Numerical Analysis
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Lecture No 32
Root finding Method 4–Newton Raphson Method 2

So we would like to consider the error analysis and convergence of Newton Raphson Method, before that we shall define what we mean by order of convergence of a given method.

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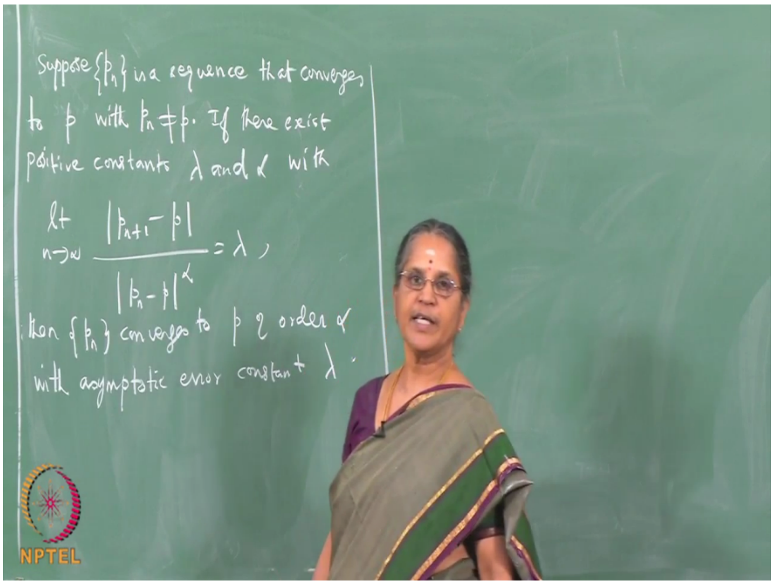

Suppose say we have a numerical method generates a sequence of iterates which we denote buy p 0, p 1, etc say p n so that p n is a sequence of iterates that we generate by a numerical method and it converges to p with p n different from p. If suppose there exist positive constants lambda and Alpha with limit as n tending to infinity of p n + 1 – p by modulus of p n – p power Alpha = lambda. Here p n is the nth iterate that we have generated, p n + 1 is the n + 1 iterate that we have generated and these are approximations to the exact value p that we want to determine then the difference between the n + 1 successive iterates – p in absolute value to the Alpha power of the difference between the nth successive approximation to p is equal to lambda positive constant then we say that the sequence p n converges to p of order Alpha, so order of convergence of this iterative method is Alpha.

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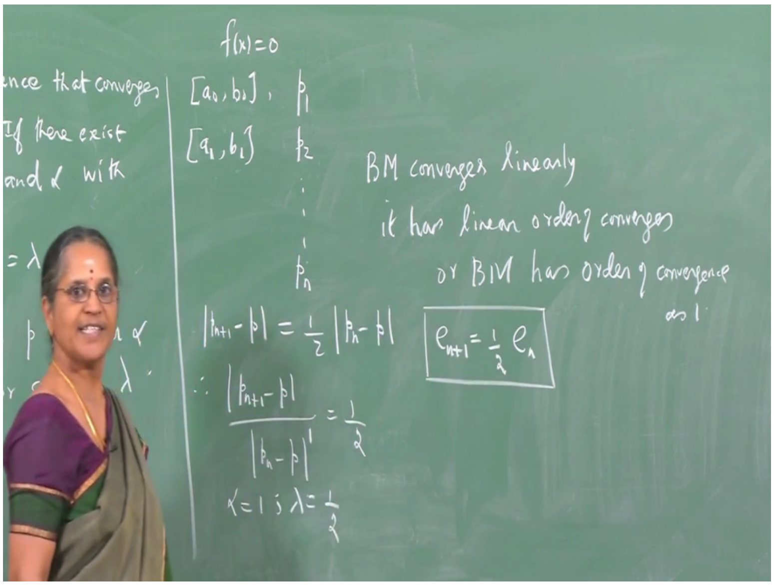

And a constant lambda there appears here is called an asymptotic error constant lambda. So let us now see what is the order of convergence of bisection method because bisection method generates a sequence of iterates. What does it generates? It starts with a 0 b 0 and it has the midpoint of this interval to be p 1 and then it takes the appropriate interval and calls it as a 1 b 1 so that this interval encloses a root of the equation and takes the midpoint of this interval and that is a 2nd approximation to the root of the equation f of x = 0. So it generates successive iterations and at every step what happens the interval that encloses a root of the equation is such that the length of that interval is half the length of the interval in the previous iteration.

