Numerical Analysis
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Lecture No 37
Root finding Methods 9
Practice Problems

So in this class we shall consider some problems based on the numerical methods that we developed to solve equations of the form f of x = 0, so let us consider the following problem.

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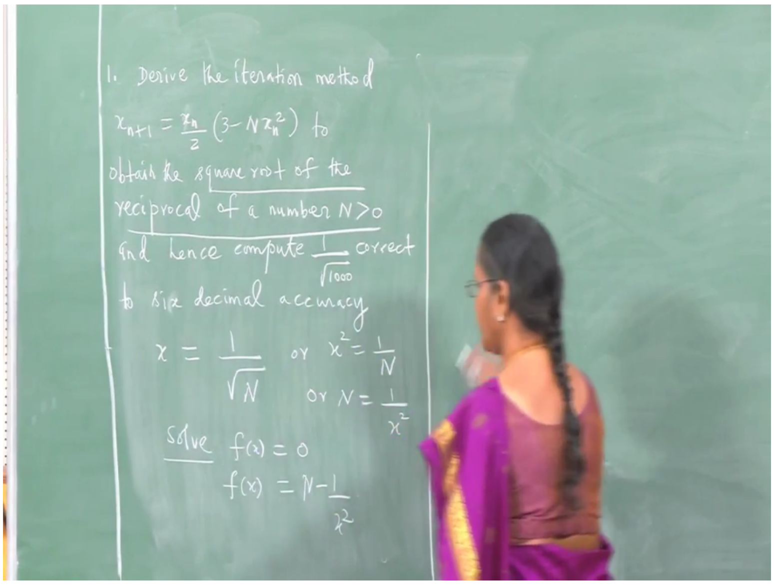

We are asked to derive the iteration method of the form x + 1 = x n by 2 into 3 – N x n square to obtain square root of the reciprocal of a number N and use this method to compute 1 by square root of 1000 correct to 6 decimal accuracy. So what is the problem? We are required to obtain square root of reciprocal of a number N, so find x such that x is 1 by root N or x square = 1 by N or N = 1 by x square, so let us solve the following problem f of x = 0 where f of x is given by N – 1 by x square. So let us apply Newton Raphson Method and solve this problem.

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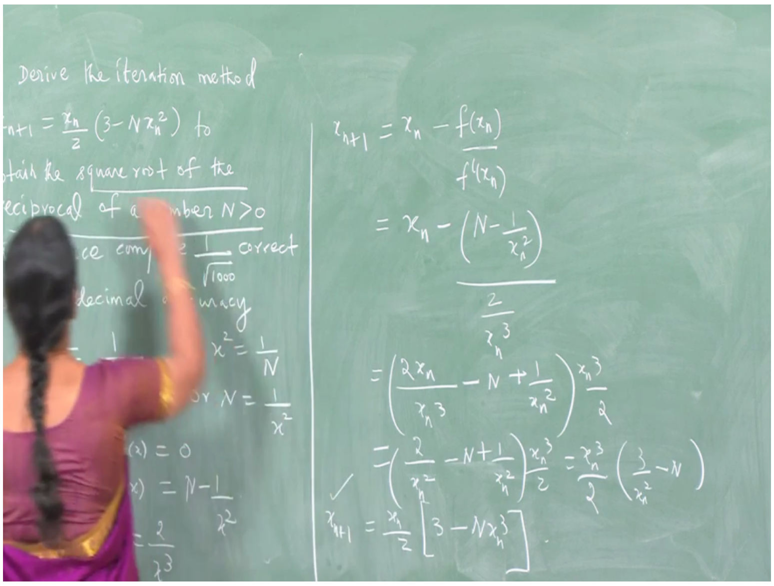

So what is Newton Raphson Method, it gives you x n + 1 = x n – f of x n by f dash of x m, why did I choose Newton Raphson Method, for a simple reason that when I look at the answer that is given in the problem, it is like x n + 1 = something. So let us try to apply Newton Raphson Method and see whether our computations finally yield us the method as this, so it will be x n – what is f of x n, it is n – 1 by x n square divided by, I require f dash of x n so let us compute f dash of x, so that will give you 2 divided by x cube. When we simplify this, we end up with 2 x n by x n cube – n + 1 by x n square multiplied by x n cube by 2. So this gives you to buy x n – N + 1 by x n square into x n cube by 2. So I have 3 by x n square – n multiply by x n cube by 2 and that simplifies to x n by 2 into 3 – n x n cube, and what is it? That is x n + 1.

