Numerical Analysis
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Lecture 21
Numerical Solution of ODE-4
Runge - kutta Methods

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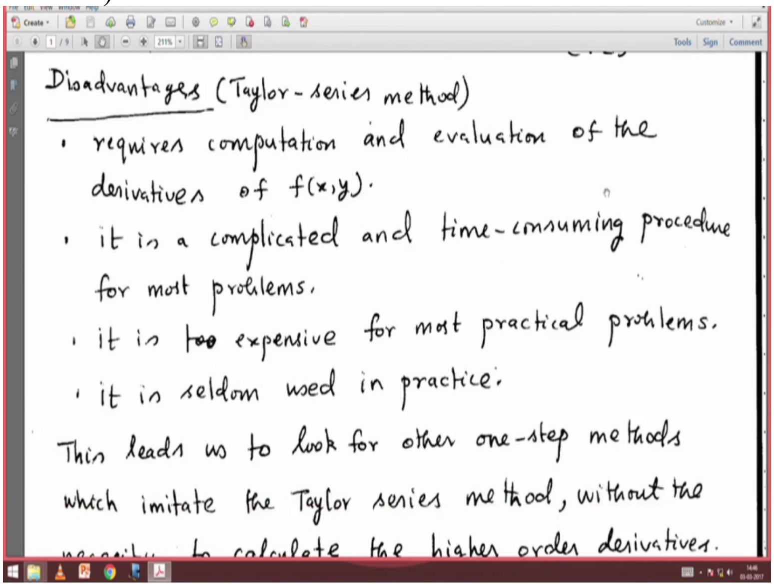

So let us recall what we have stated about Taylor series method? The Taylor series method has the desirable property of high order local truncation error but it also has some disadvantages namely it requires computation and evaluation of the derivatives of f(x,y) and it is a complicated time consuming procedure for most problems then f(x,y) is complicated. And it is too expensive for most practical problems and it is seldom used in practice because of the computational effort which are required to carry out the details.

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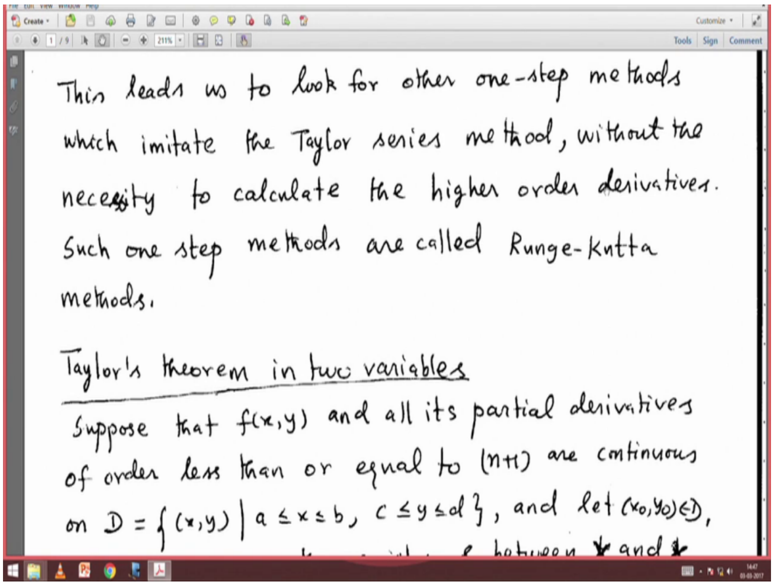

So as I said earlier this leads us to look for other single step methods which imitate Taylor Series method without the necessity of computing higher order derivatives, such methods are Runge - kutta Methods. They mimic the Taylor series method let us see how and then derive Runge - kutta Methods of different orders.

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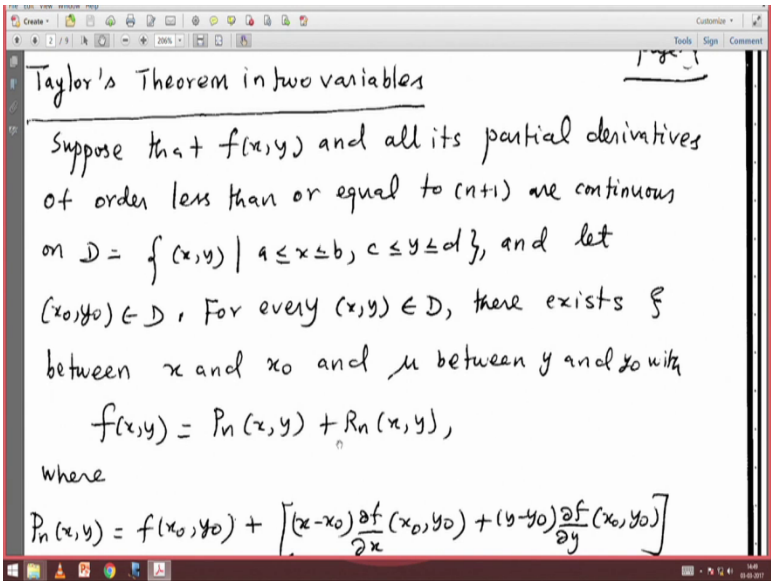

The derivation requires the knowledge of Taylor's theorem in two variables. So let us quickly go through this result. Namely Taylor's theorem for the function of two variable which says that suppose f(x,y) and all its partial derivatives of order less than or equal to n plus 1 are continuous on a rectangle R which consists of points x, y with x lying between a and b and y between c and d, and if x 0 y 0 is in d then for every x y in d there exists a Psi between x and x 0 and Mu between y and y 0 with f (x,y) equal to p n (x ,y) plus R n (x,y).

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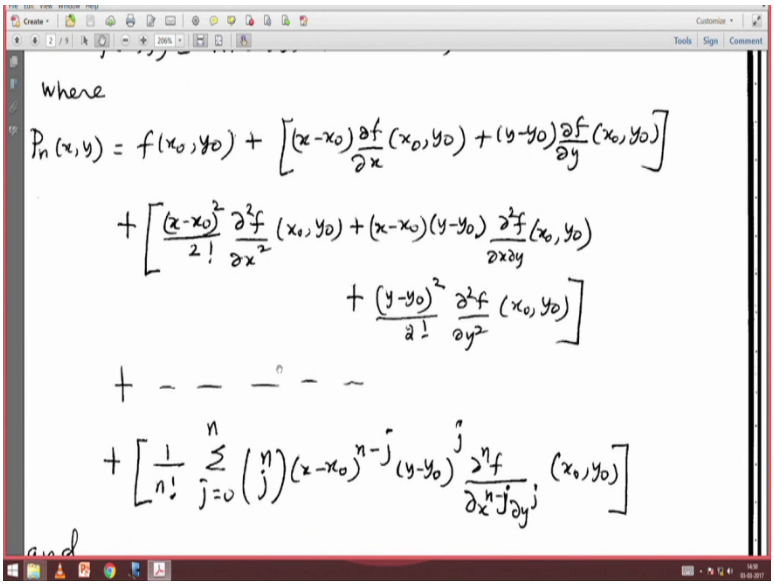

Where p n (x ,y) is f(x 0 y 0) plus x minus x 0 into partial derivative of f with respect to x evaluated at x 0 y 0 plus y minus y 0 into partial derivative of f with respect to y at x 0 y 0 plus the higher order terms plus etc plus the term involving the n th order partial derivative. That constitutes p n(x,y).

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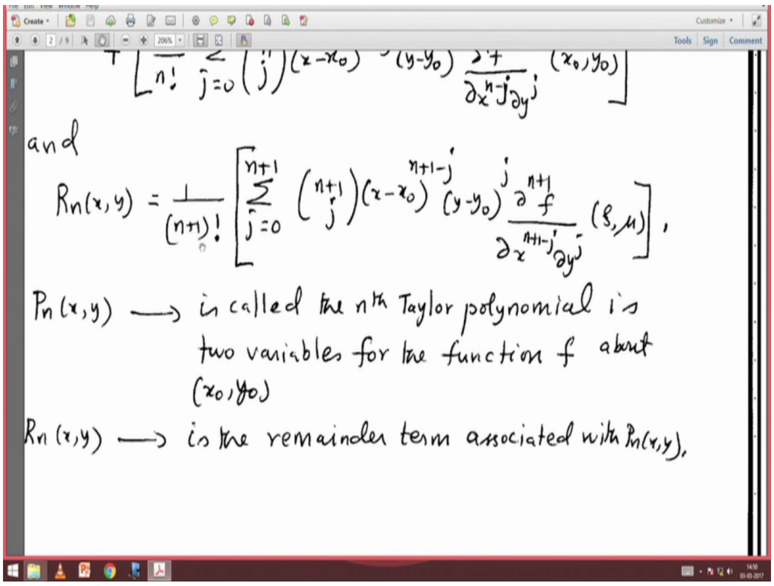

And R n(x,y) is the rest of the terms which is given by 1 by (n plus 1) factorial [sigma j equal to 0 to n plus 1 (n plus 1 is c j) (x minus x 0 power n plus 1 minus j into (y minus y 0) power j into n plus 1 th derivative of f with respect to x n plus 1 minus j times with respect to y j times evaluated at (Psi, Mu). This R n is called the remainder term associated with p n and p n (x,y) is the n th Taylor polynomial in two variables for the function f about (x 0, y 0)

This is not something new to you you have already learned that when you have discussed function of two variables in your calculus course. So let us just recall it in the same notation in which you would have learned.