And therefore we see that the root p of the equation f of x is such that distance between p n + 1 and p in absolute value is going to be half of the distance between p n and p when we apply bisection method. And therefore we have p n + 1 – p divided by p n – p = half, so we compare this with what we have in our definition. So we have mod p n = 1 – p by mod p n – p to the power of 1 = half and that gives you Alpha to be equal to 1 and lambda = half, so the bisection method has order of convergence to be 1 or we say that the bisection method converges linearly or it has linear order of convergence or we say that the bisection method has order of convergence as 1.

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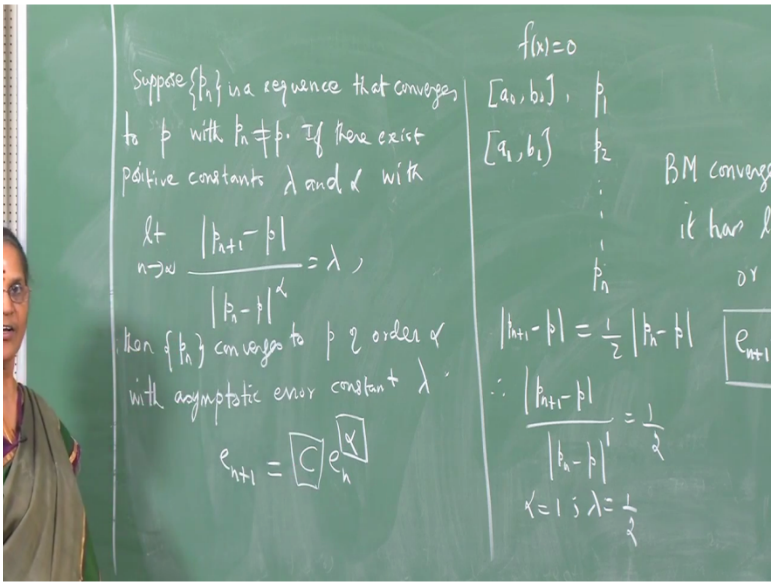

So our interest now is to see what happens to the order of convergence of Newton Raphson Method, we already have seen by means of some examples that Newton Raphson Method convergence rapidly and here we observe that modulus of p n + 1 – p is the error at the n + 1 step, and modulus of p n – p is the error at the nth step. So we have shown that there are in the nth step is related to the error at the n + 1 step by means of this. Or in general we have error at the n + 1 step to be given by some constant c times the error of the n step raise to the power of say some alpha, then we say that Alpha is the order of convergence of the method and C is the asymptotic error constant.

So when we want to discuss the order of convergence of Newton Raphson Method, we would like to obtain a relation connecting the error at the nth and the error at the nth + 1 step in this way and identify what are C and Alpha so that we can immediately make some conclusions about the order of convergence of Newton Raphson Method and also some statement about the asymptotic error constant in Newton Raphson Method, so let us work out the new details for Newton Raphson Method now.

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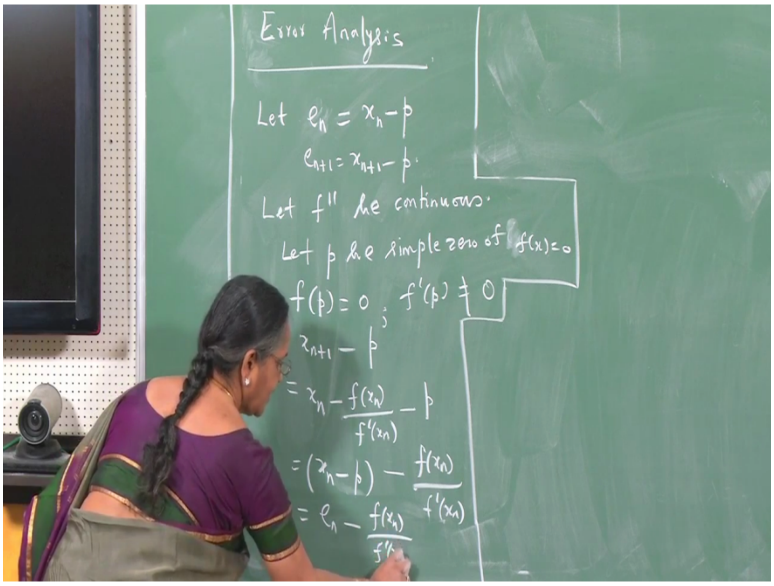