We observe that this is essentially what appears in the given problem, so what is it that we have developed? We have developed an iteration method given by x n + 1 = x n by 2 into 3 – n x n cube for finding square root of the reciprocal of a number n which is a positive number, and we are asked to compute 1 by root of 1000 correct to 6 decimal accuracy, we require an initial approximation x 0. So let us try to find out an x 0 so that when I know x 0, I can compute (())(4:07) generate the iterates x 2, x 3, et cetera and stop the computation when I have achieved the desired degree of accuracy.

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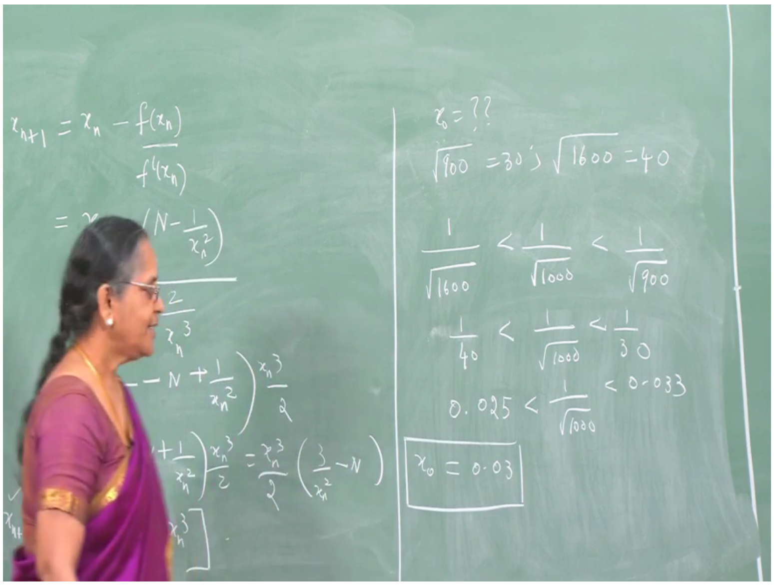

So we know that we require root of 1000, so let me start with square root of 900 is 30 and square root of 1600 is going to be 40 and so root of 1000 will lie between 30 and 40 and therefore, I require 1 by square root of 1000 so it will lie between 1 by root of 1600 and 1 by root of 900 namely will lie between 1 by 40 and 1 by 30 or it will lie between 0.025 and 0.033. So I can take my x 0 to be of value which is less than 0.03 so let me take it to be 0.033, so once I know x 0 I make use of this iteration scheme and find what x1 is and then x2 is and so on. And the problem says, determine this 1 by root of 1000 correct to 6 decimal accuracy and I leave the rest of the computations for you to complete starting, from x 0 you work out the details and then find out what is 1 by square root of 1000 correct to 6 decimal accuracy.

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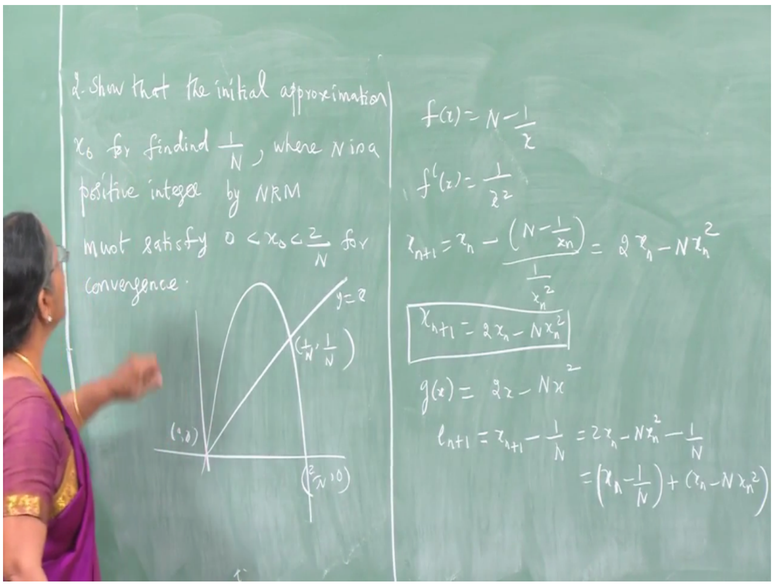

