Numerical Analysis

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Lecture No 29

Root Finding Methods–1

The Bisection Method–1

Good Morning, we begin our discussion Root finding problems, this involves development of numerical techniques for solving equations of the form f of x = 0, where f is a real valued function of a real variable and it depends on a single real variable X. Where do we come across such a question and what is the necessity for developing numerical techniques for solving these equations? Let us see some examples. The population growth for a short period of time can be modelled by assuming that the population growth varies continuously with time and the rate at which this population growth varies is proportional to the number of population present at that time say t.

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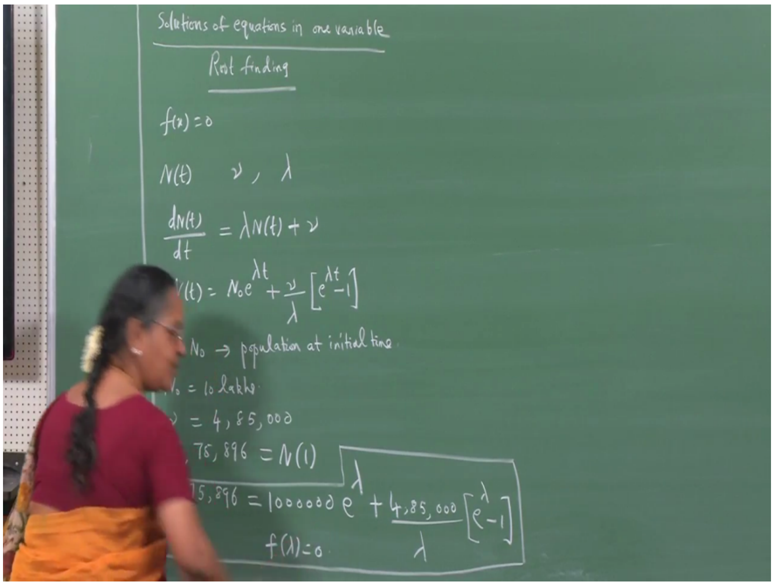

Suppose we denote by N of t it depends at time t and by Mu, the constant (())(1:50) and this Lambda denotes the constant birth rate of this population, then the population model can be given by the following differential equation namely the rate at which the population varies is equal to Lambda times N of t + the immigration that is permitted, mainly the constant rate Mu. This being a 1st order differential equation, the solution can be written down and the solution is say N 0 e to the power Lambda t + Mu by lambda into E power Lambda t – 1, and what is N 0? N 0 is the population of the initial time. Suppose say initial population N 0 is say some 10lakhs and that the immigrants in this year turns out to be say some 4,85,000 and the end of 1 year the population is say some 15,75,896 and this is of the population at the end of one year.

Then if I substitute these details here then I get 1575,896 must be equal to N 0 which is 1000000 multiplied by E 2 the power of Lambda t, so in 1 year this has happened + new immigrants the data given is 485,000 that is new, divided by lambda multiplied by e to the power of Lambda again t is 1, so e power Lambda – 1. So we observe that we have an equation involving Lambda, which is the birth rate of the population in that year and we do not know what it is and we have to determine this Lambda and we observe that this equation is of the form f of Lambda = 0. So I need to obtain a solution of this equation and determine what Lambda is. There are various physical situations where we come across the equations of different forms.

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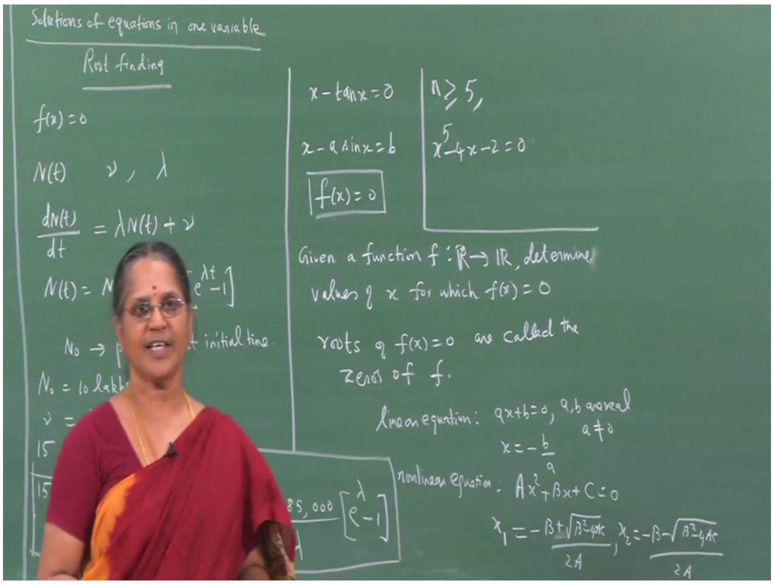