So we perform error analysis of Newton Raphson Method, so let e n be the error of the nth step, what is e n? e n is x n – p, x n is the nth approximation that we obtained using Newton Raphson Method and p is the exact root of the equation f of x = 0. Sp e n + 1 will be x n + 1 – p, let us assume that f double dash be continuous and let p be a simple 0 of the equation f of x = 0. What does that mean, f at p is 0 but f dash at p is different from 0 because p is the simple 0 of the equation. So we start with e n + 1 why? We want a relationship correcting the error at the nth + 1 step and the error at the nth step, so start with e n + 1 what is it, e n + 1 is x n + 1 – p but we are using Newton Raphson Method and the nth + 1 iterate is given by x n – f of x n by f dash of x n – p.

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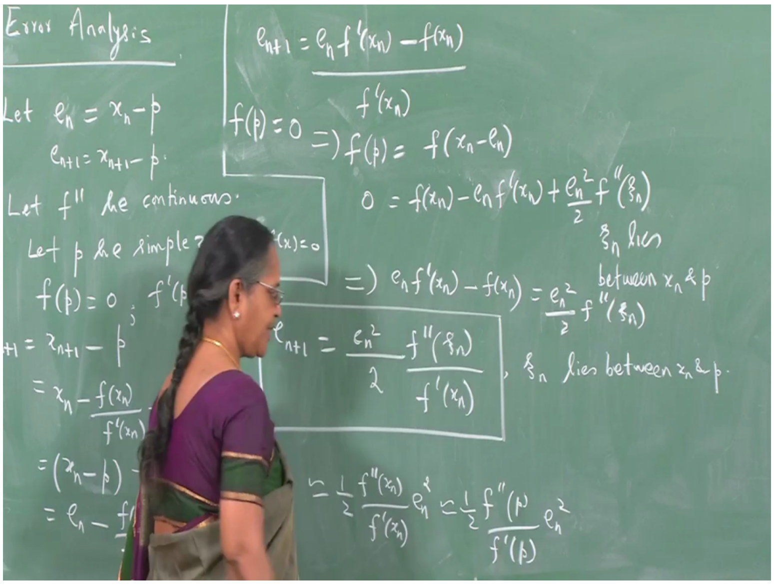

And therefore, we have x n – p – f of x n divided by f dash of x n, this is equal to e n – f of x n by f dash of x n which gives e n + 1 equal to e n into f dash of x n – f of x n divided by f dash of x n, so I would like to simplify this making use of the fact that f of p is 0. This tells me f of p = f of x n – e n, so that will be f at x n – e n into f dash of x n + e n square by factorial 2 into f double dash at x n, so I shall write the error term as e n square by 2 f double dash of Si n where Si n lies between x n and p. I know that f of p is 0 so this gives me e n into f dash of x n – f of x n to b equal to e n square by into f double dash of Si n. I observe that this is what appears in the numerator here so therefore we have e n + 1 = e n square by 2 into f dash at Si n divided by f dash at x n where Si n lies between lies between x n and p.

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So I observe that I have a relationship connecting e n + 1 and e n, what is it? E n = 1 is half of f double dash at x n divided by f dash of x n into e n square. And if x n is the nth approximation, which is correct to a desired degree of accuracy then this will be half of f double dash of p divided by f dash at p into e n square, which I can express as c times e n square. The order of convergence of Newton Raphson Method is 2, so Newton Raphson Method has quadratic convergence or we say that Newton Raphson Method has second order convergence or we say that Newton Raphson Method converges quadratically and what is the asymptotic error constant? C is the asymptotic error constant, what is that? C is half of f double prime at p divided by f dash at p and Kenny evaluate this? Yes an approximation to the asymptotic constant is half f double dash of x n by f dash of x n because x n approximates p correct to a desired degree of accuracy.