Let us now consider the following problem, we are asked to show that an initial approximation and 0 for finding the reciprocal of a number N, where N is a positive integer by Newton Raphson Method must satisfy the conditions that x 0 should lie between 0 and 2 by N for convergence. Any other value of initial approximation x 0 which lies outside this interval will make the sequence of iterates to diverge, the initial approximation for finding the reciprocal of a positive integer N was satisfy the condition that x 0 must lie between 0 and 2 by N, this is what we are asked to show. So what is it that we want to solve? We want to find that x which is the reciprocal of the positive integer n or n is 1 by x, so we need to solve the equation f of x is n – 1 by x, so we would like to apply Newton Raphson Method and it is also given in the problem solve by Newton Raphson Method.

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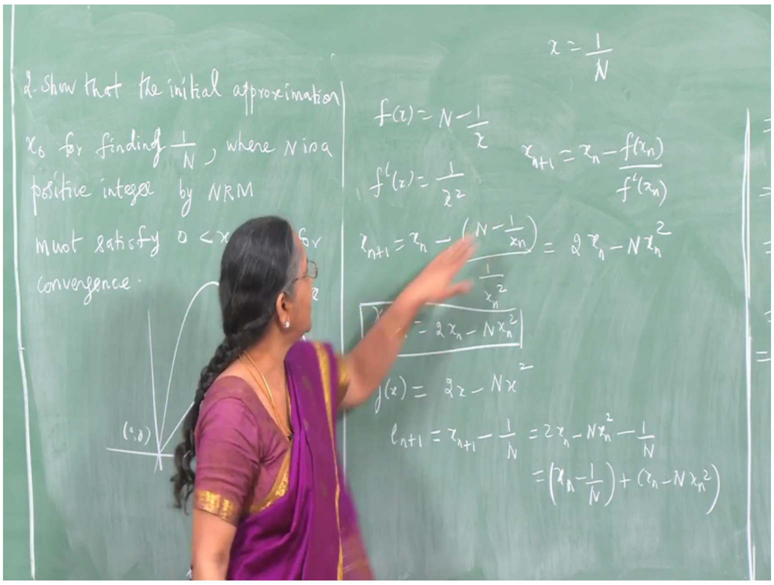

So I compute f dash of x and Newton Raphson Method is x n + 1= x n – f of x n by f dash of x n, so substitute in the method we get x n + 1 is x n – f of x n by f dash of x n, and if you simplify your end up with 2 x n – N x n, so the iteration method for solving f of x = 0 is x n + 1 = 2 x n – N x n square. So we would like to now show that this iteration method converges provided we take the initial approximation x 0 to lie in the interval 0 to 2 by N, so let us try to work out the details. So this iteration method is of the form x = g of x, this we have already seen Newton Raphson Method belongs to the class of fixed point iteration method where the right–hand side x – f of x by f dash of x is g of x.

So I take g of x to be 2 x – N x square android does the error at the n + 1 step. What is error at the n + step? It is little x n + 1 – p, what is P? It is the root to which the sequence of iterates converge to, what is it that I want? I want to find my root which is x and that is 1 by N, so here p is 1 by N, so x n + 1 – p gives me e n + 1 with p as 1 by n, so I (())(9:19) n + 1 because that is 2 x n – N x n, where – (())(9:24) so I write 2 x n as x n + x n and take one of them here as x n – 1 by n and the other here so I have a x n – n x n square.

