Numerical Analysis

Professor R USHA
Department of Mathematics
Indian Institute of Technology Madras

Lecture 17
Numerical Integration 5
Gaussian Quadrature (Three point Method)
Adaptive Quadrature

We were trying to obtain Numerical Methods which are exact for Polynomials of degree greater than n, then we make use of n points which are nodes in an interval. And we said that this is possible when we choose the nodes as the zeroes of Legendre polynomials of degree n. We use this idea proposed by Gauss and obtain Gauss integration formula when n is equal to 2. So let us now obtain Gauss integration method when n is equal to 3 and see what we get?

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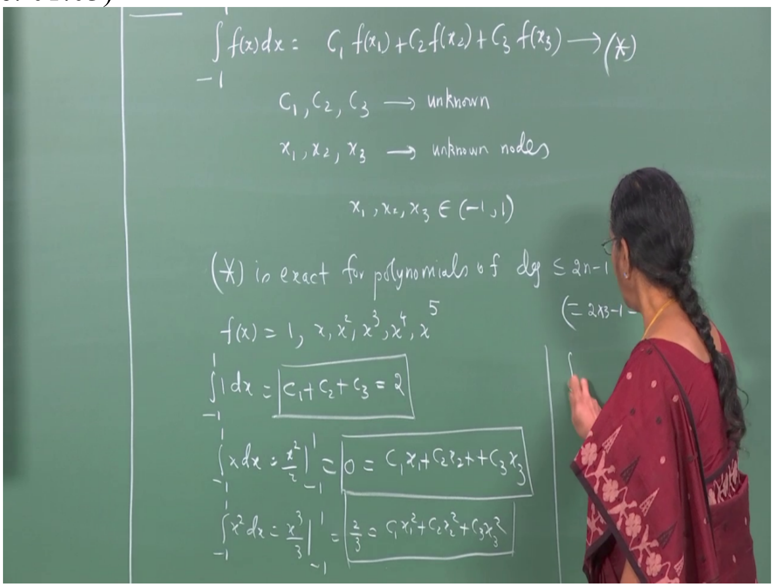

So when n is equal to 3 we require an integration formula of the form integral minus 1 to 1 f(x) dx is equal to C 1 into f(x 1) plus C 2 f(x 2) plus C 3 f(x 3) where C 1 C 2 C 3 are unknown constants to be determined and x 1 x 2 x 3 are again unknown nodes to be determined such that x 1 x 2 x 3 lie in the interval minus 1 to 1 in such a way that this integration method which I call as star is exact for polynomials of degree less than or equal to 2n minus 1 so in this case it is (minus 2 into 3 minus 1 that is 5).

So you have used only 3 nodes in the interval minus 1 t 1, so three points and informations of the function values at these three nodes here are required but you are able to get an integration method which is exact for polynomials of degree greater than 3 infact the method that we will derive is going to be exact for polynomials of degree less than or equal to 5.

So the procedure is that we want it to be exact for polynomials of degree less than or equal to 5.

So let us take it to be 1,x,x square, x cube, x to the power of 4, and x to the power of 5. So when you impose these conditions namely integral minus 1 to 1 dx will give you C 1 plus C 2 plus C 3 and that must be equal to x between minus 1 on 1 so that will give you this as equal to 2. So we can write down the other conditions namely minus 1 to 1 x dx will be x square by 2 between minus 1 and 1 that will be zero so that will give you C 1 x 1 plus C 2 X 2 plus f(x) is x plus C 3 into x 3 and that is equal to 0. That gives you the second condition. And thirdly minus 1 to 1 x square dx so that will be x cube by 3 between minus 1 and 1, so 2 by 3 that will give you C 1 x 1 square plus C 2 x 2 square plus C 3x 3 square and that must be equal to 3.

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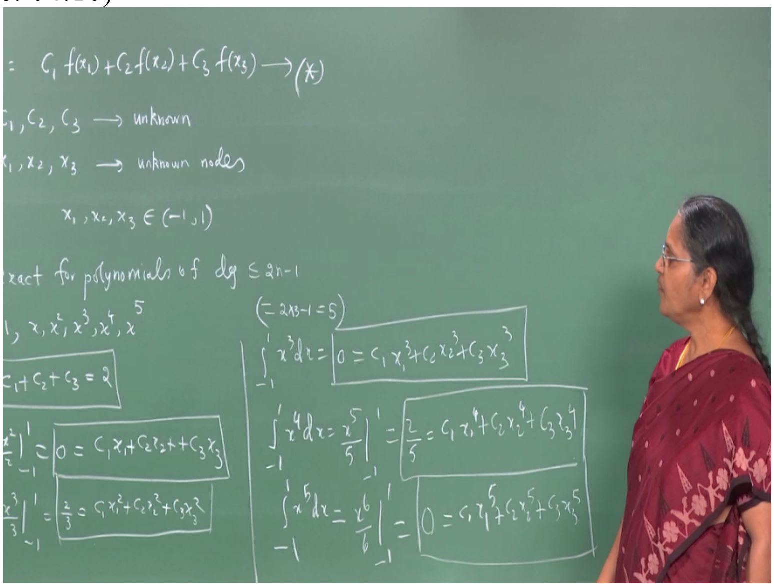

And then integral minus 1 to 1 x cube dx and that will be equal to 0 and that will be C 1 x 1 cube plus C 2 X 2 cube plus C 3 into X 3 cube. Integral minus 1 to 1 x power 4 dx will give you x power 5 by 5 between minus 1 and 1 namely 2 by 5 C 1 x 1 power 4 plus C 2 X 2 power 4 plus C 3 into X 3 to the power of 4. And the final condition is given by integral minus 1 to 1 x power 5 dx which x power 6 by 6 between minus 1 and 1 and that will be 0 and that is equal to C 1 x 1 power 5 plus C 2 x 2 power 5 plus C 3 into x 3 to the power of 5, so this gives you the 6 th equation.