So the error analysis clearly shows that the Newton Raphson Method has quadratic convergence and therefore it is much better than the order of convergence of bisection method, which converges linearly and that is why the successive iterates generated by Newton Raphson Method converges much faster than the bisection method. So we now discuss the convergence of Newton Raphson Method, why do we have to discuss the convergence of Newton Raphson Method? I recall the example that we have considered we said that there are functions where starting with some initial values we are unable to get successive iterates which converge to a root of the equation and therefore there must be some condition on the starting values so that given a function f of x = 0 and when you want to apply Newton Raphson Method.

If we ensure that our starting value namely initial approximation satisfies condition then by the theorem that we are going to prove we can be sure of the fact that the successive iterates that we generate using Newton Raphson Method will converge to root of the equation correct to the desired degree of accuracy and so we see what are the conditions that we will have to take care of when we want to apply Newton Raphson Method in finding a root of the equation.

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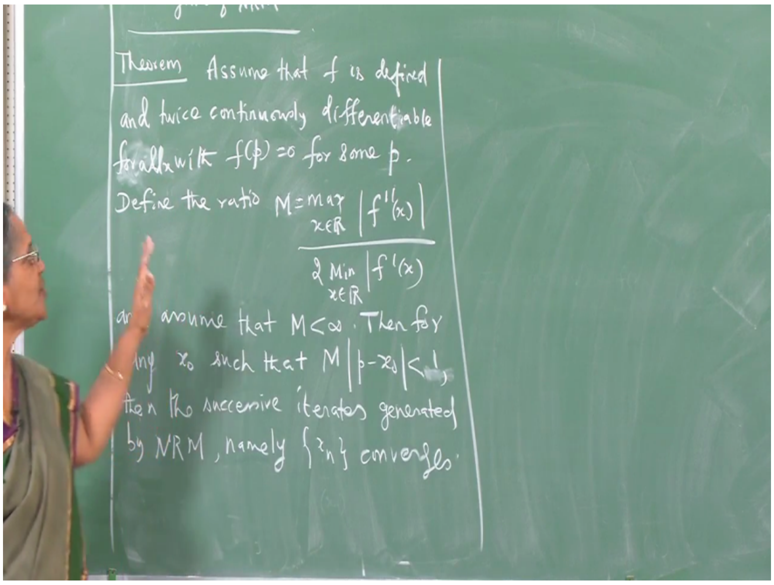

So the results says that we have a function f which is defined and it is twice continuously differentiable for all x with f of p = 0 for some p, so p is a simple 0 of f of x = 0. If M is such that it is maximum of modulus of f double dash of x by twice minimum of modulus of f dash of x and M is less than infinity, it is a finite number then starting with any x 0 which satisfies the conditions that M into Mod p – x 0 is less than 1 then the successive iterates that you generate using Newton Raphson Method namely x n will always converge to root of the equation and you will have a root of the equation correct to the desired degree of accuracy.

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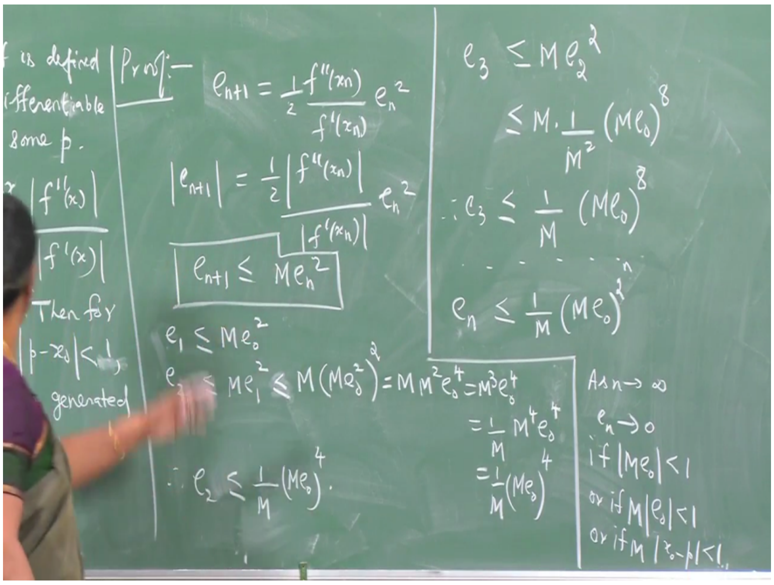