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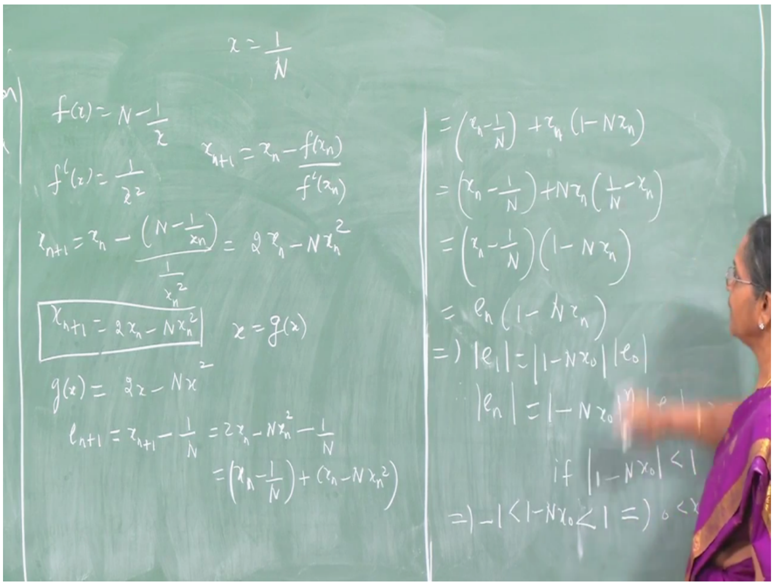

So we have x n – 1 by N + I can take out the factor x n, so x n into 1 – N x n and write this term as N x n into 1 by N – x n, so from these 2 terms I can remove the factor x n – 1 by N and the other factor will be 1 – N x n. But I know that this is the error at the nth step, so this is e n multiplied by 1 – N x n, so what is it that we have shown? We have shown e n + 1 is e n into 1 – n times x n, so we make use of this and right down what is the error of the first step in absolute value. So mod e 1 will be modulus of 1 – N x 0 into mod is 0, what will be mod e 2? Mod e 2 is mod 1 – N x 0 to mod e 1 that is mod 1 – N x 0 into mod e 0, so we can write down mode e 2 to be mod 1 – N x 0 the whole square into mod e 0.

So we continue to apply this inductively and finally we get mod e n to be modulus of 1 – N x square to the power of n into mod e 0. I want the nth step to go to 0, when the error approaches 0 what is this e n? E n is x n – p so as n tends to infinity then e n goes to 0, x n will converge to p, namely the sequence of iterates will converge to a root of the equation. So I want this e n to go to 0, when can that happen? When this quantity within absolute value is less than 1, so a quantity less than 1 is raise to the power of n, so as n tends to infinity this will go to 0 so the error at the n step will go to 0. So that requires that mod 1 – N x 0 is less than 1, what does that mean? It means 1 – N x 0 lies between – 1 and + 1.

So when I add 1 to this inequality then I end up with 0 less than x 0 less than 2 by N, which clearly tells us that the initial approximation x 0 lies between 0 and 2 by N and if this happens then the sequence of iterates converge to a root of the equation namely the value that you get will give you the reciprocal of the given positive integer.

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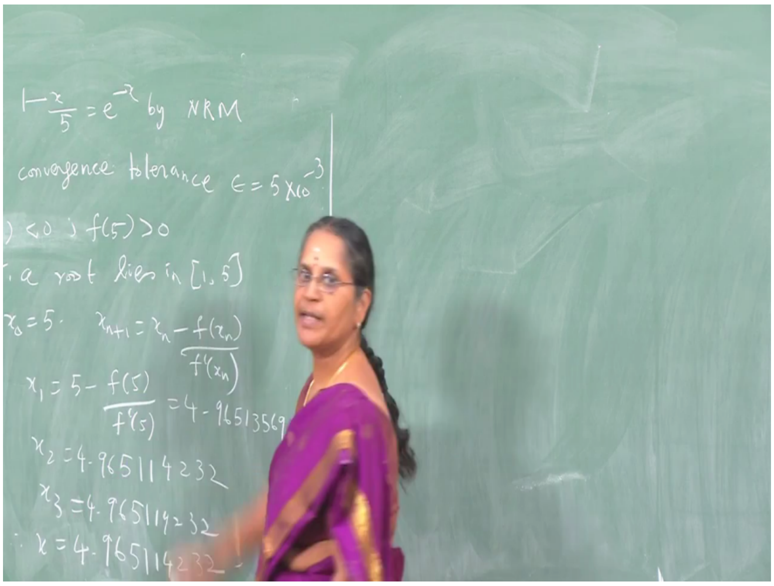