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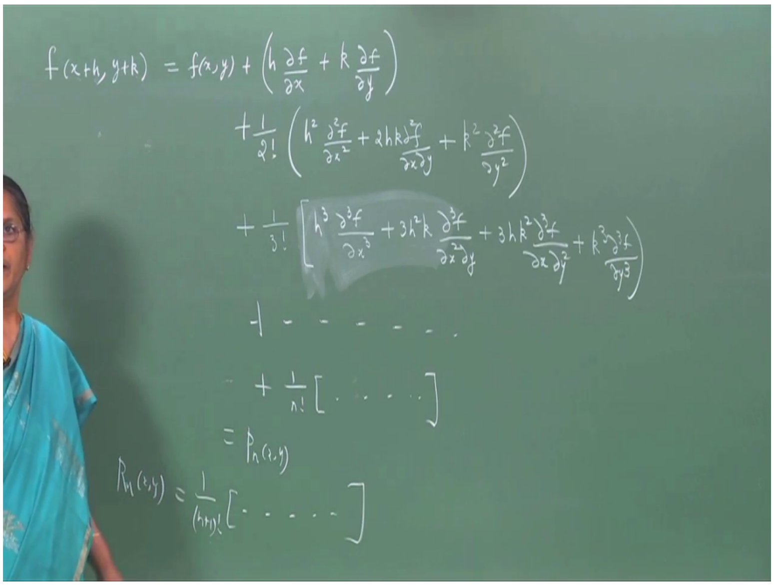

So let us look at the board and write down the details. So if I have a function of 2 variables and I give a small increment h to the first argument and an increment k to the second argument. Then using Taylor series it is given by f(x , y) plus the increment in the first argument that is h multiplied by the partial derivative of f with respect to the first variable for which h is the increment plus increment in the second argument which is k into partial derivative of f with respect to the second variable y for which k is the increment.

So this is set of terms which involve first order partial derivatives. Now we go to the next term which is 1 by 2 factorial into h square into the second order derivative of f with respect to x plus 2 h k into the second order derivative of f with respect x with respect to y which is a next partial derivative plus k square into second order derivative with respect to y twice.

Let us write down the third term which is 1 by 3 factorial h cube into the third derivative of f with respect to x plus 3 h square k into third derivative of f with respect to x twice and y once plus 3 h k square into third derivative of f with respect to x once and y twice and then k cube into the third derivative of f with respect to y twice and so on.

So if I move on upto 1 by n factorial into the terms that is what I call as p n(x, y) to this I should add R n(x,y) which will be the first neglected term after this which will involve 1 by (n plus 1) factorial into the terms which involve derivatives of f which are partial derivatives of f of order n plus 1 with respect to x with respect to y suitably. This is what we have written there.

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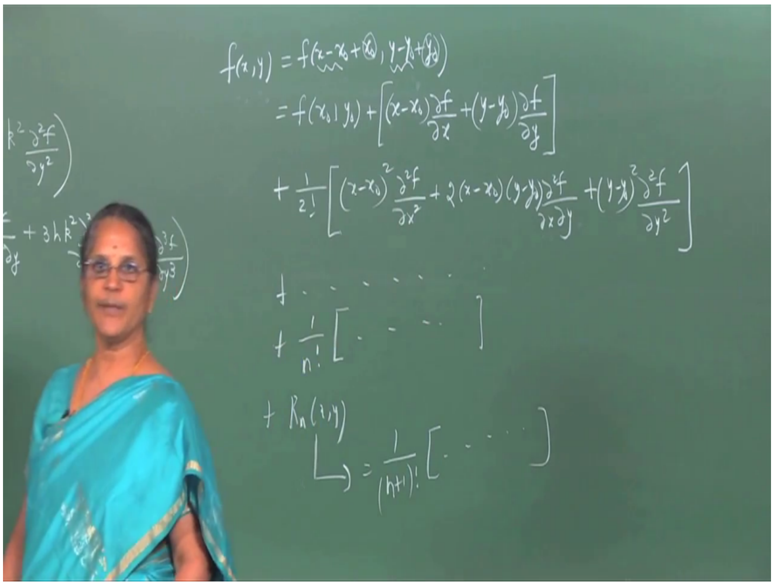

How we want f(x, y) we want to expand about the point x 0 , y 0 so I rewrite this as(x minus x 0) plus x 0 , (y minus y 0) plus y 0. So I want to expand about x 0 y 0, so my increment is (x minus x 0) to the first variable and (y minus y 0) to the second variable. So it is f (x 0, y 0) plus the increment given to the first variable is x minus x 0) into partial derivative of f with respect to x increment given to the second variable into partial derivative of f with respect to y.

So these are the first set of terms involving first order partial derivatives what will be the next term 1 by 2 factorial into h square which corresponds to square of (x minus x 0) into f with respect to x twice plus 2 h which is (x minus x 0) k which is (y minus y 0) into second order derivative with respect to x with respect to y once right? And then (y minus y 0) the whole square into f second order derivative with respect to y twice.

That will constitute the terms involving second order derivatives and so on and now I have the term involving 1 by n factorail all these terms constitute p n(x,y) which is called the n th polynomial Taylor polynomial in two variables for the function f about the point x 0 y 0. And to this I should add R n(x,y) which is the remainder and this will involve 1 by (n plus 1) factorial and terms involving the n plus 1) th derviative of f with respect to x with respect to y appropriately.

So this is Taylor expansion for a function of two variables about a point x 0 , y 0 so we will make use of this in deriving Runge - kutta Methods of appropriate orders. So let us now move on to their derivation of this method.

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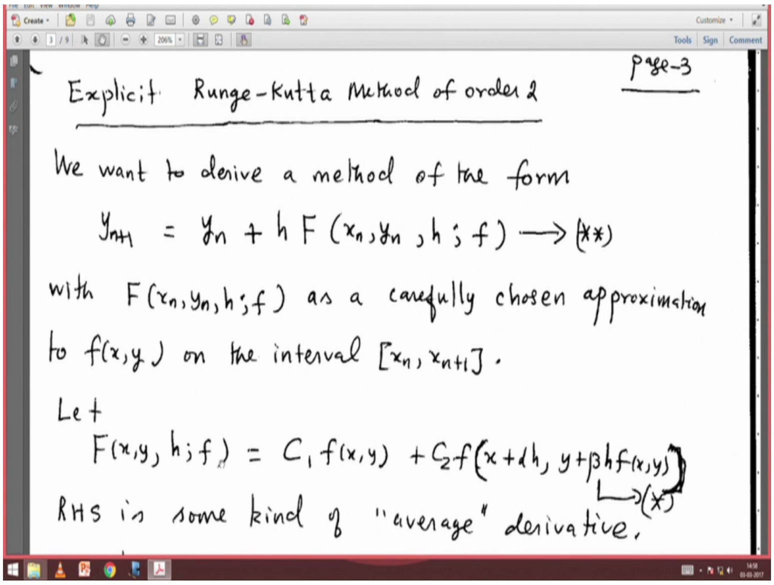

So first of all we shall derive Runge - kutta Method of order 2, so I will explain why we call this as a method of order 2 later on when we complete our derivation. It is an explicit method reason is that if we know the values of y (x 1) say y 1 then we can march one step ahead which is at a distance of h units from x 1 namely I can go to the point x 2 and compute y(x 2) namely y 2 explicitly with a knowledge that I have about the function y or f(x,y) at the point x 1.

And go ahead like this and determine the solution at each of these points explicitly and so it is referred to as an explicit method. So in Runge - kutta Method of order 2 what is it that we are interested in? So let us just look at the board to see what we want?