See for example in the case when we study diffraction of light we come across an equation of the form x – tan x = 0, where we are required to determine what this tan x is. Or when we study central orbits, we come across Kepler's equation which is of the form x – a sin x = b, where we are required to determine this x for different values of a and b. So we do come across different forms of equations involving exponential function or trigonometric functions or some equations may involve logarithmic functions or may purely be algebraic equations involving polynomials and so the problem is to obtain a solution of this equation which can be expressed in the form f of x = 0 and the numerical technique that we are going to discuss the next few classes can be used to generate approximately solutions of such equations f of x = 0 when one is unable to determine analytically the closed form solutions of such equations.

So what is the problem which is in our hand? The problem is as follows, namely given a funtcion f from R to R, real value function of a real variable determine values of x for which f of x is 0, this is what we need to find out, so let us see how we can do it. The roots of the equation roots of f of x = 0 are also called the zeros function f, so let us see some simple examples. Suppose this equation f of x = 0 is a linear equation of the form a x + b = 0, where a and b are real with a different from 0, then immediately we can write on the solution in the form – b by a.

Suppose I consider a nonlinear equation of the form say A x square + B x + C = 0 then I see that this is a quadratic equation and so you can write down the solution in the form x 1 = – B + root of B square – 4 A C by 2 A and the 2^{nd} solution is – B – root of B square – 4 A C divided by 2 A. So I am able to express the solution of this nonlinear equation involving radicals, namely square root of the sums of products of the coefficients A, B, C and appear in this nonlinear equation. Such a thing is also possible in the case of cubic equations and quadratic equations. However, for polynomial equations of the form f of x = 0, whose degree n is greater than or equal to 5, no such close formula exists so that the solution can be expressed in terms of the radicals.

In fact, for each value of n greater than or equal to 5 it is possible to find polynomial equations involving integer coefficients such that one cannot find the solution of such equations in terms of radicals. For example, consider this equation x power 5 – 4 x – 2 = 0, the coefficients are integers it is not possible to obtain a solution of this 5^{th} degree polynomial equation in terms of the radicals, so we are faced with the following question. How can we decide whether a given equation f of x = 0 has a root? And if it has a root then how can we find out? So we are faced with these 2 questions and therefore our goal now is to develop numerical methods for obtaining approximate solution of equations of the form f of x = 0 where f is a real valued function of a real variable defined on closed bounded interval and f is a continuous function.

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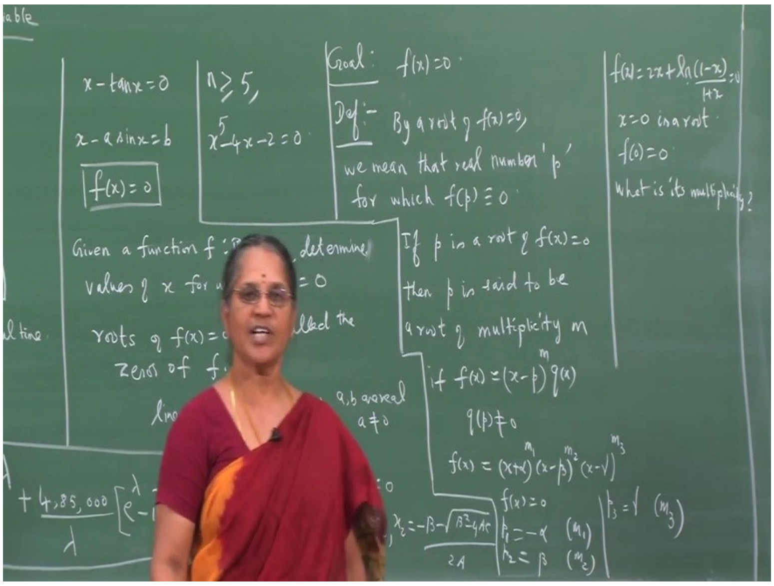

So having this goal let us start defining some simple terms which we use in our discussion, what do we mean by root of the equation f of x = 0, so by root f of x = 0 we mean that real number ‘p’ for which f of p is identically 0. And if p is a root of this equation f of x = 0, then p is said to be root of multiplicity m if f of x can be expressed in the form x – p to the power of M into some q of x where q at p is different from 0. So this definition will help us to determine what is the multiplicity of the root of that equation f of x = 0 if p is the root of that equation.