Let us now consider this problem of solving the given equation by Newton Raphson Method taking convergence tolerance epsilon as 5 into 10 to the – 3. So given f of x = 0, we determine an interval within which a root lies so that our initial approximation will lie (()) (13:20) interval and if we start with such an initial approximation and generate the successive iterates to determine a solution of this problem. So we observe that f of 1 is less than 0 and f of 5 is greater than 0, so a root lies in the interval 1 to 5 so we shall start with an initial approximation anywhere in this interval so let us take it to be say 5 and then apply Newton Raphson Method x n + 1 is x n – f of x n by f dash of x n. So knowing x 0 = 5, we obtain the right hand side and that will give you x 1 for that is 5 – f of 5 – f dash of 5 and that turns out to be this.

With this as x 1 we compute x 2 and we check whether the difference between the successive iterates satisfy the required accuracy, no, so we continue to compute x 3 with x 2 use on the right hand side for x n. x 3 turns out to be this so we observe that we have desired degree of accuracy when we compare the solutions at the 3rd step and the 2nd step and we stop our computations and conclude that x = this is a root of the given equation obtained using Newton Raphson Method and it is correct to the desired degree of accuracy, let us now consider the following problem.

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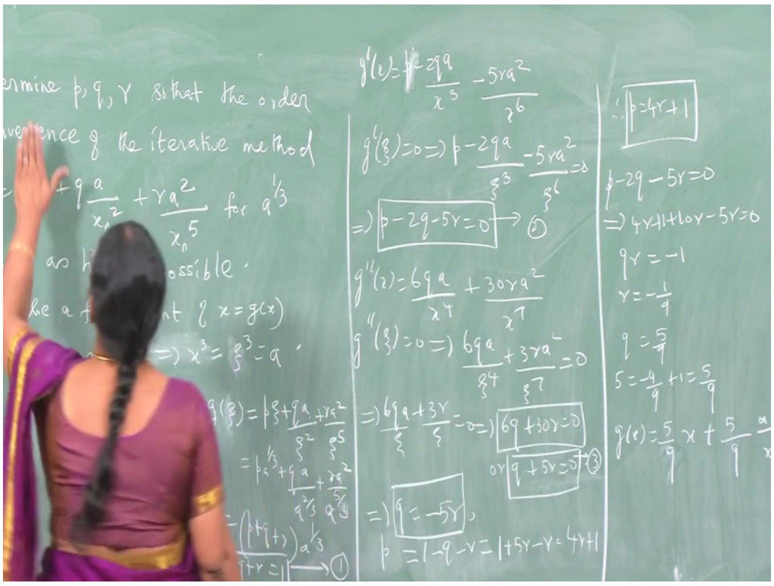

We are asked to determine constants p, q, r, which appears in this iteration method x n + 1 = p x n + q a by x n square + r a square by x n power 5 for determining cube root of a number say ‘a’ such that the order of convergence of this iterative method is as high as possible, what does that mean? We have already discussed, if a fixed point iteration method is given as x = g of x with g as a continuous function in the interval a b and has continuous derivative of the order Alpha in the open interval a b and if p is a fixed point so that g of p is p and g dash, g double dash, et cetera up to Alpha – one third and derivative of g at p is 0 and Alpha order derivative of g is different from 0 then the order of convergence of this fixed point iteration is Alpha.

So we observe that we have a fixed point iteration of the form x = g of x for solving the equation x = a to the power of 1 by 3, namely for solving or determining a cube root of a number ‘a’. So let us assume that Si is a fixed point of x = g of x, what is x? x is ‘a’ power 1 by 3 and that is Si so x cube will be Si cube and that is a so we will make use of that because Si is a fixed point for x = g of x, what is g of x? It is the right hand side p x + q a by x square + r a square by x to the power of 5. What are the conditions? We should now see whether g of Si is Si, what happens to g dash of Si, g double dash of Si and so on? How many of unknowns which we have? There are 3 unknowns p, q, r, so we required 3 conditions to determine these unknowns so we make use of the condition as follows; namely g of Si must be = Si, g dash at Si must be 0 and g double dash of Si must be = 0.