So that we have shown that e n + 1 = half of f double dash of x n by f dash of x n into p n square, so modulus of p n + 1 will be half mod f double dash of x n by modulus of f dash of x n into e n square and that will be less than or equal to M times e n square so we have e n + 1 to be less than or equal to M e n square. Let us now obtain e 1 that is less than or equal to M e 0 square then e 2 is less than or equal to M e 1 square, but I know e 1 is less than or equal to M e 0 square so this will be M into M e 0 square the whole square, so I have substituted for e 1 to be less than or equal to M e 0 square, so I will have this inequality. But what is this? This is M into M square e 0 power 4, so it is M cube into e 0 power so I shall write this as 1 by M into M Power 4 e 0 power 4 so it is 1 by M into M e 0 the whole power 4.

So we get e 2 to be less than or equal to 1 by M into M e 0 to the power of 4, now let us write down e 3, e 3 is less than or equal to M into e 2 square so it is less than or equal to M into 1 by M square into M e 0 to the power of 8 because M square in it therefore e 3 is than or equal to 1 by M into M e 0 to the power of 8, so let us now see the pattern, e 1 is less than or equal to M e 0 square, e 2 is less than or equal to 1 by M into M e 0 power 4 that is 1 by M into M e 0 to the power of 2 square, now e 3 is less than or equal 1 by M into M e 0 to the power of 8, which I can write as M e 0 to the power of 2 cube, so I continue this way and right down what is the error at the nth step, so it is going to be 1 by M into M e 0 to the power of 2 to the power of n.

And therefore this error e at the nth step will go to 0 as n tends to infinity provided M times e 0 in excellent value is less than 1, so as n tends to infinity e n will approach 0 if in absolute value mod M e 0 is less than 1 or in other words if M into modulus of e 0 is less than 1 or M into what is e 0? e 0 is the error of the initial step namely that is x 0 – p and that must be less than 1 and we observe that that is what we have to show namely for any x 0 that we choose as an initial approximation, if this is such that M times mod x 0 – p is less than 1 then the successive iterates generated by Newton Raphson Method namely x n will converge to root of the equation correct to a desired degree of increasing, so this condition must be satisfied by the initial condition.

So what does this theorem says? The theorem says if your function f is twice continuously differentiable and if p is a simple 0 of the equation f of x = 0, if you start with an initial approximation x 0 in such a way that M times mod p – x 0 is less than 1 then the successive iterates that you generate using Newton Raphson Method will converge to root of the equation correct to a desired degree of accuracy. So you now know how your initial approximation x 0 to be chosen, it must be close to the exact root of the equation in such a way that this condition must be satisfied, if this is satisfied then your Newton Raphson Method converges to a root of the equation correct to the desired degree of accuracy. So the convergence analysis clearly shows the conditions under which the Newton Raphson Method will converge to root of the equation and the condition is that the initial approximation x 0 is such that it was satisfied the condition that mod p – x 0 is less than 1 by M.

So we now give another result which is again for convergence of Newton Raphson Method, which tells you that if some conditions are satisfied then starting from any initial approximation the successive iterates that you generate by Newton Raphson Method will converge to a root of the equation correct to the desired degree of accuracy, so let us look into the result which states this.

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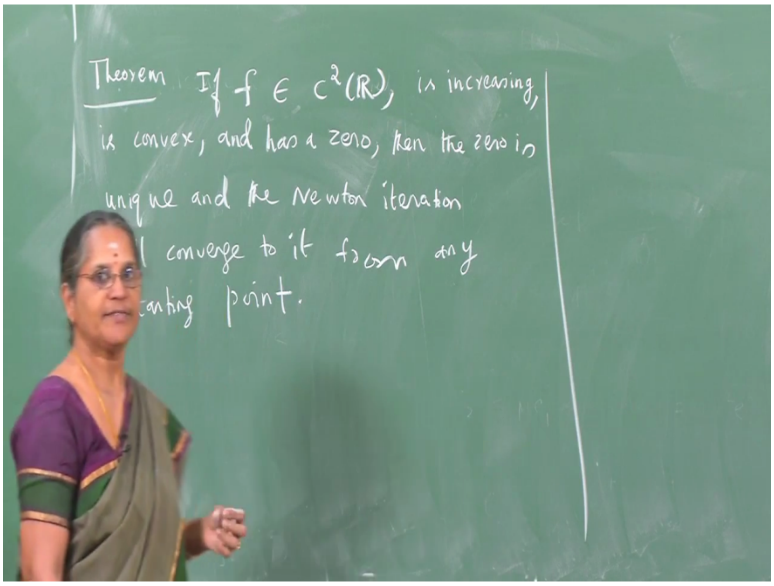