So using these 6 equations we will be able to determine the unknown constants C 1 C 2 C 3 and the Unknown nodes X 1 X 2 and X 3 . So we do the calculations which is similar to what we have done in the derivation of Gauss integration method for n is equal to 2.

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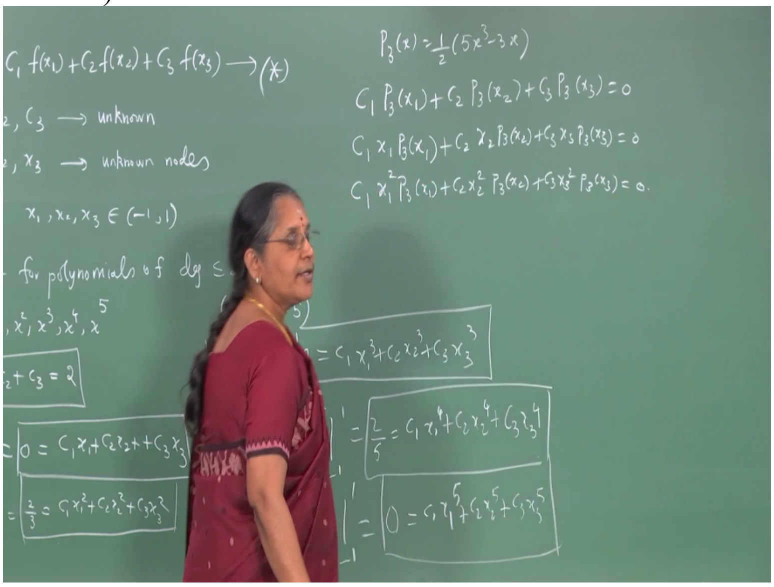

So you get C 1 into P 3(x 1) plus C 2 into P 3(x 2) plus C 3 into P 3(x 3) will be 0 by (()) (05:58) property. Then the next equation is C 1 x 1 into P 3(x 1) plus C 2 x 2 into P 3(x 2) plus C 3 x 3 into P 3(x 3) equal to 0 and you will end with the third equation as C 1 x 1 square into P 3(x 1) plus C 2 x 2 square into P 3(x 2) plus C 3 x 3 square into P 3(x 3) equal to 0.

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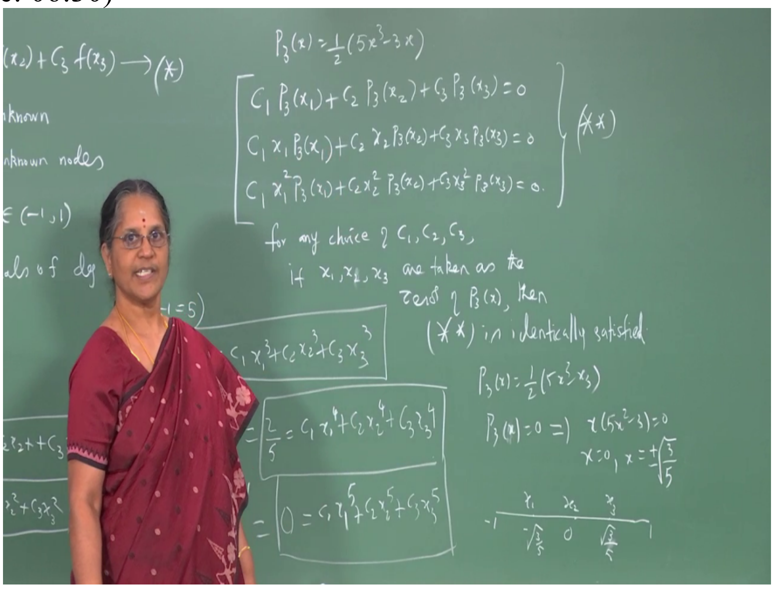

So now these equations have to be satisfied and we observe that for any choice of C 1 C 2 C 3 if you take x 1 x 2 x 3 to be the 0 of the polynomial p 3(x) which is a legender polynomial of degree 3 then this equation is identically satisfied this is again identically satisfied and the third equation will also be identically satisfied. So we see that for any choice C 1C 2 C 3 if x 1 x 2 x 3 are taken as the zeroes of p 3(x) then if I call this equation as double star, double star is identically satisfied. So let us what are the zeroes of p 3(x). We know that p 3(x) is equal to half of 5x cube minus 3x, so the zeroes of p 3(x) are given by x into 5 x square minus 3 equal to 0 so x is equal to 0 and x is equal to plus or minus root of 3 by 5.

So we have 3 zeroes namely the zeroes located at minus root of 3 by 5 at 0 and then plus root of 3 by 5 and they lie within the interval minus 1 to 1. So I take this as x 1 this as x 2 and this as x 3 and I am able to satisfy these equations identically for any choice of c 1 c 2 c 3.

So now make use of this information and determine C 1 C 2 C 3 by using any of these three equations.