So I must determine p, q, r, which appear in g in such a way that this g has the following properties namely g of Si is Si, g dash of Si is 0 and g double dash of Si = 0, so I will have 3 conditions to determine the 3 constants p, q, r, so let us do that and then continue our discussion. So take g of x to be given by the right hand side then see what happens if you impose the condition that g of Si = Si, so g of Si is p into Si + q a by Si square + r a square by Si power 5, but what is Si? It is ‘a’ power 1 by 3 so use that fact, that will give you a power 1 by 3 into p + q + r that is g Si, but that must be = Si which is a power 1 by 3, so this immediately gives you the 1st equation satisfied by p, q, r namely p + q + r must be = 1, let us now compute g dash of x and evaluate it at x = Si.

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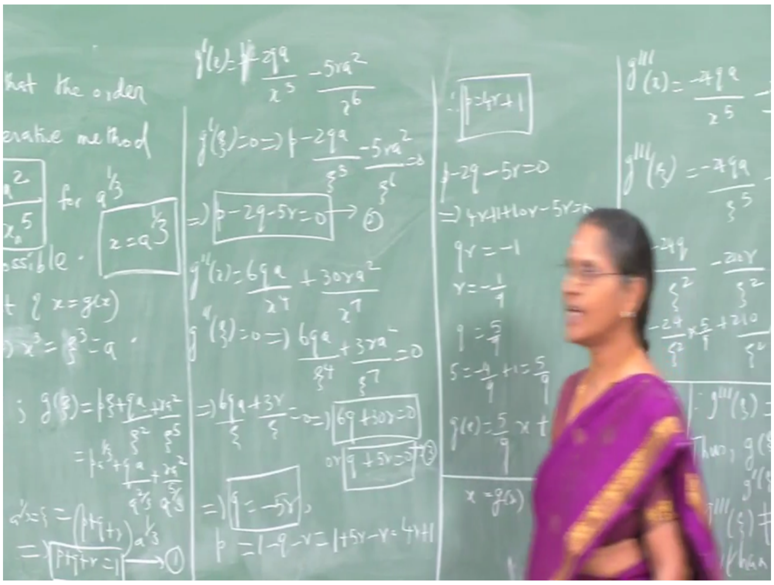

So g dash of x will be p, then – 2 cube a by x cube – 5 r a square by x to the power of 6, evaluate this at Si so g dash of Si = 0 will give you p – 2 q a by Si cube – 5 r a square by Si power 6 and that must be 0. So that immediately gives you making use of the fact that Si cube is a, so we get p – 2 cube – – 5 r is 0 that is the 2nd equation, then we require g double dash of Si to be = 0 so compute g double dash of x, evaluate it at Si and equate it to 0 that gives you 2 + 5 r = 0. So we now have 3 equations from the 3 conditions that g of Si must be Si, g dash of Si must be 0, g double dash of Si should also be = 0, so we solve for p, q and r. So this gives you q to be – 5 r, use that information here p will be 1 – q – r or use this information in p + q + r = 1 that will give you = 1 – q – r so p will be 4 r + 1.

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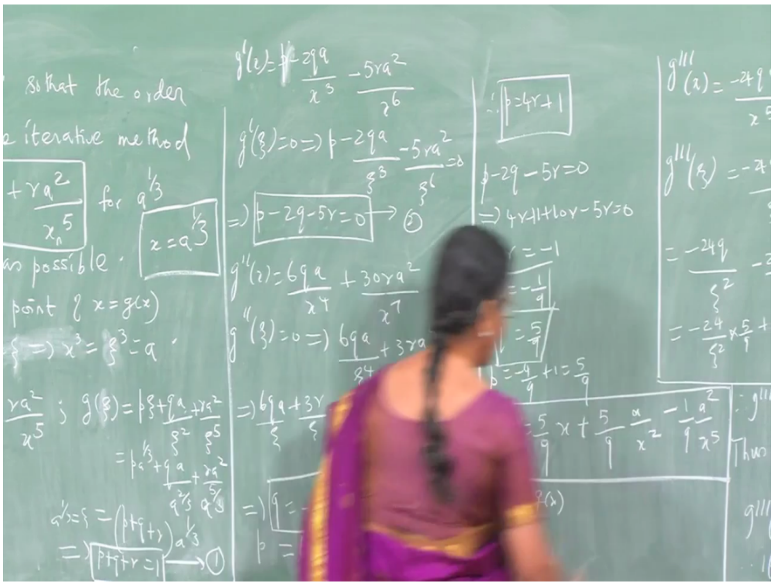