So the result is as follows it says “If f belongs to C 2 of R twice continuously differentiable, f is an increasing function and f is convex, what does that mean? A function is convex if f double prime of x is greater than 0 for all x, so if this condition is satisfied so f is convex and it has a 0 then this 0 is a unique 0 and Newton Raphson Method iterations will converge to this unique 0 starting from any initial approximation. So this gives you the condition on the shape of the function f, I again recall the example that we had considered for example demonstrated that there are functions with some shapes such that when you start with some initial approximation, successive iterates that you generate by Newton Raphson Method diverged, it did not converge.

So when you apply Newton Raphson Method and if you want to start with some initial approximation, you should also know about what the shape of the function f is. The theorem says if the function F satisfies the following conditions namely F is in c 2 of R, it is increasing, it is a convex function and it has a 0 and the 0 is then unique and Newton iteration method will generate successive iterates which will converge to this unique 0. So that completes our discussion on Newton Raphson Method namely we know how to derive this method and we have given geometrical interpretation of the method and we have used this method to generate successive iterates for a root finding problem and then we have also compared it with bisection method and showed that this is a method which converges much faster than the bisection method. And we have performed error analysis for which we understand that this method has taken order convergence.

And then the convergence analysis of Newton Raphson Method tells us that if your initial approximation is such that it satisfies a certain condition namely, the distance of this initial approximation from the root in absolute value is less than 1 by capital M, then the (sequence of iterates will converge to a root of the equation. And if the given function f is such that it satisfies some special conditions, it has some special properties as listed in the theorem, then your Newton Raphson Method iterates will converge to a root of the equation starting from any initial approximation. So now that we have understood these details about Newton Raphson Method we will work out some problems on this method, the problem is as follows.

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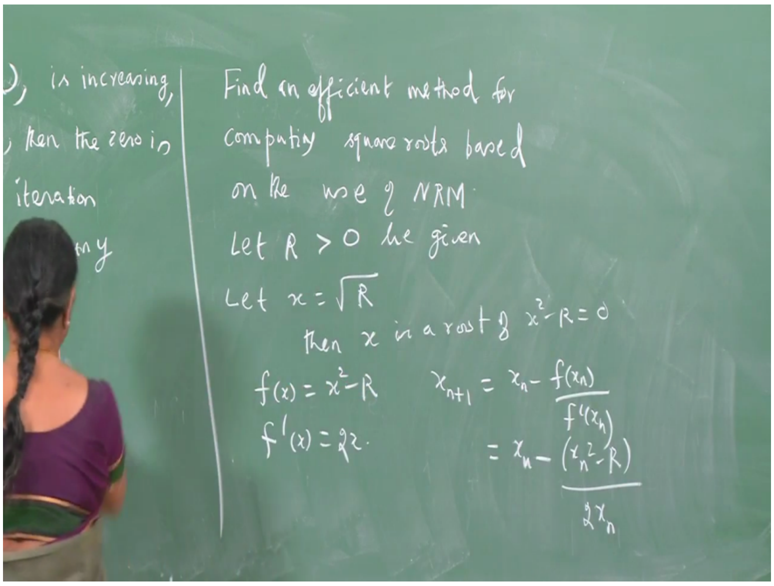

Find an efficient method which is based on Newton Raphson Method with the help of which you can compute square root of a number say R. So let R greater than 0 be given to us, so what is it that we want? We want to determine what is square root of R so call it x, then we see that x is a root of the equation x square – R = 0, so I am given F of x of x square – R and I am asked to get a root of this equation F of x = 0 and I am asked to develop a method using Newton Raphson Method with the help of which I can generate the successive iterates which will converge to the square root of R. So I determine F of x and write down Newton Raphson Method for obtaining square root of R. What is it? This is x n + 1 = x n – f of x n by f dash of x n, so it is x n – f of x n is x n square – R and f dash of x n is 2 x n.

