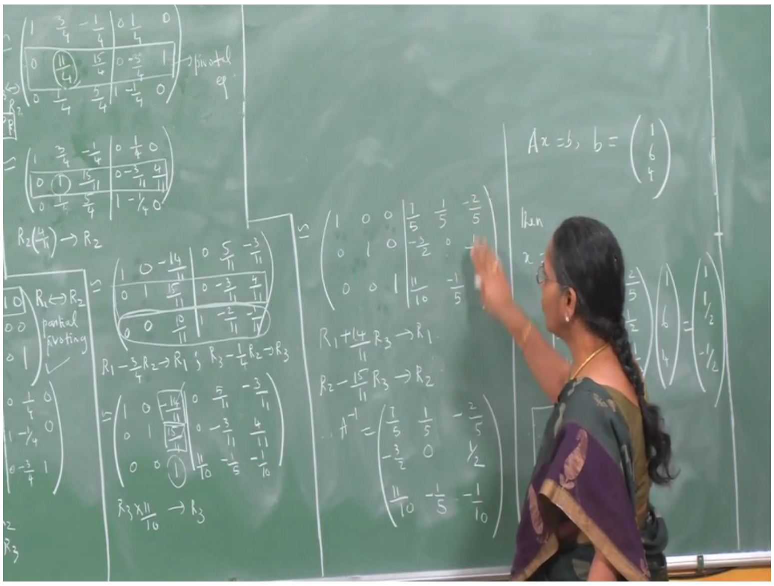

So you observe that on your left-hand side you have an identity matrix, so all the steps in the Gauss Jordan method are performed and the result is that you have an identity matrix appearing here in place of A after Gauss Jordan method has been applied. And the identity matrix which appeared on the right of A has been changed to a matrix which is a 3 cross 3 matrix and this is what your A inverse is. So you have been able to compute the inverse of a matrix A which is given by this matrix which is a 3 cross 3 matrix. So Gauss Jordan method can be efficiently used to compute the inverse of a given matrix and I hope you have understood the steps which are involved in Gauss Jordan method.

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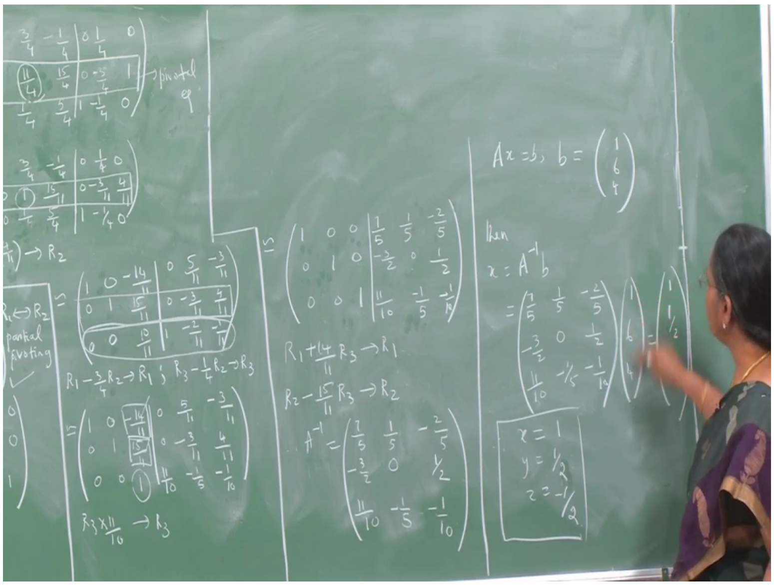

Suppose say I want to solve a system in which the coefficient matrix is this A using Gauss Jordan method. Having got the inverse the solution is going to be x = A inverse b and therefore take A inverse, multiply by the vector b which is given there and that will give you the solution so you have the values of the unknown, values of the components in the unknown factor x, y, Z to be given by this which is the solution of the given system A x = b. So you have used Gauss Jordan method in solving the system of equation A x = b by computing a inverse so that the solution can be written in the form x = A inverse b. Now, one can use calls Jordan method in computing the inverse of a matrix as follows.

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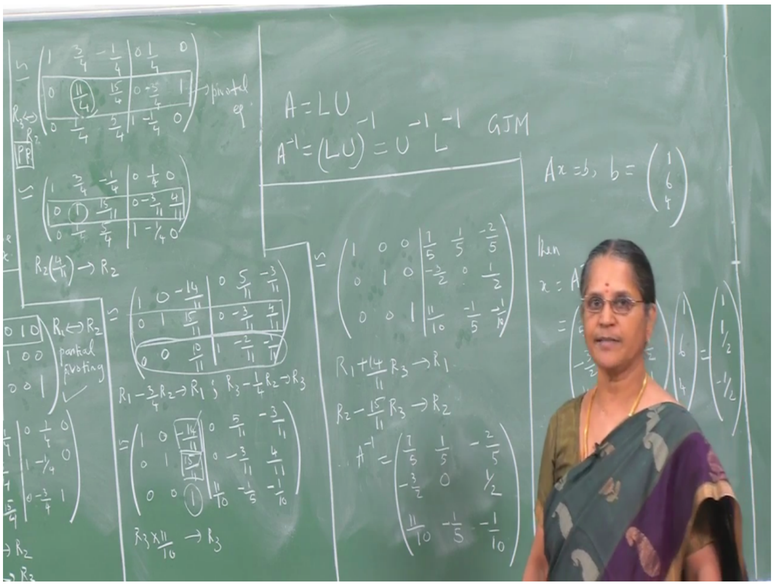

Suppose A is such that you are able to decompose the matrix in the form L into U. you already know that this is possible, when is it possible when L can be taken to be a unit lower triangular matrix in which case do little decomposition is possible, U can be taken as a unit upper triangular matrix in which case you have Crout's reduction. If the matrix A is such that it is a real positive definite symmetric matrix then A can be factored in the form L into L transpose where L is a lower triangular matrix with positive entries along the diagonal. So when you have obtained the factorisation of A in the form L into U then if you are interested in computing A inverse, it is L U the whole inverse, so it is U inverse L inverse.