So use p as 4 r + 1 and q as – 5 r in this 2nd equation p – 2 q – 5 r = 0, so that gives you r to be – 1 by 9 and hence evaluate q that is 5 and 9 and finally you have p given by 4 r + 1, so p = 4 r + 1 that is 5 by 9. You have determined p, q, r by imposing the 3 conditions namely g of x will therefore be 5 by 9 into x + q is 5 by 9 into a by x square, then r is – 1 by 9 into a square by x to the power of 5 so you have determined p, q, r such that you have now an iterative method for of training the cube root of a number a. What do we want? We want the order of convergence of this method to be as high as possible. Whatever that we have done shows that it has at least 3rd order convergence, right why because g dash of Si is 0, g double dash of Si is 0, we still have not calculated what is g triple dash at Si is, so at this stage we can make a statement that is iterative method has at least 3rd order convergence.

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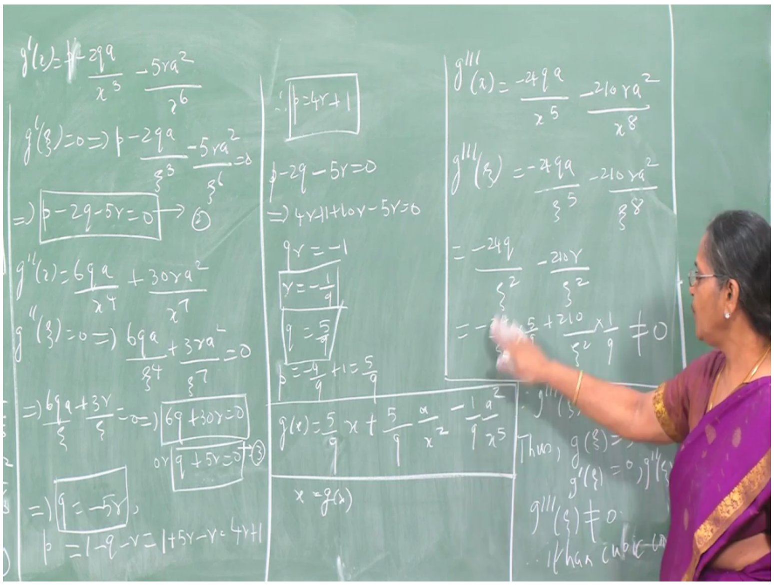

So we should now find out whether it has 3^{rd} order convergence or it has convergence of higher order, so in order to do that we compute the 3^{rd} derivative of g at Si, so let us find out g triple dash of x. So it turns out to be this and evaluate it at Si that comes in terms of q and r, we have evaluated the values of q, r, etc, so substitute for q and r we observe that it is – 24 by Si square into 5 by 9 + 210 by Si square into 1 by 9, which is different from 0 so what is it that we have shown?

We have an iterative method which is a fixed point iteration method of the form x = g of x which has the following properties namely, g of Si is Si, g dash of Si is 0, g double dash of Si is 0, but g triple dash of Si is different from 0 and therefore the order of convergence of this method is 3 that is the highest order possible. And we conclude that the method for computing cube root of a number a given by this has cubic convergence; let us now work out the following problem.

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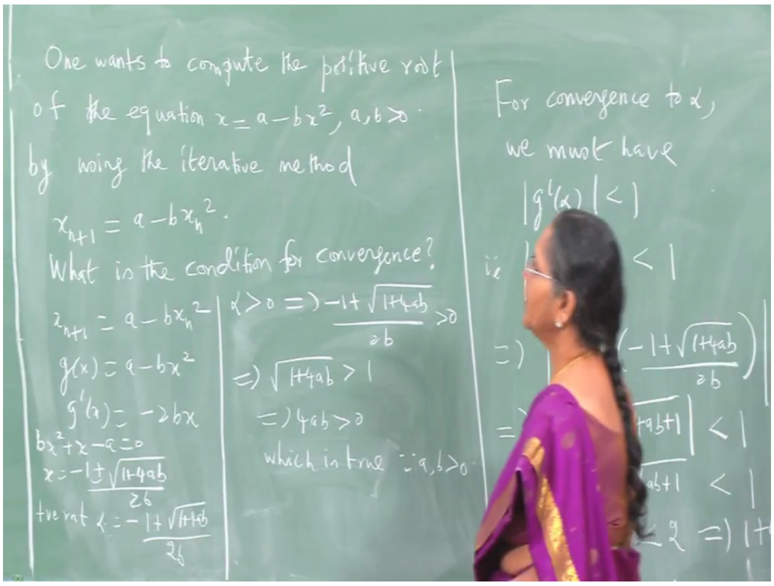