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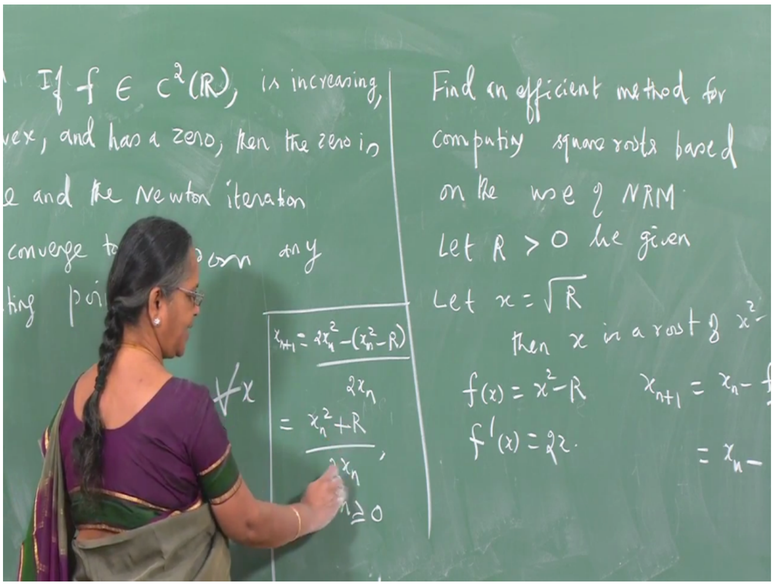

So if you simplify this then you get x n + 1 to be equal to 2 x n square – x n square – R divided by 2 x n, so it is x n square + R by 2 x n, so n greater than or equal to 0. So start with an initial approximation x 0 then you can generate x 1 by substituting for x 0 square and x 0 on the right–hand side so that x 1 is given by x 0 square + R by 2 x 0. Find x 1 and then use this again, compute x 2, et cetera, x n + 1 and then stop your iterations when the stopping criterion specified in the problem is satisfied. Suppose I say that you make use of some 20 iterations and then determine square root of a number R, then perform 20 such iterations and then you stop your iterations and say the result that you obtain at the 20th iteration is square root of R. So we have developed a method for computing the square root of a number R and that is given by x n + 1 is equal to this.

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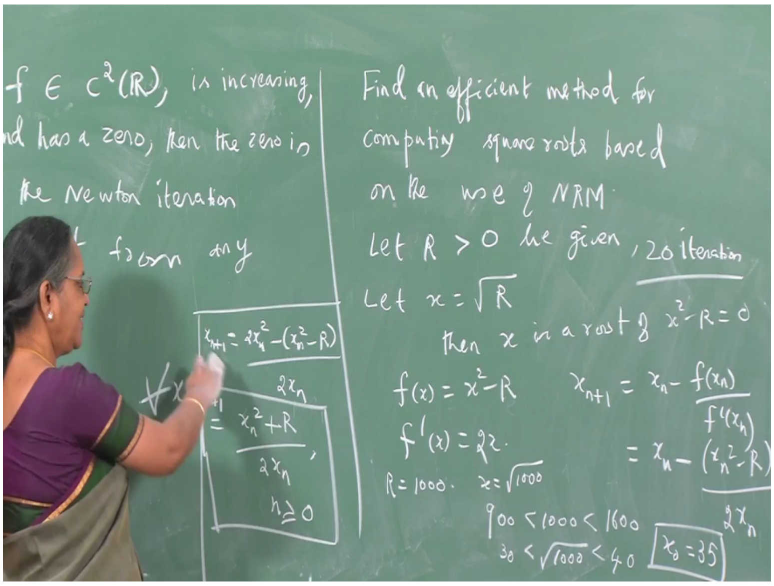

Okay, so now I ask you to obtain square root of say 1000 and so I am interested in finding root of 1000, so I must use this iterative method and find what square root of 1000 is, so I need an initial approximation. So I know that this number 1000 lies between 9000 and 1600, so taking square root 30 less than square root of 1000 is less than 40, so I can take an initial approximation for square root of 1000 to be any number which lies between say 30 and 40, so let me take my initial approximation x 0 to be say 35 and generate what x 1 is, x 2 is et cetera, and then compute 20 iterations as specified in the problem and get what the value of x n + 1 right at the 20th iteration and then call that as an approximation to square root of 1000. So I want you to work out the details now that we have developed a method and I have given you an initial approximation, you should be in the position to apply this method and generate the successive iterates.