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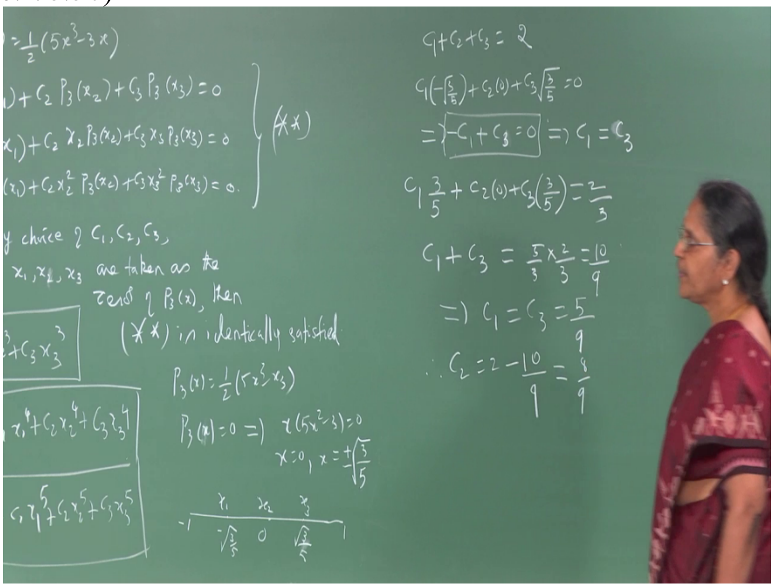

So let us take f(t) equation, that says that C 1 plus C 2 plus C 3 must be equal to 2 the second tells me C 1(x 1) that is minus root of 3 by 5 minus C 2 into (0) plus C 3 into root of 3 by 5 must be 0 so this tells you that minus C 1 plus C 2 is 0. That gives you another equation for C 1 C 2 the third one tells C 1 into x 1 square so 3 by 5 C 2 x 2 square plus C 3 into x 3 square which is again 3 by 5 that must be equal to 2 by 3 that is the third equation from which we obtain here the equation given by C 1 plus C 3 is equal to 5 by 3 into 2 by 3 so it is 10 by 9. So C 1 plus C 3 must be equal to 10 by 9. And this gives you that C 1 is equal to this is C 3 this tells you that C 1 is equal to C 3. And therefore C 1 plus C 3 is 10 by 9 so C 1 must be equal to C 3 must be equal to 5 by 9.

Therefore C 2 must be 2 minus C 1 plus C 3. So this is going to be 8 by 9 so we have been able to obtain the integration method when n is equal to 3 first we have determined C 1 C 2 C 3 and X 1 X 2 X 3 so it tells us that when n is 3 the nodes are the zeroes of p 3(x) is equal to 0. Earlier we saw where n is 2 the nodes must be chosen such that they are the zeroes of p 2(x) is equal to 0. So its clear if n is some k then the nodes chosen in such a way that they must be the zeroes of p k(x) is equal to 0. And the corresponding constants C 1 C 2 etc C k can be determined and the method will be exact for polynomials of degree greater than k in fact they will be exact for polynomials of degree less than or equal to 2k minnus 1.

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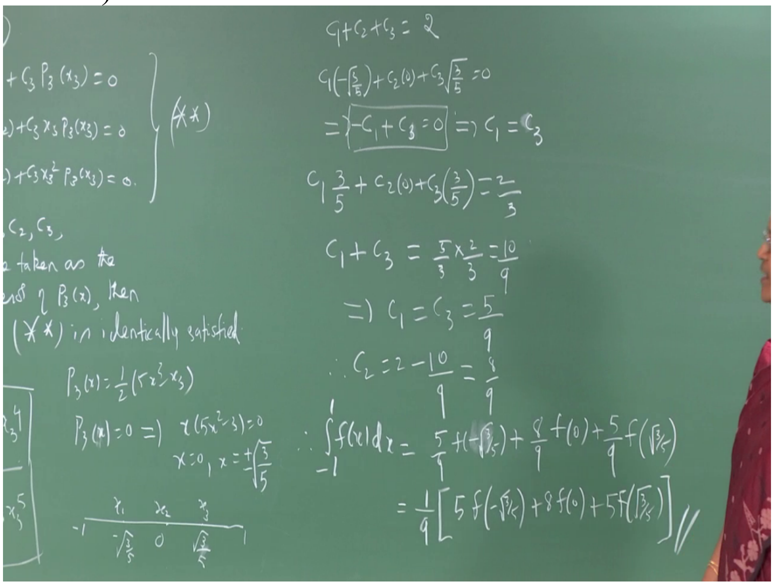

So let us write down the details when n is 3 so this tells us that integral minus 1 to 1 f(x) dx is equal to C 1 which is 5 by 9 into f(x 1). What is x 1? X 1 is minus root of 3 b y 5 C 2 is 8 by 9, what is x 2? 0 and C 3 is again 5 by 9 f(root of 3 by 5). So the method is 1 by 9[5 into f(minus root of 3 by 5 plus 8 into f (0) plus 5 into f(root of 3 by 5)] this is our three point Gaussian Quadrature method.

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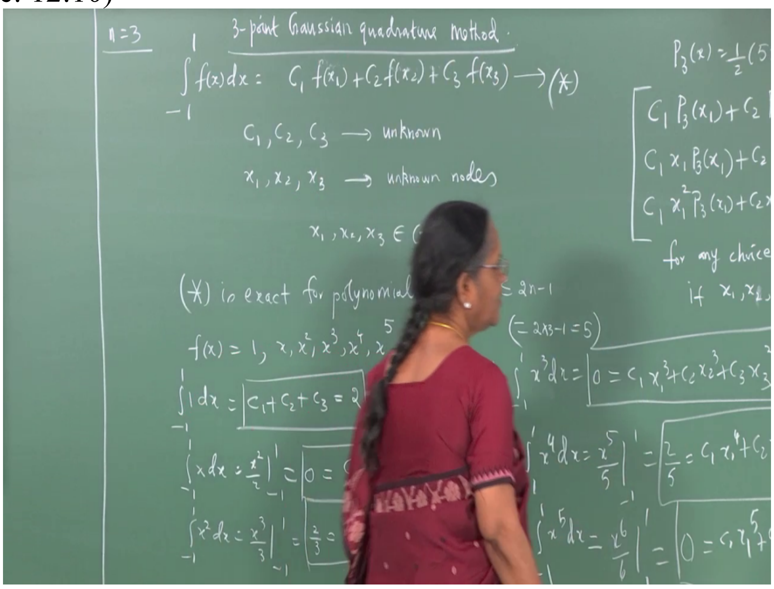

So we have derived the three point Gaussian Quadrature method.