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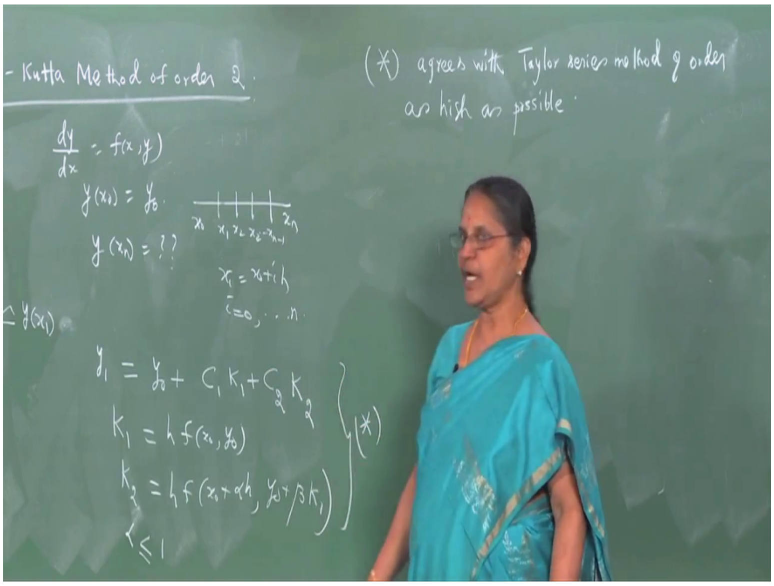

So we would like to derive RK method we call it as RK method Runge - kutta Method of order 2 and this method we solve a first order differential equation of the form dy by dx equal to f(x,y) with y(x 0) equal to y 0 and go upto the computation of our solution at x n. So we have an interval x 0 to x n which we devide into number of sub intervals of the form x 0 to x 1 x 1 to x 2 etc such that x i is x 0 plus i h for i is equal to 0 1 2 3 upto n.

So at the first step we would like to compute y(x 1). This is approximated by y 1 , so y 1 approximates y (x 1). And I would like to derive this method in such a way that y(x 1) is going to y 0 plus some constant c 1 times k 1 plus c 2 times k 2 where c 1 and c 2 are constants to be determined and k 1 and k 2 are functions such that k 1 is h into f(x 0, y 0) and k 2 is h into f(x 0 plus alpha h y 0 plus beta into k 1). So you observe that k 1 involves evaluation of the function f which appears here at x 0 y 0 and we know both x 0 and y 0 so immediately this can be obtained.

And what is k 2 that again involves evaluation of the same function but now at a point (x 0 plus alpha h,y 0 plus beta k 1) where again alpha and beta are constants to be determined and where does this point where does this x 0 plus alpha h lie we are computing the solution at x 1 namely y 1.

So I want this point x 0 plus alpha h to be a point between x 0 and x 1. So its immediately clear that alpha is less than or equal to 1. This point can be x 1 also when alpha is 1. So I want the point to be somewhere between x 0 and x n. Already I have taken the function value would x 0 y 0. So I would like to evaluate this function at some point x 0 plus alpha h. I do not know what this alpha is now I have to determine not only alpha I also have to determine the constant beta.

So we now have 4 constants c 1 , c 2 , alpha, beta with the restriction that alpha must be less than or equal to 1 such that these 4 constants have to be evaluated. What is the condition with the help of which you are going to determine these constants. The condition is that the method that we have written so I call this as star right? This method star agrees with Taylor series method of order as high as possible.

Remember Taylor series method involves derivatives of certain functions but I can replace the derivatives in terms of the partial derivatives of f of different orders. On the other hand this method involves only the function evaluations at certain points in the interval x 0 to x 1 when you want your solution at x 1 knowing your solution at x 0.

So the requirement is that the constant c 1, c 2, alpha, beta must be determined in such a way that methods star mimics or it agrees with Taylor series method of order as high as possible. So let us obtain what this star is and what is Taylor series method of order as high as possible that matches with this method star and then determine the constants c 1, c 2, alpha and beta.

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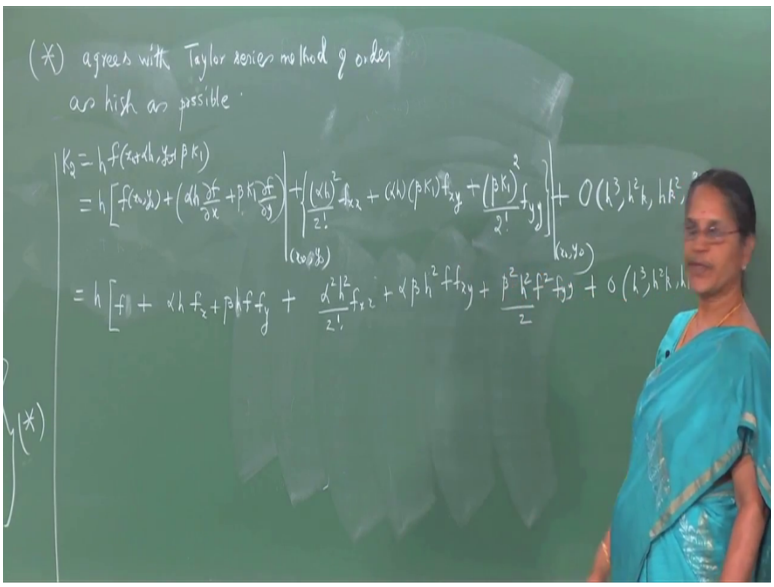

So let us first take k 2 so what is k 2? k 2 is h into f (x 0 plus alpha h , y 0 plus beta into k 1) So I have a function of two variables and each variable is such that an increment is given to the first argument an increment is given to the second argument so I use Taylor's theorem for two variables and write this as f(x 0, y 0) plus apha h into partial derivative of f with respect to f plus beta k 1 into partial derivative of f with respect to y then I take alpha h the whole square by factorial 2 into f with respect to xx twice then 2 h k is the increment for h and k divided by factorial 2 so I simply have increment alpha h multiplied by increment beta k 1 into partial derivative of f with respect to x with respect to y.

And then the next term will be square of the increment given to the second variable what is the square of the increment given to the second variable? It is beta k 1 so beta k 1 the whole square divided by factorial 2 into f yy that is the next term. So these constitute terms involving the second order derivatives, plus terms of order of h cube h square k and then h k square and k cube. In case we require then we will write down let us see what happens as our computations are carried out.

So remember that terms on the right hand side all these have to be evaluated at the point x 0 y 0. So let us simplify this further so h into f(x 0,y 0) I shall simply write it as f remembering that it should be evaluated at x 0 y 0 in then the next term plus alpha h into f x plus beta what is k 1 I already have K 1 as h into f. So I substitute k 1 is h into f multiplied by f y and then I write down the next set of terms which give me alpha square h square by factorial 2 fxx plus alpha beta h into what is k 1 h f so that will give me h square f into f x y plus beta square k 1 square is h square into f square into f yy divided by 2 plus order of terms involving h cube h square k h k square k cube and so on.

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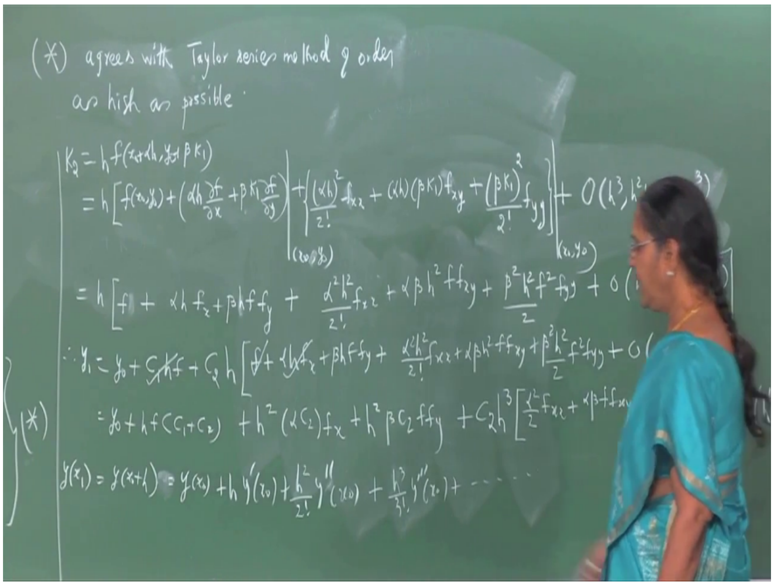

So this is what we get for k 2. So I shall substitute in this method, so we have therefore y 1 to be given by y 0 plus C 1 into k 1 is h into f plus C 2 into k 2 ; k 2 has been computed so k 2 is h into [f plus alpha h fx plus beta h f yy plus alpha square h square by factorial 2 f xx plus alpha beta h square ff xy plus beta square h square by 2 h square f yy plus order of these terms that is k 2.