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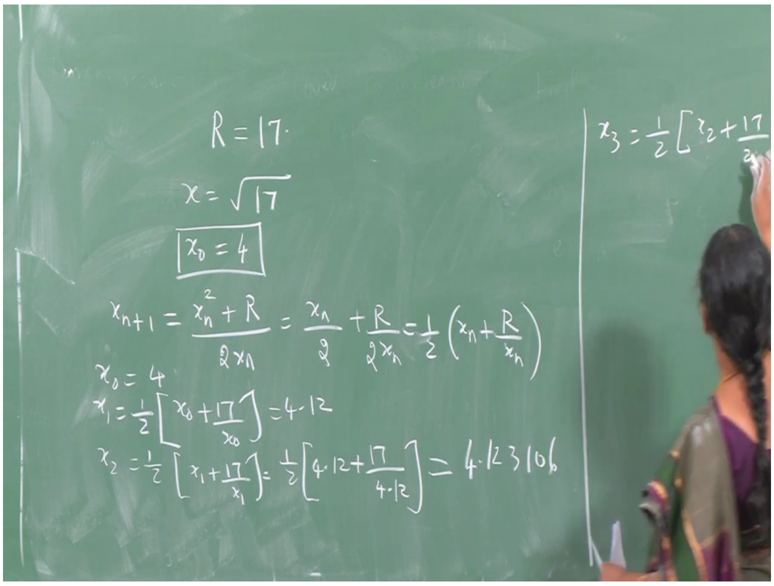

If suppose I had given the same problem and I have asked you to compute square root of R where R is 17, so I am interested in solving what x = root 17 is. Then in that case I can choose an approximation x 0 to be 4 and generate the successive iterates, x n + 1 is x n square + R divided by twice x n, so which I write as x n by 2 + R by x n. I start with x 0 which is 4, so my x 1 will be half of... so this I can write as half of x n + R by x n. So this will be half of x 0 + 17 by x 0 that will give you 4.12, x 2 is half of x 1 + 17 by x 1, so it is half of 4.12 + 17 by 4.12 and that gives me 4.123106.

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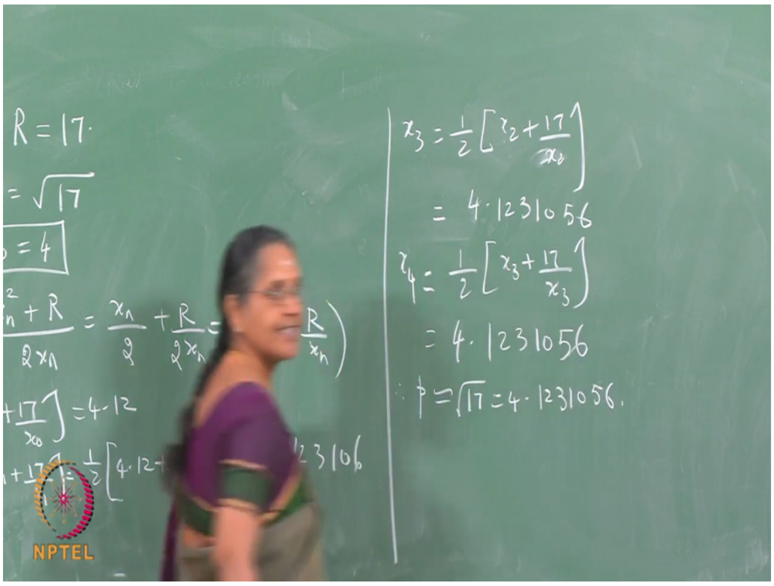

Then compute x 3 which is half of x 2 + 17 by x 2, and I know my x 2 is this, when I substitute I get x 3 to be 4.1231056. Let us work out another iteration, x 4 is half of x 3 + 17 by x 3 and when I substitute for x 3, this gives me 4.1231056, so I observe that x 4 – x 3 is such that their result is correct to 7 decimal places and therefore, I take my approximation to square root of 17 as 4.1231056. So the iteration method that we developed helps us to obtain square root of a real number. So I shall give some more problems of this type in the assignment and you can try to work out the details.