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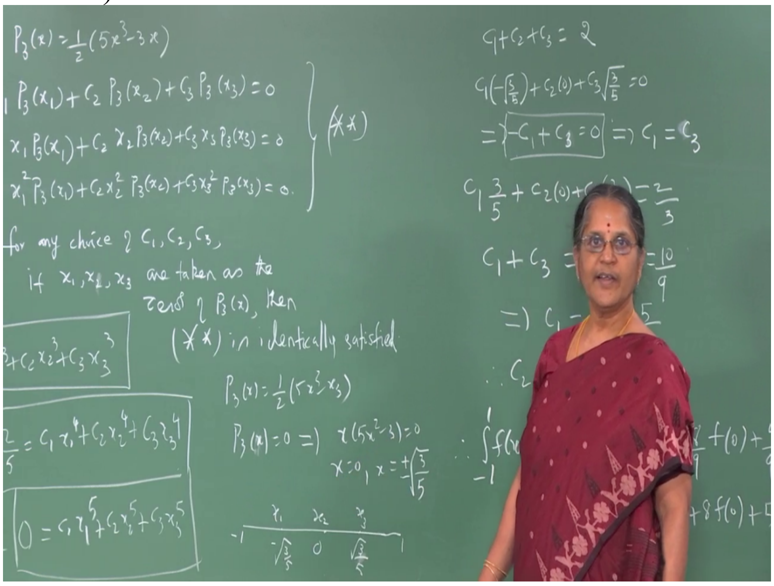

So we again observe that the nodes lie in the interval minus 1 to 1 so we have obtained an open type formula because the end points are not included and another observation is that the nodes along with minus 1 1 in this interval are such that these points are not equally spaced, however these three nodes are such that two of these nodes x 1 x 3 lies symmetrically on either side of this node namely which is located at the origin, so we have sacrificed the requirement that the nodes must be equally spaced but this has yield at the method which is such that the integration method is exact for polynomials of degree greater than n when we make use of the information of the function values at n nodes which lie within this interval that is the beauty of Gaussian Quadrature techniques. So one can go on and derive right Gaussian Quadrature methods for any value of n and you know where the nodes should lie you only have to solve for the constants C 1 C 2 C 3 etc C n depending upon what you choose your n as?

So let us see this by means of an example.

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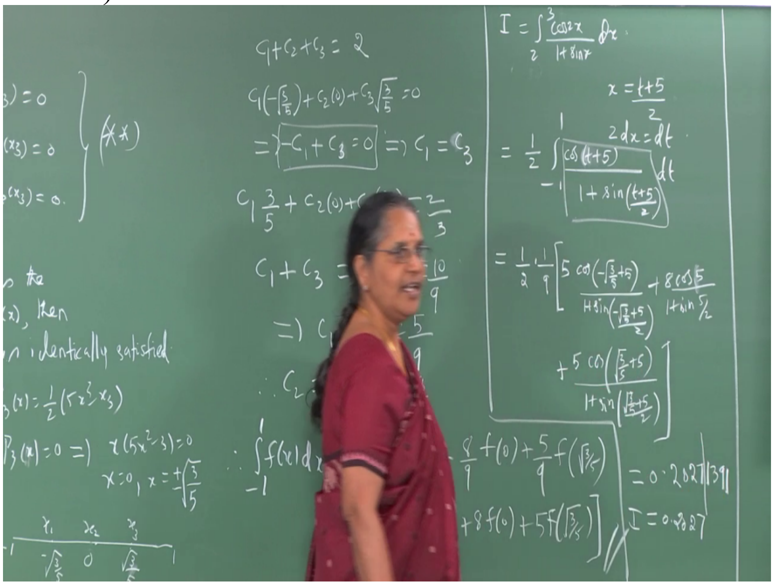

So find the value of the integrals say I which is integral 2 to 3 cos 2x by 1 plus sin x dx. So obtain this using Gauss three point integration method we have already done this using two point method. Let us work without using three point method. Since the inteval of integration is from 2 to 3 we must change this interval to minus 1 to 1. So we use the transformation that x is equal to t plus 5 divided by 2 so that 2 dx is equal to dt. So this will be equal to integral half of minus 1 to 1 cos of 2 into 2x is t plus 5 so I can substitute cos (tplus 5) divided by 1 plus sin (tplus 5) by 2 into dt.

So I have integration between minus 1 and 1 of some function right some integrant so I directly apply Gauss three point formula so I have half into if I apply Gauss method it is going to be 1 divided by 9 into 5 times the function value at minus root of 3 by 5 the function is this. So cos of minus root of 3 by 5 plus 5 divided by 1 plus sin of minus root of 3 by 5 plus 5 by 2 and then I have plus 8 times f(0) so cos 0 plus 5 divided by 1 plus sin 0 plus 5 by 2 and then the next term 5 cos (root of 3 by 5) plus 5 divided by 1 plus sin (root of 3 by 5) plus 5 by 2. So this gives you the value of the integral you can get these values and show that this turns out to be 0.20271391. So that is the value of the integral whenyou use a three point Gaussian method and suppose you are asked to get evaluate this integral correct to say 4 decimal accuracy then the solution will be i is equal to 0. 2027 you see what the next digit is its 1 less than 5 and therefore the solution correct to 4 decimal places is 0.2027.