So which again can be smplified and written as y 0 plus h into f into C 1 plus C 2. So we have these two terms then the next term is h square into alpha C 2 into f x then the next term h square into beta C 2 into ff y. Then I observe that the terms are C 2 into I have h square appearing in all the three h outside so I shall take C 2 into h cube multiplied by write down the rest of the terms [alpha square by 2 f xx plus alpha beta ff xy plus beta square by 2 into f square f yy] that is the next set of terms.

And then I have terms now of what already I have h cube h square k etc that must be multiplied by h so they will be all the of the order of h power 4 h cube k h square k square and h k cube, so such terms will appear. So this is what our method is when we use Taylor expansion for a function of variables.

What is that we want? We want this method to agree upto Taylor series expansion upto order as high as possible. So let us write down the Taylor expansion namely I want to compute what is y(x 1) which is y(x 0 plus h) because y 1 approximates y(x 1). So I would like to get a Taylor expansion of y about the point x 0.

So it is y(x 0 plus h) which is x 1 that will be equal to y(x 0) plus h into y dash (x 0) plus h square by factorial 2 into y double dash(x 0) plus h cube by factorial 3 into y triple dash (x 0) plus etc this is Taylor series. So we shall now substitute for these derivatives and write down the solutions.

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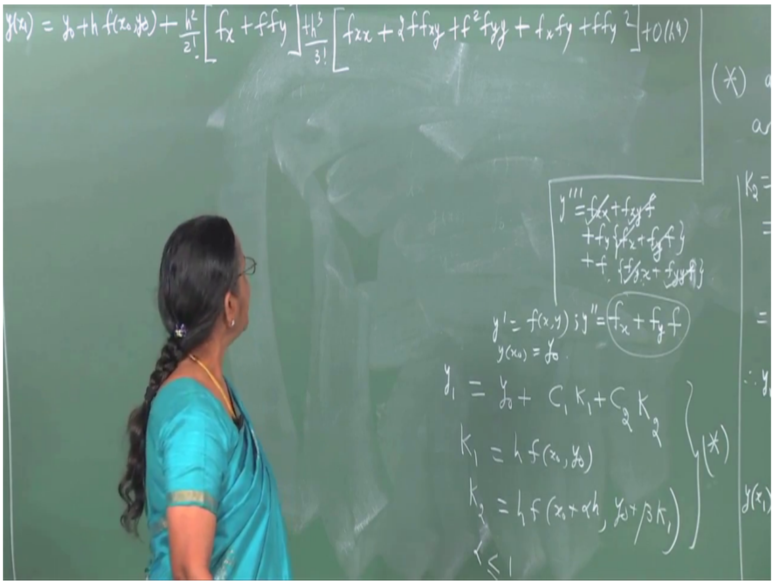

So we now write down what y(x 1) is? So y(x 1) will be equal to y 0 plus h into y dash we know the differential equation is y dash is equal to f(x, y) and the initial condition is y(x 0) is y 0 and we want y dash(x 0) so it is f evaluated at(x 0 y 0).

Then the next term h square by factorial 2 what do I have y double prime when y prime is f (x ,y) y double prime is d by dx (f (x,y)) f(x,y) is a function of two variables x and y, y itself is a function of y so it will be f with respect to x partially f with respect to the second argument y into y with respect to x which will be y prime and that y prime is f.

So I know y double prime now and I have to evaluate it at x 0. so the next term will be x square by factorial 2 into y double prime at x 0. So it is [f x (x 0 y 0) plus ff y at x 0 y 0]. Now let us go to the next term so I have h cube by factorial 3 into I require the third derivative namely y triple prime. So we already have computed this third derivative in the previous class, let us work out the details.

So I need to get d by dx of this the term is a function of x and y and I require the derivative d by dx of this , so this is going to be the partial derivative of this with respect to x plus this with respect to y into y with respect to x.

Now I apply the product rule right? Then partial derivative of f with respect to x f with respect to y into y with respect to x. Then the next term f into partial derivative of f y with respect to x plus partial derivative of f y with respect to y into y prime that is f. So this will give you the third derivative so we shall substitute there so I have f xx when I have f into f xy and f into f yx f being continuous the mixed partial derivatives are equal so I can combine this and write it as 2 f into f xy.

Then I have f square into f yy so let me write that also and now we are left with the terms f x into f y which comes from here then f into f y square which comes from here so we have taken into account all the terms. So plus order of h to the power of 4, So we have written down Taylor expansion about the point x 0 in terms of the partial derivatives of f and all these terms on the right hand side we have to be evaluated at x 0 y 0 similarly all the terms which appear in the method star which we have developed also have to be evaluated at x 0 y 0.

We now enclose this condition; the method that we have derived here must match with Taylor series method which is written here upto order as high as possible. What is the maximum order of the Taylor series method with which the given method star which we are deriving will match. So let us look at the first term in y 1 here that is y 0, in the Taylor expansion it is y 0. So that is matched, move on to the next term.

I have a h f into C 1 plus C 2 in the method that we are deriving and here we have a h into f so in order that these two match we must have C 1 plus C 2 to be equal to 1. If C 1 plus C 2 is equal to 1 then the term here is the same as the term in Taylor expansion. Fine we move on to the next term. So we take the coefficient of f x here which is h square alpha C 2 and we observe the corresponding term in the Taylor expansion.

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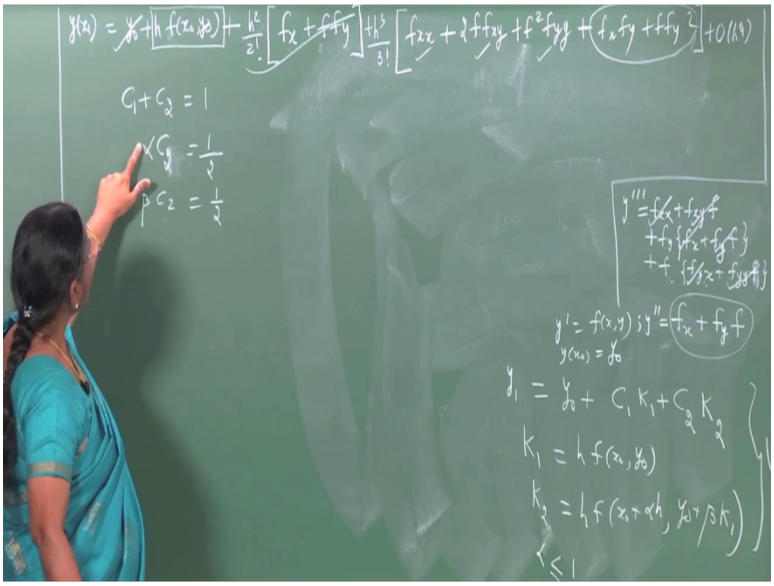

We have h square by 2 into f x so if we equate the coefficient of h square f x from both we get half to be equal to alpha into C 1 ; alpha into C 2 so alpha C 2 must be equal to half. Then we take the next term ff y into h square the coefficient is beta C 2 and I have ff y h square and its coefficient is equal to half, so we have matched the coefficient of h square terms in both the methods.

We now move on to the coefficient of h cube term we observe that we have a f xx term here as well as here f into f xy term which appears in both of them f square f yy that again appears in this term. These coefficients involve C 2 in it right? I can take the corresponding terms f xx whose coefficient will be alpha square by 2 into C 2 and equate it to the corresponding coefficient here which is 1 by 3 factorial I can do it for each of these terms but I observe that I have two extra terms alright which I cannot match with the method that I have because my method does not have that term in it.

One may say that if I choose my C 2 to be equal to 0 these terms do not appear at all. But I cannot make this C 2 to be 0 right? In that case what will happen when C 2 is 0 then we get inconsistent values here alpha into 0 will be half beta into 0 will be half. So C 2 cannot be made 0 that is clear. So C 2 has to take a non zero value. So when C 2 takes a non zero value here because of the presence of these extra two terms I cannot match order of h cube terms in the Taylor expansion with the order of h cube terms in my method that is not possible.

So I can match the method that we are deriving namely star with the Taylor series method only upto order of h square term. That is the maximum order possible here is a Taylor series method of order 2 namely terms upto order of h square in Taylor series can be matched with the method that I am looking for namely star.

..So the h cube terms cannot be matched and that will be the first neglected term in the Taylor expansion. So I look at the equations which have helped us to match upto Taylor method of order 2. We observe that there are 4 constants c 1 c 2 alpha beta but there are only three equations so one of the constants can be arbitrarily chosen. So let us choose say alpha to be equal to 1.