Suppose if f of x is a polynomial equation and we can factor the polynomial in the form x + Alpha to the power of m 1, x – Beta to the power of m 2 and x – Gamma to the power of m 3 then it is immediate that this equation f of x = 0 has a root of multiplicity m 1 at x = – Alpha and it has a root of multiplicity m 2 at x = Beta, it has a root of multiplicity m 3 at x = Gamma. So in this case it is very simple provided the polynomial can be factored in this form. Suppose I consider say an equation of the form f of x = 2 x + log 1 – x by 1 + x and I want you to find the root of this equation and determine x multiplicity, then let us look at the equation we observed that x = 0 is a root of the equation because it is identically satisfied f of 0 is 0 that is clear, it is immediate, so x = 0 is a root.

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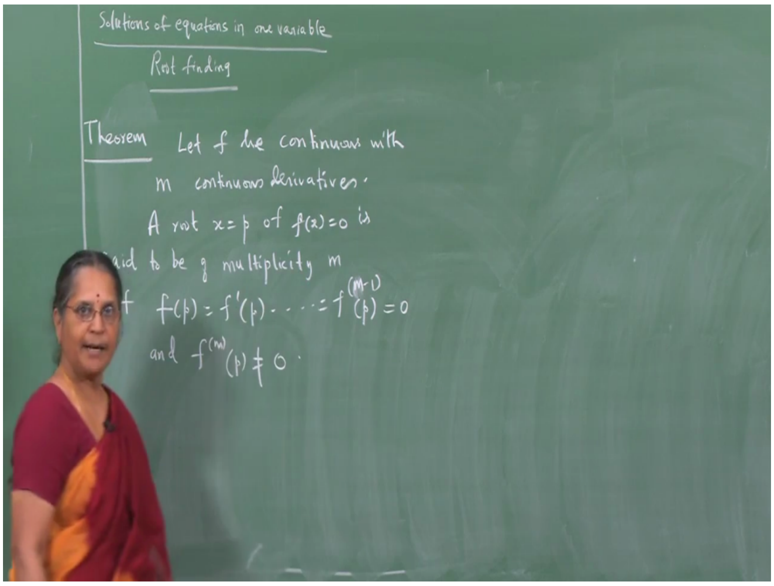

The question is its multiplicity, so let us try to answer that and we have the following theorem comes to our rescue. It states, let f be continuous with m continuous derivatives, a root x = p of f of x = 0 is said to be of multiplicity m if f at p, f dash at p and so on… up to m – 1th derivative at p are 0 and the mth derivative at p is different from 0. We just have to check whether this happens, given an equation of the form f of x = 0, where f is continuous and f possesses continues m derivatives and if we see that p is the root of the equation f of x = 0, check whether the 1st m – 1 derivatives at p are 0 and the mth derivative is different from 0, if this happens then we conclude that x = p is root of this equation of multiplicity m.

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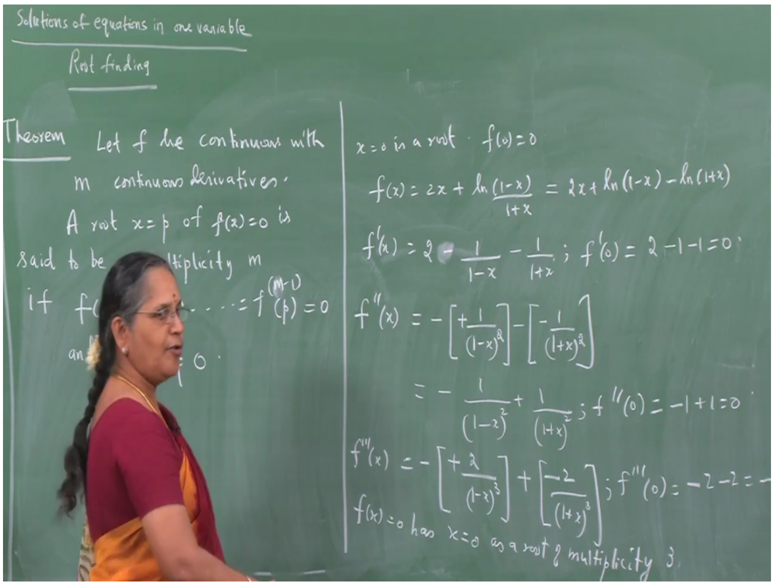