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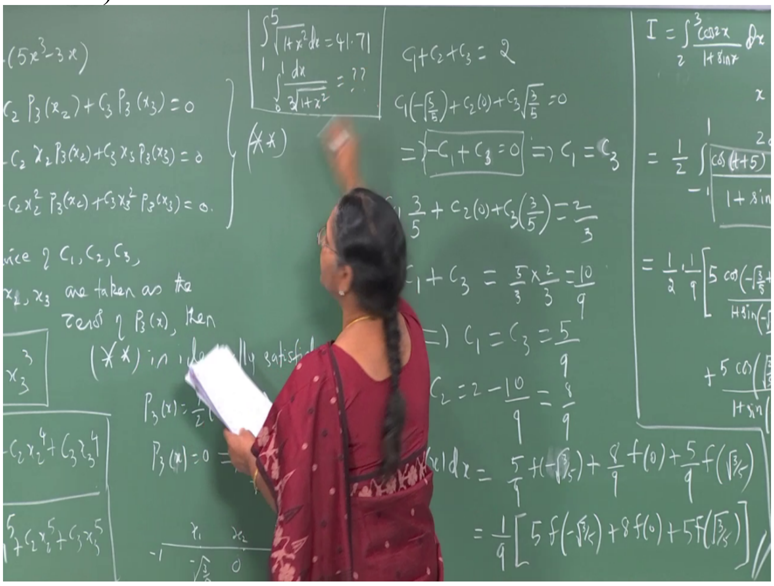

So we use this technique of three point Gaussian integration method and solve the following show that integral 1 to 5 root of 1 plus x square dx is equal to 41.71 evaluate integral 0 to 1 dx by cube root of 1 plus x square, evaluate using Gaussian three point formula and then you also take following as a homework and try it out derive five point Gaussian quadrature formula. That completes our discussion on Gaussian quadrature methods. So in all the methods that we derived so far we took the step size h to be a constant. There are situations where inner interval there is a sub interval where the function variations are very large whereas some other portion of that interval the function variation are very slow in such cases it is important that we take the step size in that interval where the function varies rapidly to be very small to account for a large variations in the function in that interval and in another part of the interval the step sixe need not be as small as we take here, because the variation of the function in that inteval is very very slow.

So it is important that an integration method or a quadrature rule that should be derived so that it automatically changes its step size and accordingly perform the integration using the rule that we want to use and then gives us the value of the integral correct to the desired degree of accuracy so we would like to now such an integration method when we do this the resulting method that we are going to give now or explain is capable of taking appropriate step size in some intervals where the integration is to be performed so that it accounts for the variations in the function whether it is a large variation or a small variation.

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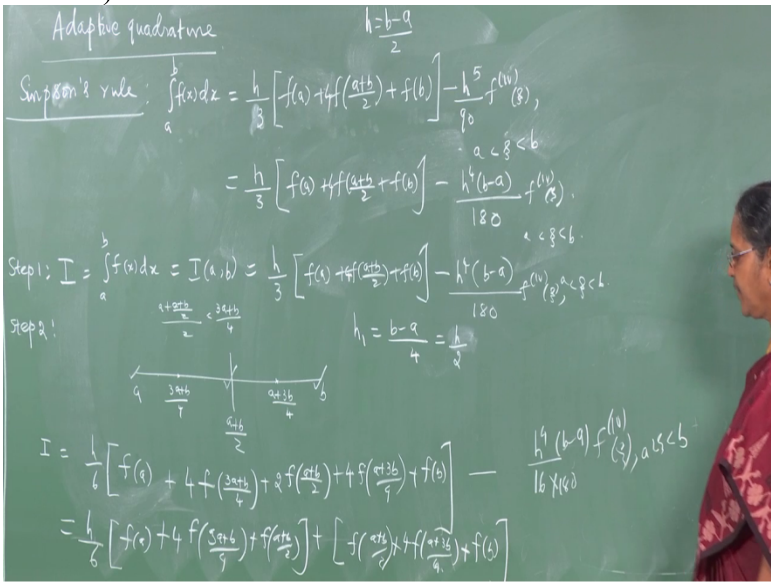

Such integration methods are called adaptive quadrature rules or methods. So let us now derive the adaptive quadrature technique which is based on Simpson's rule. So Simpson's rule tells us that integral a to b f(x) dx is h by 3 f(a) plus 4f(a plus b by 2) plus f(b)] and the error involved is of order of h power 5 where h is b minus a by 2. So when I use that then the error term can be written as h power 4 into (b minus a) by 180 into fourth derivative at Psi where Psi lies between a and b. So let us now derive simpson's rule based adaptive quadrature method.

So in step 1, I apply Simpson's rule with step size h and write down the rule. In step 2, I divide the interval a b into 4 equal sub intervals of width h 1 which is b minus a by 4, so it is half the step size of the earlier step. So the four points are going to be a a plus b by 2 and the midpoint of that interval then a plus b by 2b and the midpoint of the sub interval. So we apply Simpson's rule on opposite Simpson's rule in this interval a to b which is divided into 4 equal sub intervals. So you get h by 6 I would have written this as h 1 by 3 but h 1 is h by 2, so it is h by 6 into some of the end ordinates f(a) plus f(b) plus 4 times the intermediate ordinates which are odd suffix and two times the value of the function at the E 1 suffix ordinates.

And now this can be expressed in the form h by 6 into f (a) plus 4f(3a plus b by 4) plus f(a plus b by 2) and then plus f(a plus b by 2) plus 4 f(a plus 3b by 4) plus f(b)].

So when I have divided this into 4 equal parts what is the error that is incurred h is now h by 2 and so h power 4 will give you power 4 by 16 into 180 into b minus a into 4 th derivative evaluated at Psi where Psi lies between a and b. So now I write this into two parts and I see that the first term corresponds to Simpson's rule in the interval a to a plus b by 2 and the second term gives me Simpson's rule in a plus b by 2 to b and the error incurred is minus h power 4 into b minus a by 16 into 180 into the 4 th derivative at Psi for a less than Psi less than b.