So U is an upper triangular matrix, L is a lower triangular matrix and therefore computing inverse of these matrixes by Gauss Jordan method will be easier than rather than computing directly inverse of the given matrix A. So decompose the matrix A in the form L into U and then take A inverse to be U inverse into L inverse then the computation of inverse of an upper triangular matrix or a lower triangular matrix are simpler and easier then computation of inverse of the given matrix A. So, one can easily obtain the inverse of a given matrix by first applying decomposition of the given matrix and then taking inverse of the given matrix to be the product of U inverse into L inverse.

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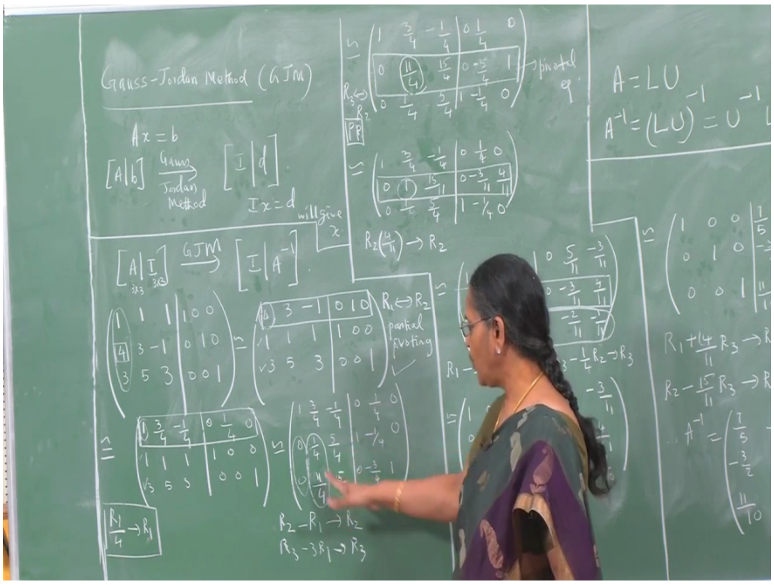

So I shall give some problems where you can use this method to compute the inverse in the assignment sheets. So with this our discussion on the class of direct methods for solving a system of equation is complete, the only thing that remains to be discussed is the error analysis for direct method. Why do we say we need to perform error analysis for direct methods, when we compute or solve this problem using Gauss Jordan method then we take these entries say 5 by 4, 11 by 4, 15 by 4 and represent it in the form of a number with decimal places. Depending upon the machine precision, the number of decimal places that it takes is going to be such that it is going to either chop of the numbers or there will be rounding of those numbers as a result round of errors are incorporated.

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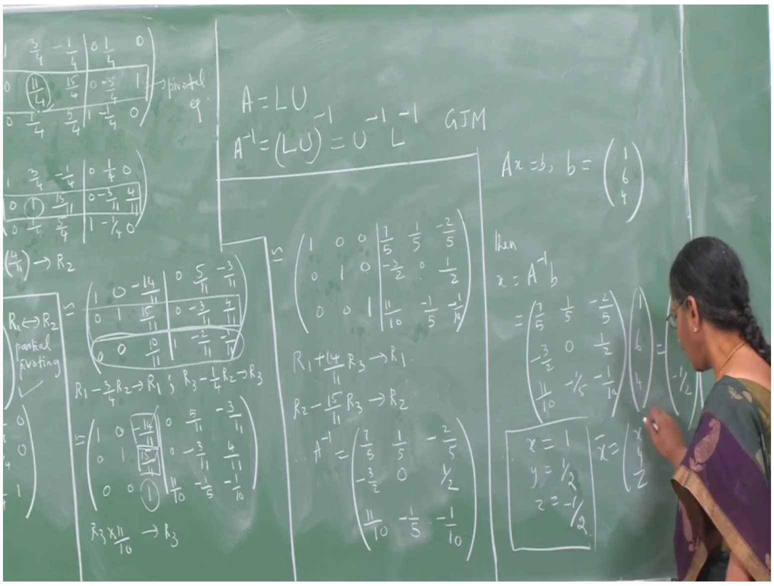

This happens at every step of the computations and as a result the round of errors are incorporated in each step of computation and the final solution that you get may not represent the actual solution of the given system A x = b because there are round of errors which are incorporated at each step of the computation. So we would like to know whether small deviations or changes that occur at any step namely the small errors that are incorporated at each step result in large deviation in the solution that we obtain finally.

And we obtain the solution as a vector, this x is a vector having components x, y, Z so there are going to be errors in the vector which come out as a solution for the system A x = b and therefore we must have some knowledge of measuring this error in a vector and therefore we introduce the notion of norm of a vector norm of a matrix which will enable us to give the size or the magnitude of the error in the vector that we may obtain as a result of solving a particular system using a computer. So we need to discuss the errors and we need to discuss what is the magnitude of the error that is incorporated at each step of our computations while solving system of equations in the case of direct methods.

So we shall do error analysis for direct methods now and then find out what is going to be the magnitude of the absolute error, what is the magnitude of the related error, can we give abound on the absolute error and the relative error in case to make use of direct method for solving a system of equations.