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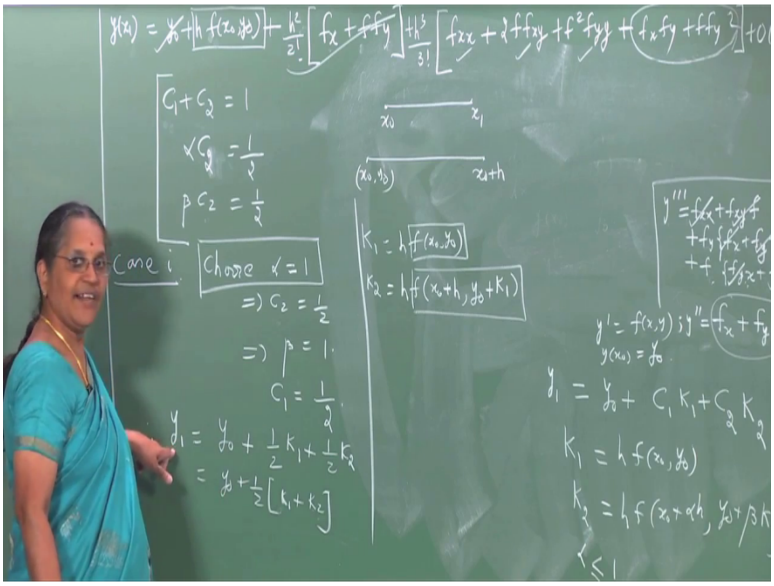

So let us choose alpha to be equal to 1 which will give you C 2 to be equal to half and therefore beta will be equal to 1. So when alpha is 1, beta 1 C 2 half and since C 1 plus C 2 is 1 is immediately gives you that C 1 is also equal to half. So we start with the choice namely alpha equal to 1 this is one way of choosing any one of the constant can be arbitrarily chosen such that consistent solution of this system can be obtained.

So this is the first case namely the choice has been made like this. So with this we obtain our method to be y 0 plus C 1 which is half into k 1 plus C 2 which is again half into k 2 or it is y 0 plus half (k 1 plus k 2). What are k 1 and k 2 we see that k 1 is h into f (x 0 y 0) and what is k 2 it is h into f(x 0 plus alpha h) what is alpha; alpha is 1 so plus h , y 0 plus beta k 1 beta is 1 so plus k 1.

So I have to evaluate the function where at the point x 0, y 0 and at the point which is this point? x 0 plus h and the corresponding y argument is y 0 plus k 1 and I know what k 1 is. But any way since this will be some number k 1 is some number which represents a slope of a certain line right? And x 0 plus h will be a number what is this x 0 plus h is x 1 so when I start with the point with x 0 and move to x 1 what am I doing? I am considering k 1 to be h times a slope at x 0, y 0 and k 2 which is h times the slope at the point x 1, y 0 plus k 1.

So I have two slopes and I am taking the average of these slopes k 1 and k 2 and adding it to y 0 to get the value of y 1. It is a sort of an average of the slopes that is being taken and the method that we have derived is this with k 1 and k 2 given by this and we do not have any derivative evaluation of a function we simply evaluate a function f what is this function? The function f which appears on the right hand side of the given initial value problem at different points namely one of them is at x 0 y 0 the other one is x 1 at which we want the solution , y 0 plus k 1 and k 1 is nothing but h f(x 0 y 0).So we simply have to use this method and arrive at our solution at the point x 1 and what we get is an approximation to the solution y(x 1) at the point x 1.

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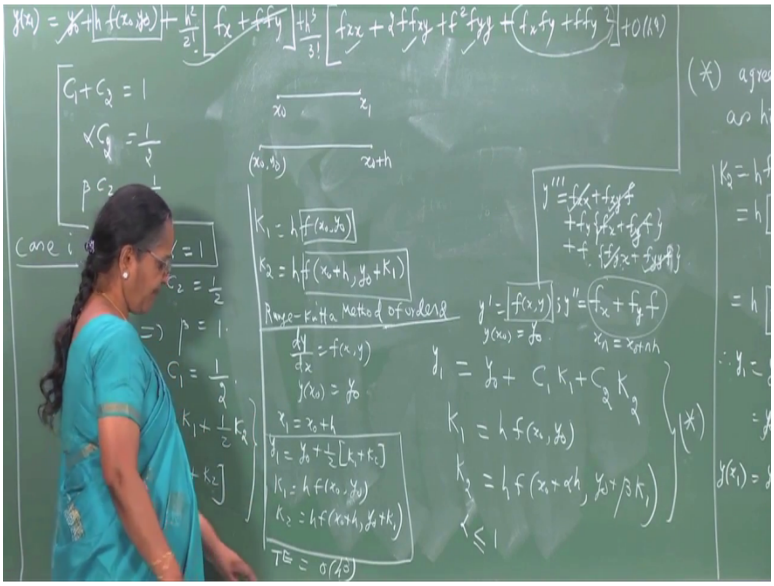

So we summarize and write down our Runge Kutta method which we have derived it is an explicit method it is a single step method so it solves the initial value problem dy by dx is equal f(x ,y) subject to y(x 0) equal to y 0 and when I require the solution say at the next point which is x 0 plus h then its solution is y 0 plus half of k 1 plus k 2 where k 1 is h into f(x 0 y 0) and k 2 is h into f(x 0 plus h, y 0 plus k 1).

So this is my method and what is the property of this method? This method matches with a Taylor series method of order 2 because upto h square term we have matched so it matches with Taylor series method of order 2. So when you consider Taylor series method of order 2 what is the truncation error it is of order of h cubed.

Since Runge Kutta Method method matches with Taylor series method of order 2 we call this as Runge Kutte Method of order 2. Therefore what is the truncation error? The truncation error is the order of the first neglected term namely h cube so the local truncation error of this method is going to be of order h cube.

So we use Runge Kutta Method of order 2 to compute the solution at the first point x 1 by taking the step size h to be small the accuracy can be proved. Ok what was our goal? Our goal was to go upto the point x n such that x n is x 0 plus n h we have just given the solution at x 1.

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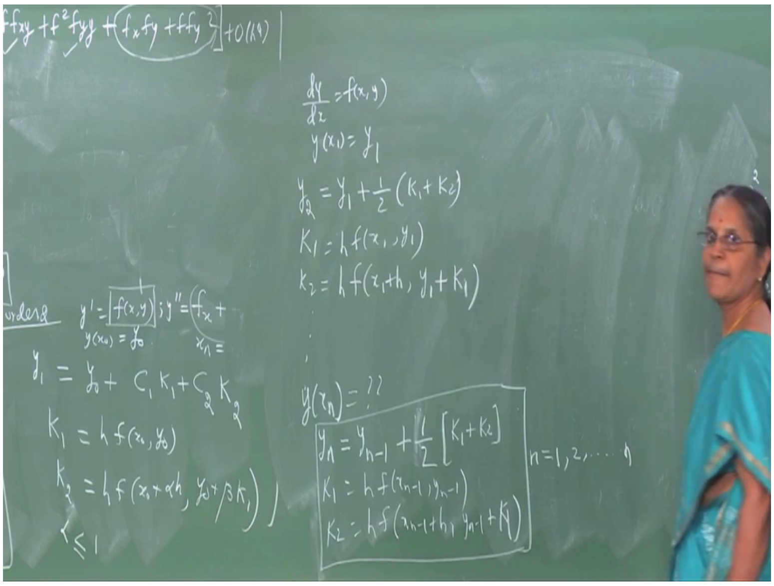

So what do we do and solve the initial value problem dy by dx is equal to x is equal to f(x , y) subject to the condition y(x) equal to y 1. What is this y 1? This y 1 is what we have already computed. And so what is the solution y(x 2) is approximated by y 2, and what is it? It is y 1 half of k 1 plus k 2, and what are k 1 and k 2? K 1 is h f(x 1 y 1) and k 2 is h f(x 1 plus h, y 1 plus k 1).

So at every step you evaluate 2 functions the method involves just two function evaluations and the method is of order 2. So I can continue this and finally come to the solution at the point x n. So to get this we again apply RK method of order 2 and get y n to be equal to y n minus 1 plus half (k 1 plus k 2) and what are k 1 and k 2?

K 1 is h into f(x n minus 1) (y n minus 1) and k 2 is h f(x n minus 1 plus h, y n minus 1 plus k 1). So if you apply this you obtain the solution at the point x n or one can say that the solution of the initial value problem given by Runge Kutta method of order 2 is this for values of n equal to 1,2,3 etc upto n.