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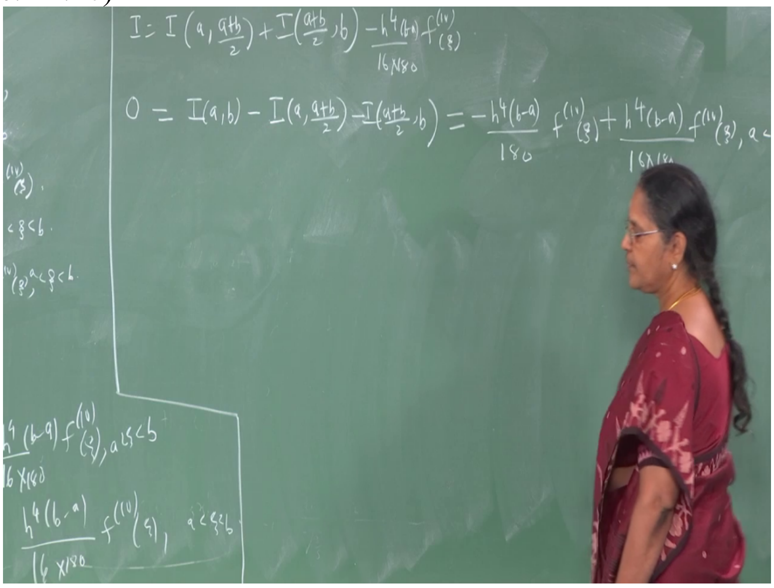

So I write down this as I is therefore I(what have I done I have applied Simpson's rule in a to a plus b by 2 , so I shall call this as a to (a plus b by 2) and the next term corresponds to (a plus b by 2), b so I have applied simpson's rule in that sub interval. So earlier I was written as I (a,b) where you had applied simpson's rule in that interval a, b by dividing it into 2 equal sub intervals. Now the same I is expressed as I (a, a plus b by 2) plus I (a plus b by 2,b) So now we observe that and the error incurred is minus h power 4 by 16 into 180 into b minus a into the 4 th derivative at Psi, so that I minus I will give me 0 and that is equal to I (a,b) minus I(a, a plus b by 2) minus I (a plus b by 2,b) and that will be equal to what is the error that is incurred, the error that is incurred is minus h power 4 into b minus a by 180 into the fourth derivative at Psi we can even take here Psi 1 and this as Psi 2 and replace the Psi in such a way that the maximum of the fourth derivative of f at Psi 1 and fourth derivative of f at Psi 2 is given by the fourth derivative of f at Psi, so I am replacing that here as Psi. And then what does the error from the next term give because I am subtracting that is plus h power 4 into b minus a by 16 into 180 into the fourth derivative at Psi for Psi lying between a and b.

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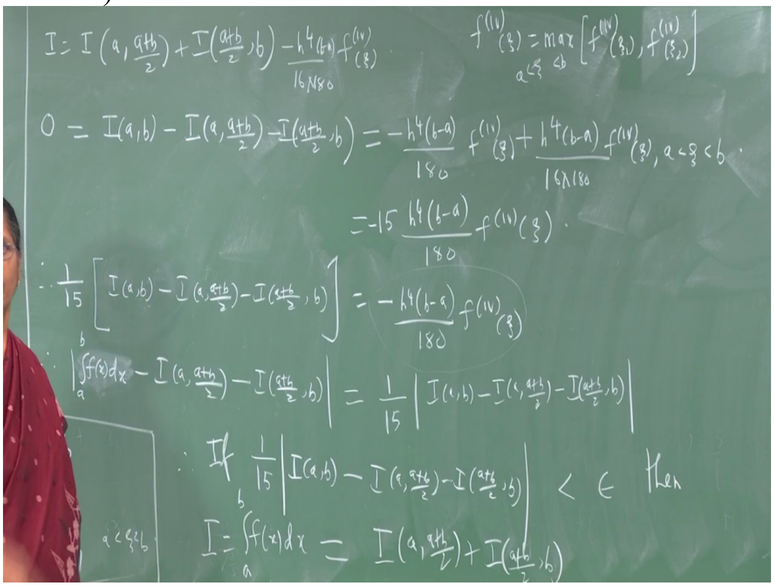

So If you want you can replace this by 4 th derivative at Psi 1 and this as 4 th derivative at Psi 2 and say that the 4 th derivative at Psi is maximum of the [fourth derivative of f(Psi 1) and fourth derivative of f(Psi 2)] for Psi lying in a to b. So we have replaced it by the fourth derivative of f(psi) so what does this give this gives you minus 16 plus 1 so minus 15 times h power 4 b minus a by 180 into the fourth derivative at Psi or in other words you get 1 by 15 of [I (a,b) minus I(a,a plus b by 2) minus I (a plus b by 2,b)] is minus h power 4 into b minus a by 180 into fourth derivative (Psi). What was this do you recall? this is essentially the error that you get here in evaluation of this integral i by applying simpson's rule and that says that error is 1by 15 th of the difference between applying Simpson's rule in an interval a, b and then applying Simpson's rule in the interval after dividing a,b into 4 equal parts that is what this step says.

And therefore we have absolute value of the difference between a,b and (a, aplus b by 2) minus I (a plus b by 2, b) this differnce in absolute value is going to be what or I can replace this I (a,b by the integral itself , integral a to b f(x) dx, so the difference in the exact integral minus the value that you get by applying Simpson's rule is such that it is going to be 1 by 15 [I (a, b) minus I (a, aplus b by 2) minus I (a plus b by 2, b) so this is what you get. So therefore you will now specify the test if this quantity 1 by 15 into absolute value of I (a,b) minus (a, aplus b by 2) minus I (a plus b by 2, b)] if this quantity on the right hand side in absolute value comes out to be say less than Epsilon namely the accuracy that you prescribe then what does it mean then itm means you can take your value of integral I, integral a to b f(x) dx to be equal to I [(a, aplus b by 2) plusI (a plus b by 2, b). Namely by applying the Simpson's rule to each of the half intervals into which you have divided the interval (a,b) into and your solution or your result will be correct to the desired degree of accuracy because you have made sure that the error that is incurred is less than epsilon.