So let us see whether can make use of this result and determine the multiplicity of the root x = 0 of this equation, so we have determined what the root is, namely x = 0 is a root and therefore f of 0 is 0, so let us compute f prime. What is f of x? It is 2 x + log 1 – x by 1 + x, I rewrite this as 2 x + log 1 – x – log 1 + x. Now I compute f dash of x and that will b 2 + 1 by 1 – x into derivative of 1 – x – 1 by 1 + x, so let us find what is as prime 0 is, so that is going to be 2 – 1 – 1 and that is 0. Let us see what happens to f double prime x so this will give me – – 1 by 1 – x the whole square into – 1, then – – 1 by 1 + x the whole square. So this gives me – one by 1 – x the whole square + 1 by 1 plus x the whole square and so if I compute f double prime at 0 that again is 0.

Let us now see what is f triple prime of x, so this will give – 2 by 1 – x the whole cube into – 1 – 2 by 1 + x the whole cube and we evaluate f triple prime at 0 and that will give you – 2 – 2 which is – 4 and it is different from 0. We see that equation f of x = 0 has a root at x = 0 and f prime 0, f double prime 0 are all zeroes but f triple prime 0 is different from 0. The given equation f of x = 0 has x = 0 as a root of multiplicity 3, so now we know what we mean by root of the equation f of x = 0 and how we can determine the multiplicity of this root. We now move onto techniques for finding such a root of a given equation f of x = 0 and the technique belong to 2 categories.

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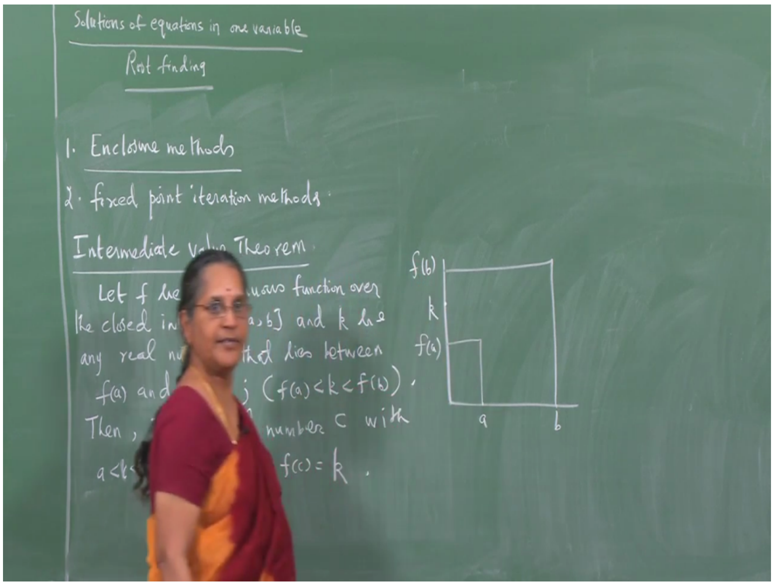

Namely the methods are either Enclosure methods or they are fixed point iteration methods, so we shall 1st focus our attention on an Enclosure Method. The enclosure methods are based on the results from intermediate value theorem, so let us just recall intermediate Value Theorem and see how the result given their helps us to find an interval that encloses a root of the given equation f of x = 0. Intermediate value theorem states that if f is a continuous function over the closed interval a b and k is any real number that lies between f of a and f of B, then there exists a real number c, which lies between a and such that f of c = k. So if f is a continuous function defined in this interval a b and k is any real number, which lies between f of a and f of b then there exists a real number c between a and b such that f of c = k.