So when n is 1 y 1 is y 0 plus half k 1 plus k 2 with appropriate function values for k 1 k 2, where n is 2 you get y 2 is y 1 plus something and continue this you end up with a solution at the point y n. So explicitly you determine the solution at each of the nodes into which you have divided the entire interval into n equal sub intervals. And you go step by step to compute this solution, so it is a single step explicit method of order 2 and is called Runge Kutta method of order 2 matches with Taylor series method of order 2 and so it has truncation error of order h cube.

So we now move on to higher order Runga Kutta methods but before that we will see whether there are any other possibilities so the derivation is presented here. I thought it will be easier to follow when I discuss it on the board and that is why we worked out the details on the board. But all these details are also available in the material that is displayed on the monitor.

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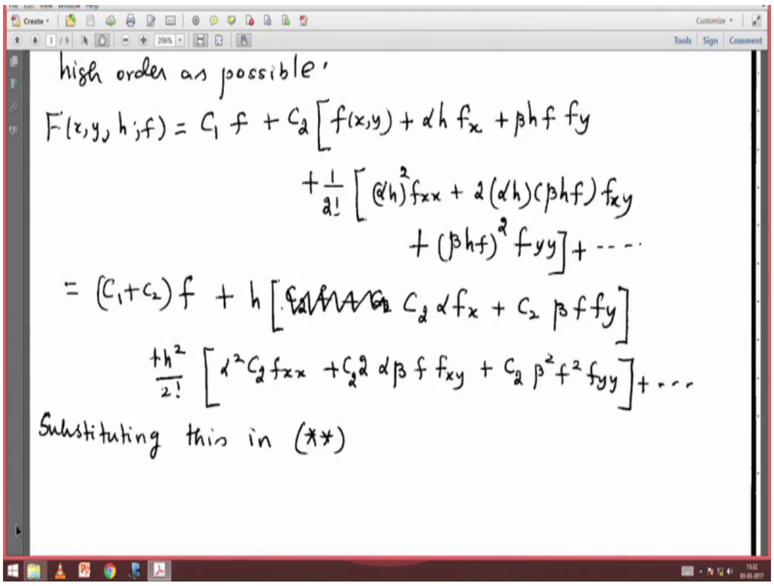

So if you look at Case 1 there we have chosen C 2 to be equal to half namely whatever that is given here has been considered and the RK Method of order 2 has been obtained. It is also known as modified Euler's Method and I shall explain the details of why it is called a modified Euler's Method later on, when I complete the discussion on the Runge Kutta Methods.

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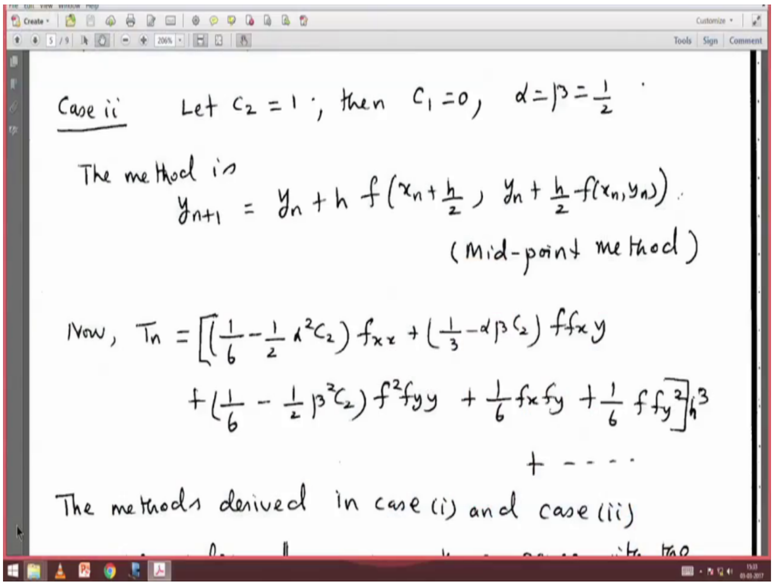

There is another possibility of solving this system of equations which we have written here namely C 1 plus C 1 is 1; alpha C 2 is half and Beta C 2 is equal to half. So let us look at the monitor and see what are the choices. I can also make C 2 to be 1 in which case C 1 will be 0 and alpha and beta will each be equal to half.

Then in that case I end up with a method which is y n plus 1 namely y (x n plus 1) is y n plus h into f (x n plus alpha h alpha is half so x n plus h by 2 then , y n plus beta k 1 so h by 2 into f(x n y n). We call this as the midpoint method because we started with our initial condition at x n we want to determine the solution at x n plus 1.

We evaluate the function f at the midpoint which is x n plus h by 2. So we call this as the midpoint method. And if you write down the truncation error in that method which is the difference in the order of h cube term that appears in our method star minus the order of h cube terms which appear in the Taylor expansion and that is what is presented as t n in this material.

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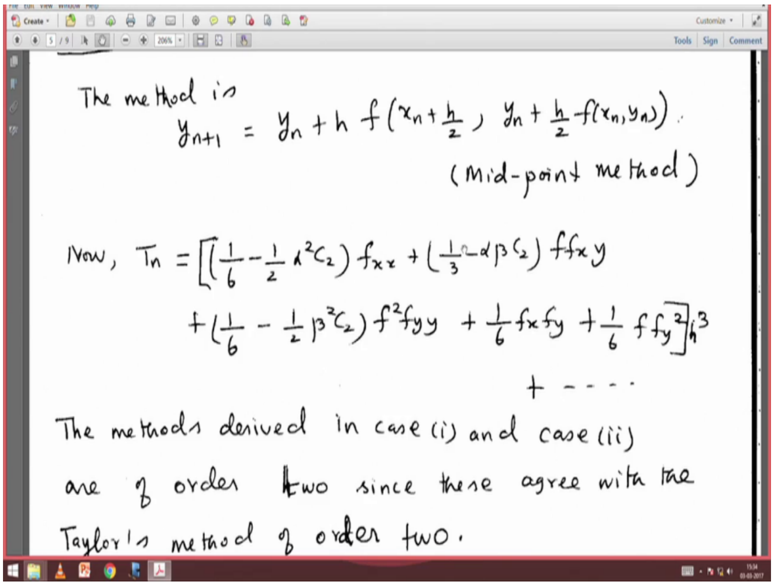

The midpoint method that we derived is again a Runge Kutta method of order 2 because we have been able to match this method upto second order in Taylor expansion of the function and therefore its again a second order method.

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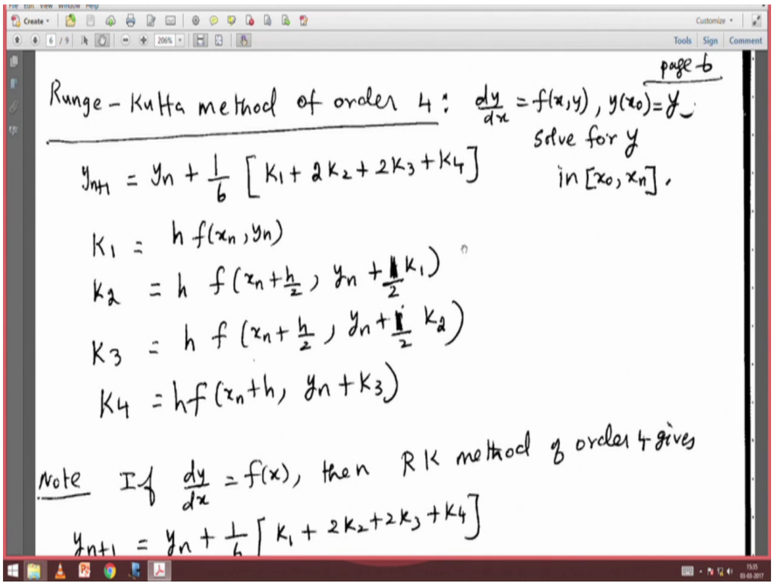

Now we move on to Runga Kutta method of order 4. Now it is clear when I say I want to move on to Runga Kutta method of order 4 it means that we would like to derive a method which involves four function evaluations and the method must match with Taylor series expansion upto order of h to the power of 4.

So we write down this method which solves this initial value problem dy by dx is equal to f(x,y) y(x 0) equal to y 0. My method is going to be y n plus 1 is equal to y n plus 1 by 6 k 1 plus 2 k 2 plus 2 k 3 plus k 4 , where k 1 k 2 k3 k 4 are function evaluations done at some intermediate points between x n and x n plus h.