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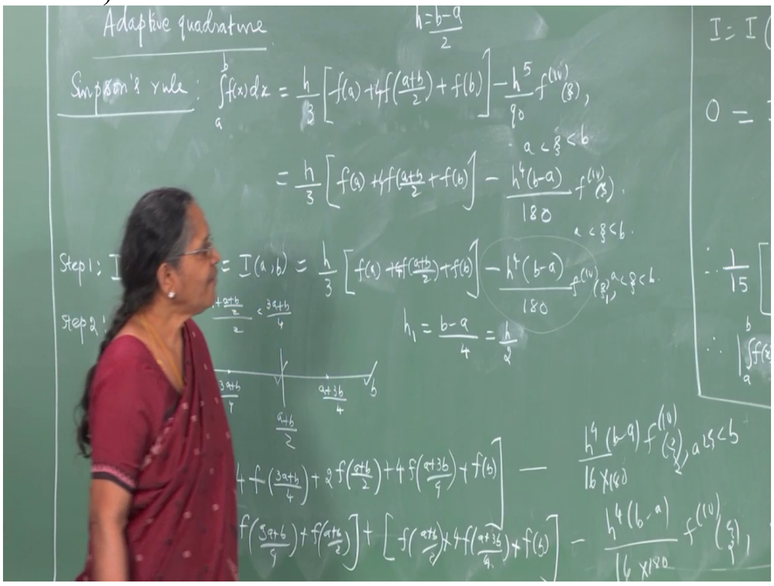

So all this should be done in step 2 so I repeat step 2 again. Step 2 is equivalent to doing the following divide the interval a, b in such a way that it is split into 4 equal intervals and composite Simpson's rule is applied in that interval. The result tells you that it is equivalent to applying Simpson's rule in each of these half intervals a to a plus b by 2 and a plus b by 2 , b.

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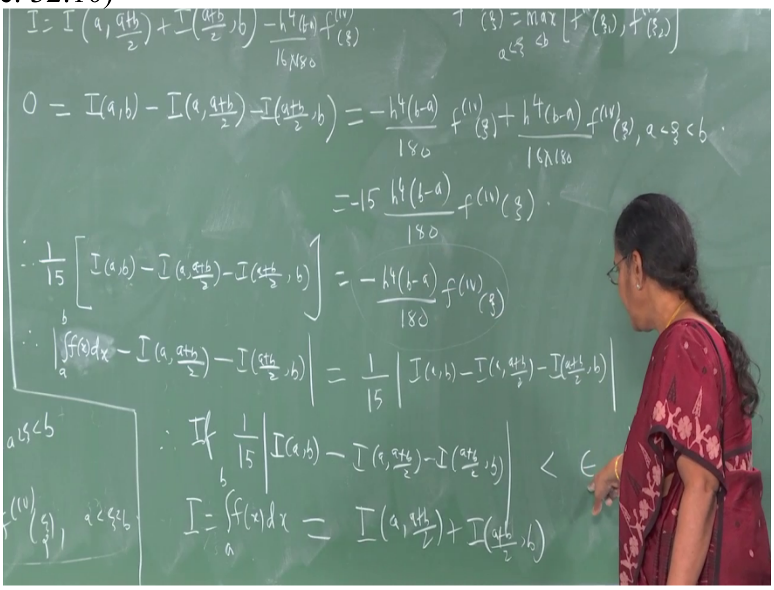

And then you write down the expression for the error and compute what is the difference between I (a, b) and this then 1 by 15 of this gives you an error which is exactly the expression that you have over here. And therefore if you can make that error small according to what you require namely correct to the desired degree of accuracy namely if 1 by 15 of this turns out to be less than epsilon at the end of your computations then you are through namely you have obtained the value of the integral correct to a desired degree of accuracy namely this. So the question arises what if this does not happen? Suppose `I call this as star.

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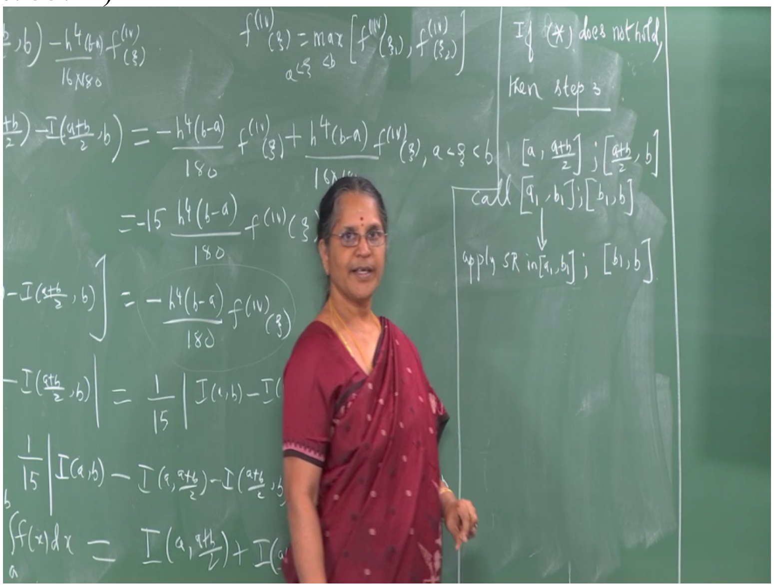

So if star does not hold so you observe that in absolute value 1 by 15 of this is not less than Epsilon then what do you do? You perform step 3 what is your step 3? Step 3 is take the interval [a to a plus b by 2] and the interval [a plus b by 2, b] call this as say interval [a 1, b 1] and [b 1 b] you just name this and now you apply Simpson's rule in this interval [a 1, b 1], alright? And also apply Simpson's rule in [b 1, b] so now you have to do two things at this step. Your interval now is [a 1, b 1] to start with. So there are two parts to it, one is you should apply Simpson's rule to interval [a 1, b 1] and the second part is apply Simpson's rule to [b 1, b]. Let us first do the first part, so apply Simpson's rule to [a 1, b 1] then what are you doing?