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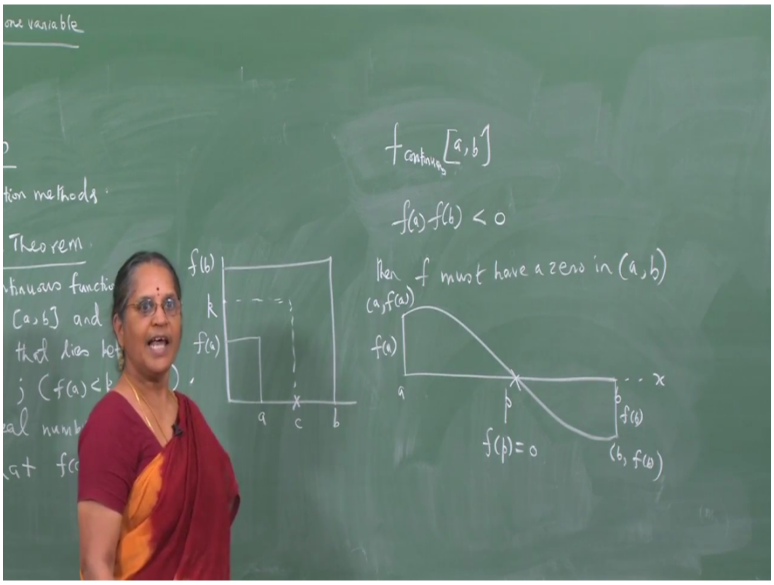

Let us Now see a consequence of intermediate value theorem, which says if f is a continuous function in the closed interval a b and f of a into f of b is negative then f must have a 0 in the interval a b, let us try to understand this result. So we have a continuous function f defined on the closed interval a b and it is such that f of a into f of b is negative. So without loss of generality, let us take f of a to be positive and f of b to be negative. So if I draw the graph of the function passing through the point a, f of a and b, f of b then I observe that the graph crosses the x axis at some point say p, this is the x axis. What does that mean? At p what is the value of f? It is 0, so f of p is 0 and therefore p is a 0 of the function f.

Where does the 0 occurs, it occurs in the open interval a b, so this is a very nice result which will help us to locate the interval within which the root of the equation lies and that is why we say the first class of methods that enables us to develop a numerical algorithm for determining approximate solution as an closure methods, the method help us to find out an interval that encloses a root of the equation f of x = 0. So we describe one such enclosure Method now which is called the bisection method.

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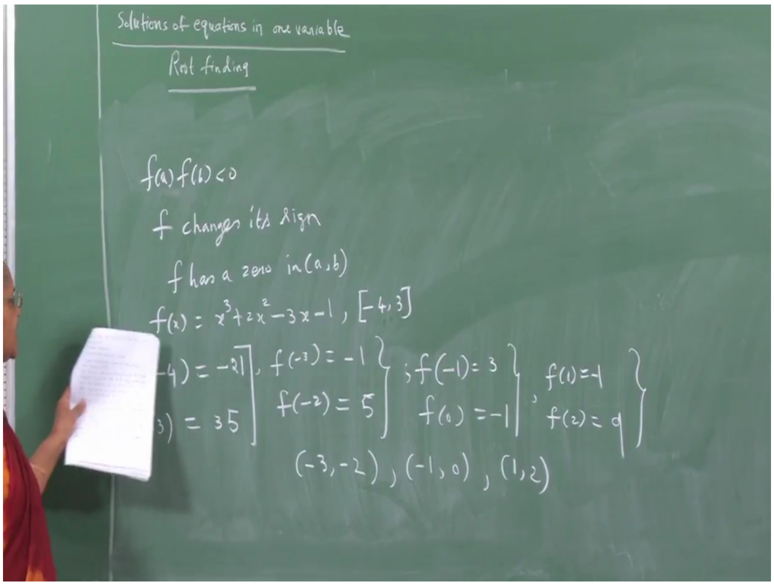

So what happens when f of a into f of b is negative? F changes its sign as it moves from a to b and therefore f has f 0 the interval a b. Let us take the following example, suppose f of x is x cube + 2 x square – 3 x – 1 and it is defined in the interval – 4 to 3, so let me evaluate f at – 4 and that turns out to be – 21 and f at – 3 is – 1, and f at 3 if I evaluate at the other endpoint, it turns out to be 35. So I immediately conclude that f of a into f of b is negative, so there is a route between – 4 and 3 for this equation f of x = 0. Let us see whether there are some more zeroes in this interval for this equation. f of – 3 is – 1 and if I compute f of – 2, it is 5 and so again I conclude that there is a root between – 3 and – 2 because f of a into f of b is negative.