What is x n plus h? That is x n plus 1, what is x n plus h by 2? That is a midpoint of this interval x n to x n plus 1. So function evaluations are done at points which lie between x n and x n plus 1. And it involves these 4 function evaluations k 1 k 2 k 3 k 4 and method which is given here is the fourth order method such that it agrees with the Taylor series method of order h to the power of 4.

Just one important observation we need to solve for y in the interval x 0 to x n plus1. So we divide the interval x 0 to x n plus1 into n plus 1 equal parts by means of points x 0 x 1 x 2 etc x n x n plus 1. So any x n can be written as x 0 plus i h where i runs from 0 to n plus 1. That is why the solution given here as y n plus 1 equal to y n plus 1 by 6 into sum of these 4 terms.

So we do not go into the details of the derivation of this method. Here the method has to be derived similar to what we have done in the derivation for second order method. So write down the method expand by Taylor series expansion for function of two variables. Now you have to go upto order of h to the power of 5 for our method.

Also write down Taylor series method of order h power 5 and then match that terms upto order of h power 4 in Taylor series method and also in our method then you will end up with the coefficients which appear here to be 1 by 6 2 by 6 and 1 by 6 which we can call as C 1 C 2 C 3 C 4 and also the coefficients that may appear here has coefficient of h as half coefficient of k 1 to be half.

Here again coefficient of h to be half, here 2 to be half and here it is 1 and the coefficient of k 3 is 1, so that the terms of Taylor series of method of order 4 agree with our method upto h to the power of 4 terms.

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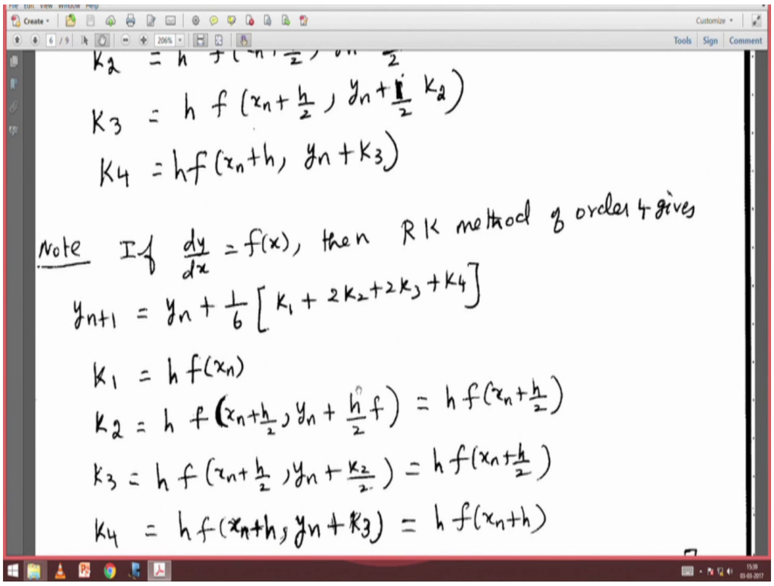

So we essentially have derived Runge Kutta method of order 4. In particular I would like to see what happens if the right hand side f(x,y) only depends on x there is no dependence of f on y , then in that case what does this Runge Kutta method of order 4 give. So it is y n plus 1 equal to y n plus 1 by 6 k 1 plus 2 k 2 k 3 plus k 4. But now k 1 will be h into f(x n) k 2 will be h into f(x n plus h by 2), y n plus what k 1? K 1 is h into f k 1 by 2 which is this.

What is this? f(x n plus 2), y n plus h by 2 f, function of x,y is simply a function of x alone so this is essentially h into f(x n plus h by 2) similarly k 3 will be h into f(x n plus h by 2) and k 4 will be h into f(x n plus h).So we get y n plus 1 to be y n plus 1 by 6 k 1 which is f (x n) plus k 2 which is 2 h f(x n plus h by 2) 2 k 3 which is 2 h f(x n plus h by 2) and finally k 4 which is h into f(x n plus h).

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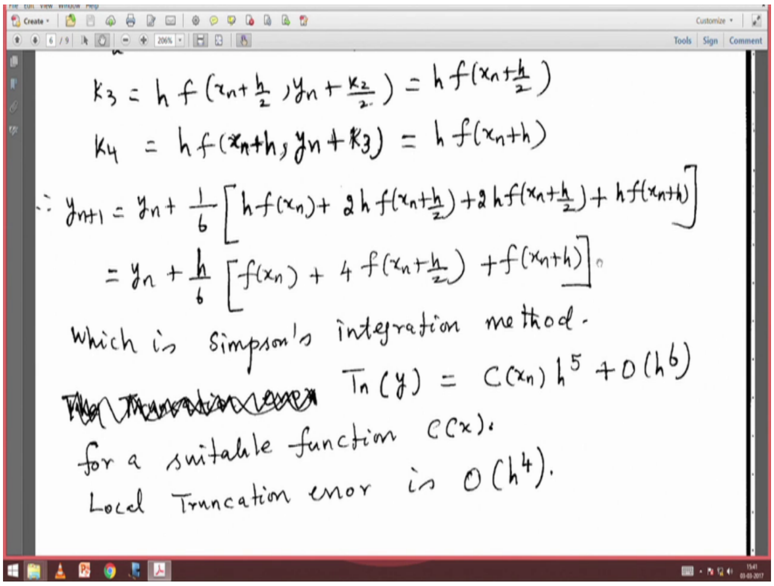

So you get y n plus 1 to be y n plus I can remove this factor h from here so h by 6 f(x n) plus 4 times f (x n plus h by 2) plus f(x n plus h).And do you recall what this method is? This is essentially Simpson's integration method which divide the interval x n to x n plus 1 into two equal subintervals by when we want to apply Simpson's rule then the Simpson's rule for integration between x n to x n plus h tell us that it is equal to h by 6 into sum of the n ordinates plus 4 times the intermediate ordinates. And that is what appears here.

So Runge Kutta method of order 4 reduces to simply y n plus 1 equal to y n plus a term which comes from the integration of f(x,y) between x n and x n plus 1 by applying Simpson's one third rule to that integral. And the error in this case will be of order of h to the power of 5.

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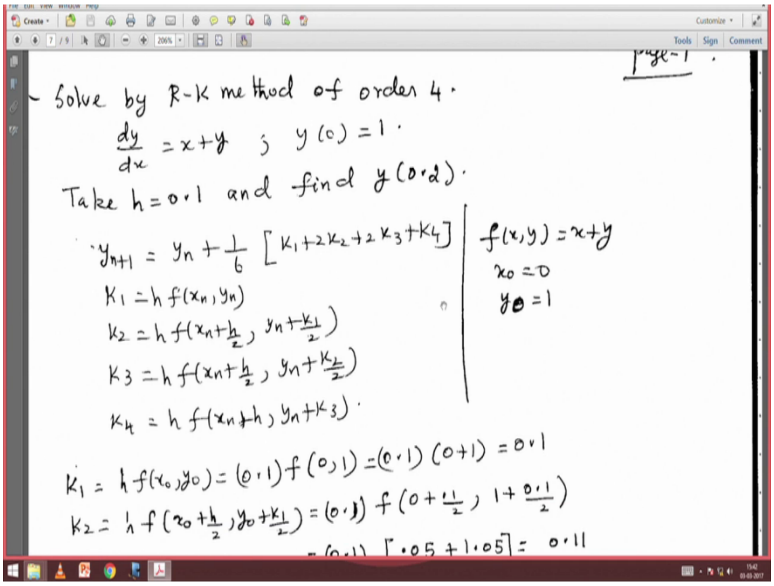

So we shall conclude this discussion by taking a simple example of RK method of order 4 and applying it to say this equation. Dy by dx is equal to x plus y y(0) is 1. So I want to take h as 0.1 and would like to find out what is y(0.2). So I write down this method I know f(x y) is x plus y x 0 is 0 y 0 is 1, so substitute in the expression for k 1 k 2 k 3 k 4 the values are given here.