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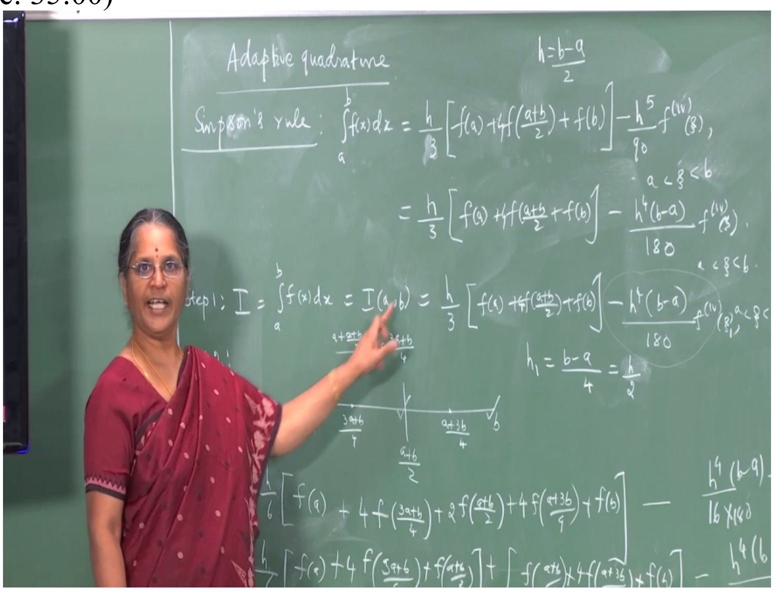

You are essentially doing something like step 1 in an interval (a, b) alright?

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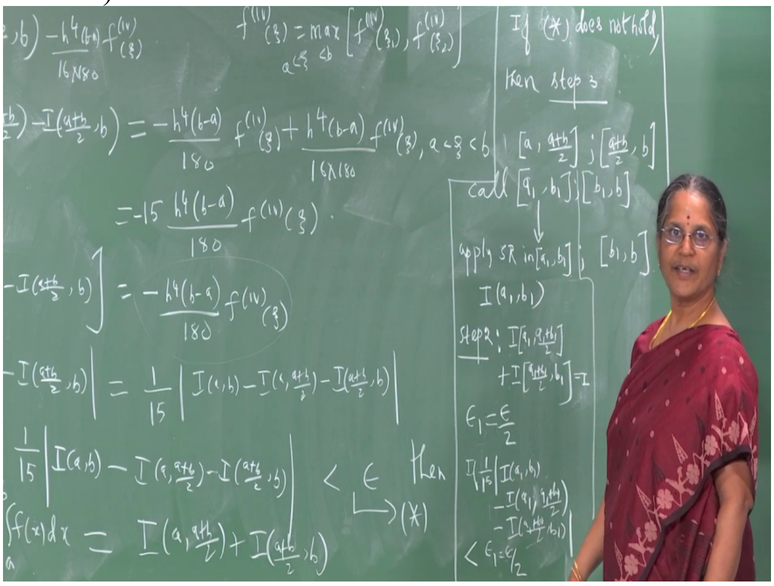

And then if you do that then you will have I (a 1, b 1) and then what should you do? You should perform step 2 in the interval (a 1, b 1) right? What does that mean? So perform step 2 to the interval (a 1, b 1) what will you get? You will get I(a 1, a 1 plus b 1 by 2) plus I (a 1 plus b 1 by 2, b 1) that will be your integral I then you perform. So essentially the first part of step 3 is repetition of step 1 and step 2 but now the interval integration is [a 1, b 1].

Now check whether the condition accuracy conditions star is satisfied where in [a 1, b 1] that now take your epsilon as epsilon by 2 so take your epsilon say as epsilon 1 which is epsilon by 2.

And check if 1 by 15 of I [a 1, b 1] minus I (a 1, a 1 plus b 1 by 2) minus I (a 1 plus b 1 by 2, b 1) in absolute value is less than what epsilon 1 that is epsilon by 2. Check whether this happens now you know if that happens right? If the error is less than epsilon by 2 then you can take integral I [a 1, b 1] f(x) dx to be equal to this I (a 1, a 1 plus b 1 by 2) plus I (a 1 plus b 1 by 2, b 1) alright? If that does not happen then what should yo do ? You should take (a 1 plus b 1 by 2) and call it as some a 2 , b 2 and perform steps 1 and 2 sorry a 2 b 2 and do the test given by star with what accuracy such that the error is less than some epsilon 2 where epsilon 2 is epsilon 1 by 2 namely epsilon by 4 and continue this process this is for the first part [a 1, b 1].

Similarly you should do everything for the second part [b 1, b]. If suppose the condition is not satisfied even in one of the intervals right? You should continue to apply this procedure of repeating step 1 or step 2 to that particular interval again and again so continue this procedure so you may ask will it converge due you think that the error will reduce yes we observe that the error that you commit every time is 1 by 15 of the error that you have here. So every time when you perform these steps the error gets reduced by 1 by 15 and therefore at some stage after finite number of steps your error will be less than the prescribed epsilon with which h you have started and you have an adaptive quadrature technique that takes care of the step size appropriately and that is what is adaptive quadrature method.