Let us now evaluate f at – 1 and that turns out to be 3 and f at 0 is – 1, so again the sign changes, so there is a root between – 1 and 0 and let us see what is f at 1, it is – 1 and f at 2 is 9. And there again what happens, the sign changes and therefore there is a root between there is a root between 1 and 2, so f is defined in the interval – 4 to 3, so within that interval I observe that there is an interval – 3 to – 2, – 1 to 0 and 1 to 2 in each of which there is 0 of this function f. So how have we determined these zeros? We have made use of the result which is a consequence of the intermediate value theorem, so at this stage we shall see some enclosure methods that will enable us to obtain approximate solutions of equations of the form f of x = 0, one such method is the bisection method. So we shall now describe this method and try to see how we can use this method to find an approximate solution of the equation f of x = 0 namely root of the equation f of x = 0.

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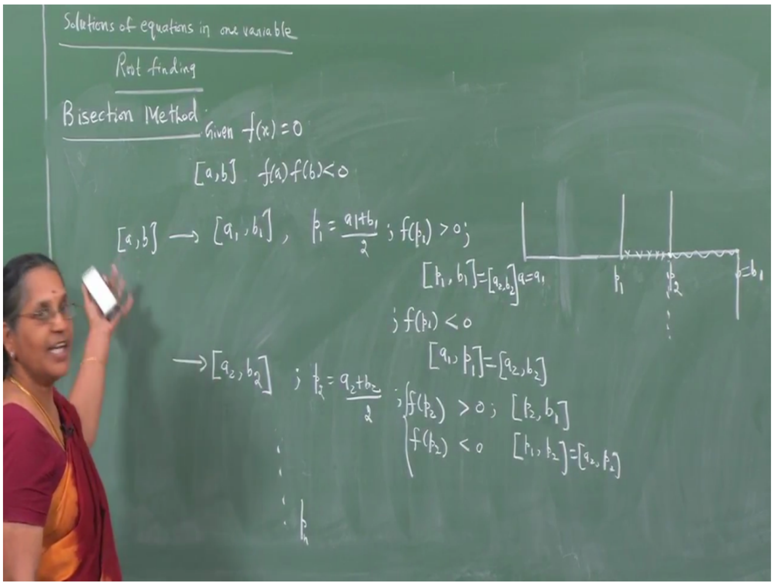

Let us consider the bisection method, so given the equation f of x = 0 and an interval a b in which f is defined such that f of a into f of is less than 0, then we try to determine root of this equation f of x = 0 by the following way. Namely, we denote this interval a b by a 1 b 1 and find the midpoint a 1 + b 1 by 2 and call it as p 1. We know that root of the equation lies in this interval because f of a into f of b is negative, so I have called this as a 1 and this as b 1. I determine p 1, which is the midpoint of this interval. I know that f of a into f of b is negative. Having determined p 1 I check what is f of p 1, f of p 1 is going to be either positive or negative. If suppose f of p 1 is positive then in that case I have f of p 1 into f of b 1 to be negative, so this interval encloses a root of the equation f of x = 0.

So in this case I have my interval p 1 b 1 that encloses a root of the equation. Or if f of p 1 is negative then in that case where does the root lie, f of a 1 into f of p 1 is negative and therefore the root lies in the interval a 1 to p 1, so in this case the interval that encloses the root is a 1 to p 1. Whatever be the case, I am able to determine another interval either p 1 b 1 or a 1 p 1 in which a root of the equation lies. So I shall call the interval as a 2 b 2, so what do I do? I call p 1 as a 2 and b 1 as b 2 in this case, if it is this case then I call a 1 as a 2 and p 1 as b 2 so I now move to the next step. What is the next? At this step I have an interval a 2 b 2, I know what are they I have already determined are given out in this way. So at this stage I have an interval a 2 b 2, which encloses the root of this equation.

So what do I do? I am determining the midpoint of this interval which is a 2 + b 2 by 2 and evaluate what is f at p 2. Namely, if it is greater than 0 then I observe that f of p 2 is greater than 0 and so a root lies in this interval namely p 2 to b 1. On the other hand, if f of p 2 is negative then in that case suppose f of p 2 is negative, then I observe that f of p 1 into f of p 2 is negative, so root lies in this part of the interval, namely it is between p 1 and p 2, but I have called the p 1 as a 2 so it lies between a 2 and p 2. Now I check what happens if f of p 1 is less than 0 and I take a 2 b 2 as this interval, again I compute p 2 which is the midpoint and check whether a root lies in this part of the interval or in the other part of the interval, so every time I determine an interval which encloses a root of this equation.