Here we are required to compute the positive root of the equation x = a – b x square where a and b are positive using an iterative method given by x n + 1 = a – b x n square, we are asked to find out the condition on convergence of this iterative method, so let us look at this method. We observe that it is of the form x = g of x with g of x given by a – b x square and we have to solve g of x = x and we have to solve this quadratic equation b x square + x – a = 0, which gives you x = – 1 + or – root of 1 + 4 a b by 2 b. We want this root to be a positive root so if I called that route as Alpha then Alpha must be – 1 + root of 1 + 4 a b by 2 b and since this has to be a positive root, this – 1 + root of 1 + 4 a b by 2 b must be positive.

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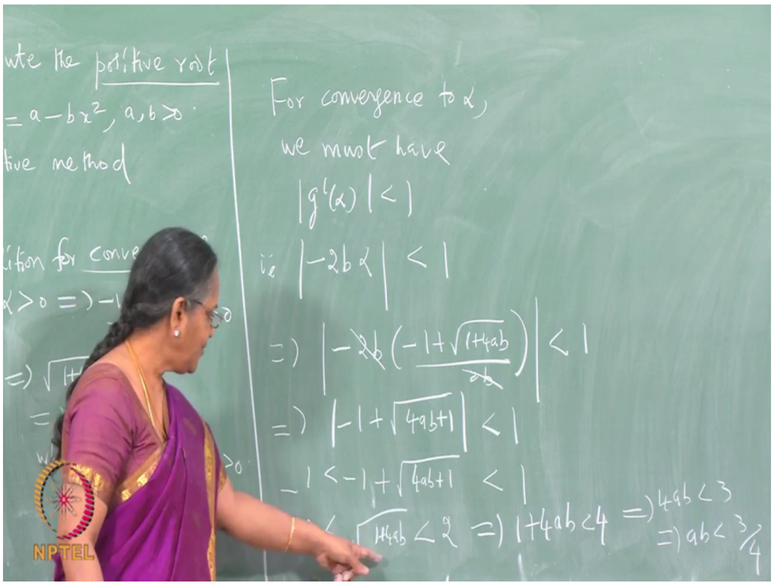

Since b is positive, this implies that root of 1 + 4 a b must be greater than 1 or 4 a b must be greater than 0 which is true because a and b are given to be positive. So now let us find out the condition for convergence, we have already seen this condition in our results which we have shown earlier. If you are given a method which is x = g of x for convergence modulus of g dash of x must be less than 1 at this root and here the root is Alpha so we must for convergence modulus of g dash of Alpha to be less than 1, so we compute g dash of x from here which is – 2 b x. So at x = Alpha we have mod – 2 b Alpha to be less than 1, so this implies modulus of – 2 b into we already have shown that the positive root is given by – 1 + root of 1+ 4 a b by 2 b, so we substitute for Alpha as that and we must have absolute value of this to be less than 1 for convergence.

So this gives you – 1 + root of 1 + 4 A b must be less than 1 in absolute value, which implies – 1 less than – 1 + root of 1 + 4 a b and that must be less than 1. So adding 1 throughout this inequality gives you 0 less than root of 1 + 4 a b less than 2, which shows that 4 a b must be less than 3 and therefore, a and b are such that the product a b must be less than 3 by 4. So if you are given an equation of the form x = a – b x square where a and b are positive satisfying the condition that a b is less than 3 by 4 then the iterative method given by x n + 1 = a – b x n square which is a fixed point iteration method is guaranteed to converge to a positive root of the equation given by Alpha = – 1 + root of 1 + 4 a b by 2 b that is what the problem illustrates. We stop here and then continue our discussion in the next class.