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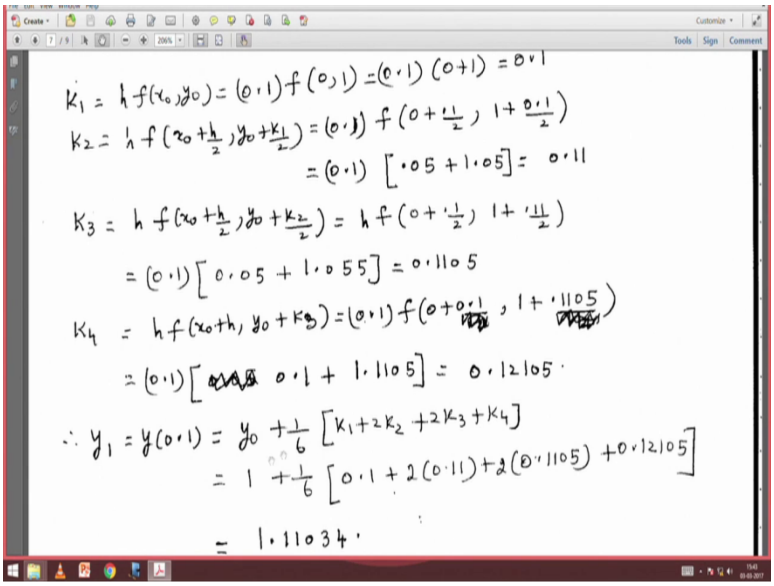

Evaluate k 1 k 2 k 3 k 4 and then substitute in the method. So you get y 1 to be equal to 1.11034.

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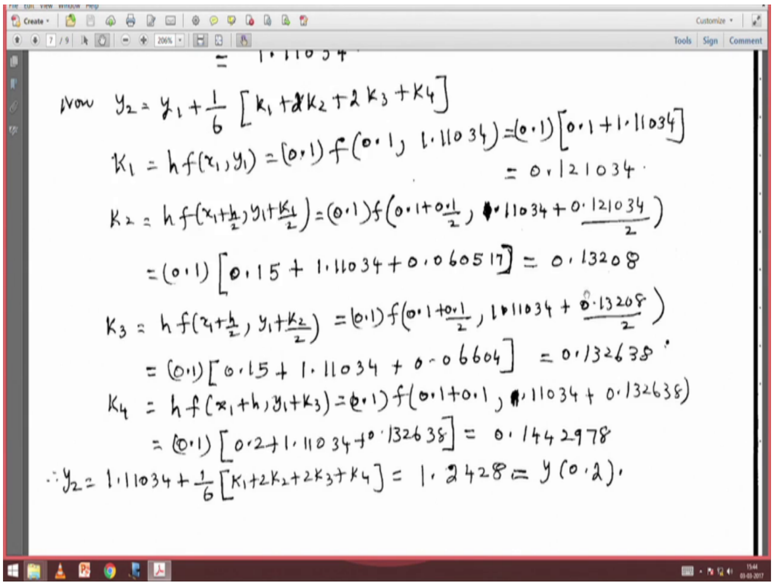

And then start with the method and with the initial condition y(x 1) is y 1 given by 1.11034 obtain y 2 namely y(0.2). Let me y 1 plus 1 by 6 into k 1 plus 2 k 2 plus 2 k 3 plus k 4. Again evaluate k 1 k 2 k 3 k 4 by substituting these values and you get y 2 to be equal to 1.24248.

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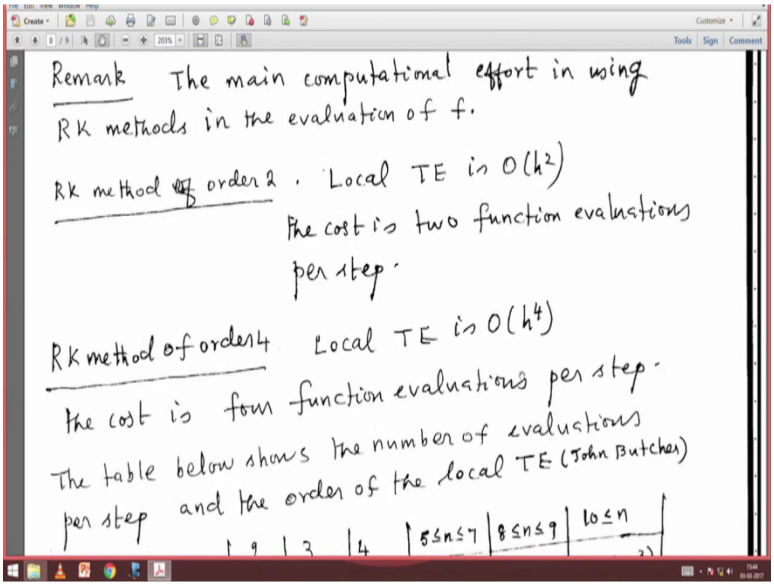

So the main computational effort in using RK method lies in the evaluation of the functions f. So the local truncation error in a second order method is of order of h square the cost is just two function evaluations plus step. And an the method of order 4 it is order of h to the power of 4.

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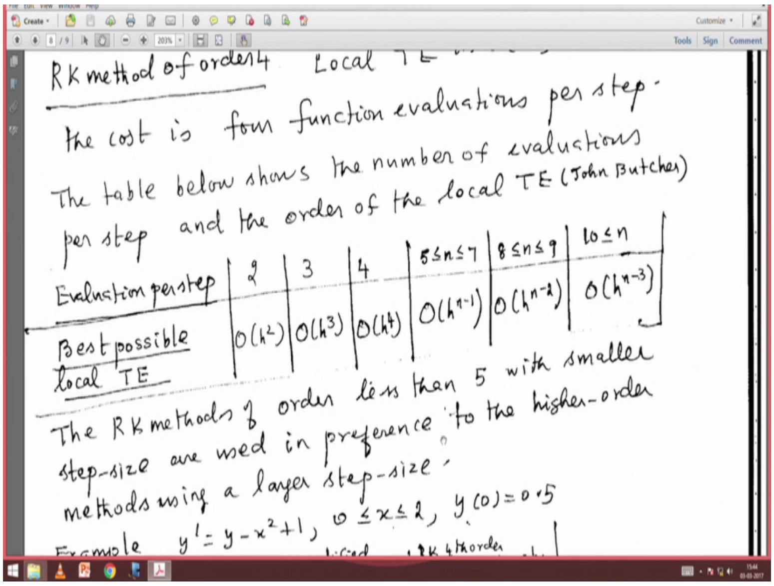

So the table below shows you the number of evaluations per step that you require when you consider the order of local truncation error. This was given by John Buccher. So you observe that in the second order evaluation of function per step when you use a second order method the best possible local truncation error is of order of h square.

If it is 3 then truncation error is of order of h cube and so on. But you observe that as the order of the method increases which lies between 5 or 7 or 8 or 9 or n greater than or equal to 10 the best possible truncation error that you obtain is only of order of h to the power of n minus 3.

So when n is equal to 10 you only end up with a method whose truncation error is of order of h to the power of 10 minus 3 namely h to the power of 7. So the RK methods of order less than 4 with smaller step size are preferable to higher order RK methods using a larger step size.

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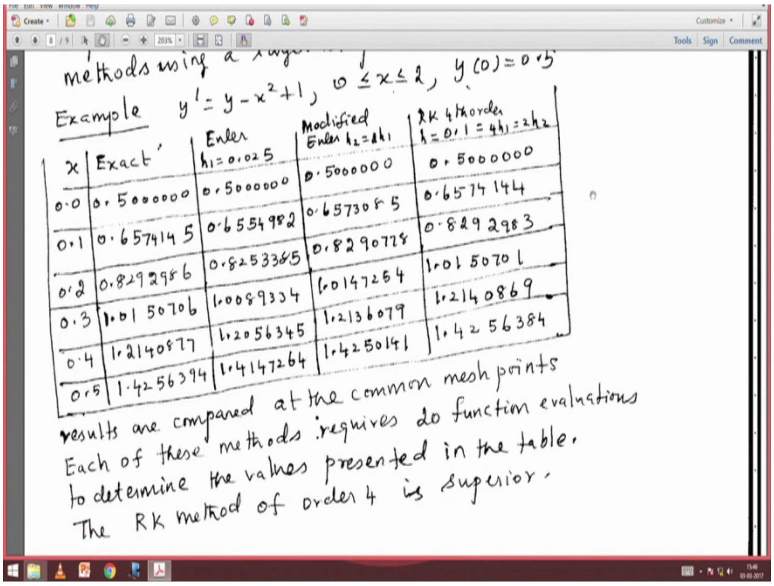

So the values of the function y which are solutions to this differential equation subject to initial condition by various methods are presented, namely Euler's method and then Runge Kutta method of fourth order and modified Euler's method which is nothing but Runga Kutta method of order 2. They have been presented at a set of points. And the RK method of order 4 seems to be superior to the other two methods.

So in practical applications one uses Runge Kutta method of order 4 in solving initial value problems which are governed by the first order differential equations. So we conclude this topic on Runge Kutta methods which are single step explicit methods and in the next class we shall try to derive methods which are known as Predictor Corrector Methods namely there is an opportunity for us to Predict the solution of an initial value problem and this predicted value can be corrected and successively re corrected till the desired degree of accuracy is reached. So we shall look into the details in the next class.