And every time I obtain the interval in such a way that its length is half of the length of the interval in the previous step. See, in the 1st step a 1 b 1 is the interval, the width is b 1 – a 1, in the 2nd step what is my a 2 b 2, it is either this or this. What is p 1? It is the midpoint of the interval a 1 b 1 and so what is the length of this interval a 2 a b 2, it is half of the interval a 1 b 1. In the next step I have a p 2, what is the length of the interval which encloses a root? It is half the length of this interval so it is 1 by 2 square times the length of the original interval and that is at the 2nd step, so in the 1st step the length of the interval at that step is half of the length of the original interval. In the 2nd step the length of the interval with encloses the root is 1 by 2 square times the length of the interval a 1 b 1 namely the original interval.

In the 3rd step, the interval is going to be further halved so it is going to be 1 by 2 cube times the length of the interval a 1 b 1 and so on. So every time I determine an interval that encloses a root and find out what the midpoint is, namely I bisect the intervals by that midpoint and find that part of the interval in which a root lies. So I continue to do this and every time I obtain the midpoint and I call that midpoint p n, which are obtained at the nth step as an approximation to a root of the equation, why what is the reason?

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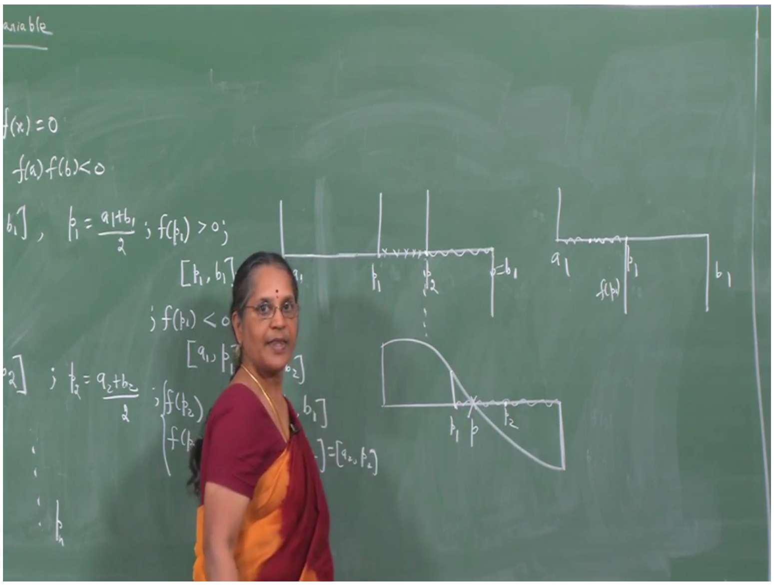

Let us see f of a 1 into f of b 1 is negative, so the actual root lies here, suppose I called this as p. What is the 1st case, f of p 1 is positive suppose p 1 is this midpoint, f of p 1 is positive then where does the route lie, it lies in this portion of the interval and you see that I have come closer to the root of the equation the actual root p of the equation. In the next time since f of p 1 into f of b 1 is negative, root lies here and therefore the midpoint p 2 of this interval is here. And I evaluate what is f of p 2 and I see f of p 2 is negative, f of p 1 is positive so root lies in this portion. I choose the midpoint that is p 3, I evaluate f of p 3 that is negative, this is positive so root lies in the interval p 1 to p 3 right and I determine the midpoint which I call as p 4.

If f of p 4 is positive and root lies between p 4 and p 3, so every time I reduce the length of the interval enclosing put of the equation (())(36:58) enclosure to the root of the equation. The method that we have described now is what is called the bisection method and it is an enclosure method because at every step we obtain an interval that encloses a root of the equation f of x = 0. Now the question is, where we stop, how many steps should we continue to arrive at a root that is closer to the exact value of the root of the equation. This is just an approximation, so the bisection method generates a sequence of iterates over the root of the equation, the question is does this sequence converge?

If it converges, does it converge to the exact of the equation f of x = 0. And what is the rate at which this convergence takes place, is it faster or slower? All these questions have to be answered and we will continue in the next class.