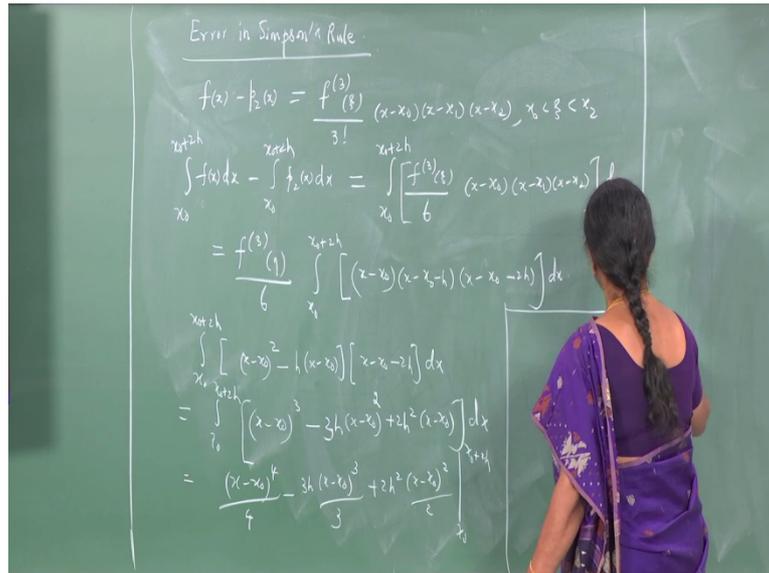


**Numerical Analysis**  
**Prof R Usha**  
**Department of Mathematics**  
**Indian Institute of Technology Madras**  
**Lecture 14**  
**Numerical Integration 3**  
**Error in Simson's Rule**  
**Composite Trapezoidal Rule Error**

(Refer Slide Time: 00:15)



We now compute error in Simpson's rule, So we have expression for error in interpolation in the interpolatory function  $f(x)$  by a quadratic polynomial so it will be a third derivative by factorial 3 into  $(x - x_0)(x - x_1)(x - x_2)$  where  $\xi$  lies in the interval  $x_0$  to  $x_2$ . So we would like to get the error in evaluating the definite integral by approximating  $f(x)$  by  $p_2(x)$ , so that will be  $\int_{x_0}^{x_0+2h} f(x) dx - \int_{x_0}^{x_0+2h} p_2(x) dx$  and we want this expression which is error in Simpson's rule and that will be  $\int_{x_0}^{x_0+2h} \left[ \frac{f^{(3)}(\xi)}{3!} (x - x_0)(x - x_1)(x - x_2) \right] dx$  integrated with respect to  $x$ .

So again by using mean value theorem for integrals we have the third derivative some  $\eta$  by 6 into  $\int_{x_0}^{x_0+2h} [(x - x_0)^2 - 1(x - x_0)] [x - x_0 - 2h] dx$  I use the fact that the points are equally spaced so it is  $[(x - x_0)(x - x_0 - h)(x - x_0 - 2h)]$  and this must be integrated with respect to  $x$ .

So let us consider  $\int_{x_0}^{x_0+2h}$  then this will give you  $[(x - x_0)^2 - 1(x - x_0)] [x - x_0 - 2h]$  and that must be multiplied by  $\left[ \frac{f^{(3)}(\eta)}{6} \right]$  and

integrated with respect to x. So x 0 to x 0 plus 2h [(x minus x 0) the whole cube minus 2h into (x minus x 0) so (x minus x 0) the whole cube minus 2h into (x minus x 0) the whole square then the next term minus h into (x minus x 0) the whole square so this becomes 3h. Then plus 2h square into (x minus x 0). So this must be integrated with respect to x. This will give me (x minus x 0) power 4 by 4 minus 3h into (x minus x 0)the whole cube by 3 plus 2h square into (x minus x 0) the whole square by 2] between x 0 and x 0 plus 2h.

(Refer Slide Time: 03:38)

Error in Simpson's Rule.

$$f(x) - f_2(x) = \frac{f^{(3)}(\xi)}{3!} (x-x_0)(x-x_1)(x-x_2), \quad x_0 < \xi < x_2$$

$$\int_{x_0}^{x_0+2h} f(x) dx - \int_{x_0}^{x_0+2h} f_2(x) dx = \int_{x_0}^{x_0+2h} \left[ \frac{f^{(3)}(\xi)}{6} (x-x_0)(x-x_1)(x-x_2) \right] dx$$

$$= \frac{f^{(3)}(\xi)}{6} \int_{x_0}^{x_0+2h} [(x-x_0)(x-x_0-h)(x-x_0-2h)] dx$$

$$\int_{x_0}^{x_0+2h} [(x-x_0)^2 - h(x-x_0)] [x-x_0-2h] dx$$

$$= \int_{x_0}^{x_0+2h} [(x-x_0)^3 - 3h(x-x_0)^2 + 2h^2(x-x_0)] dx$$

$$= \left[ \frac{(x-x_0)^4}{4} - \frac{3h}{3} (x-x_0)^3 + \frac{2h^2}{2} (x-x_0)^2 \right]_{x_0}^{x_0+2h}$$

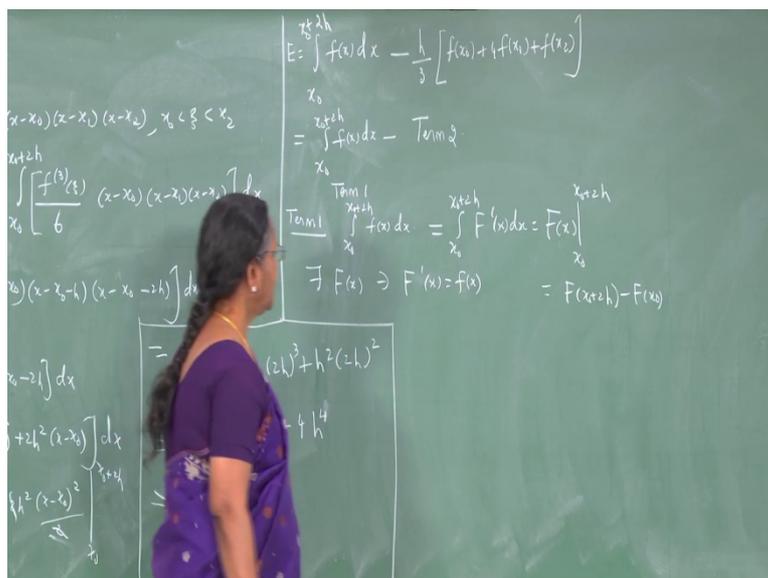
$$= \frac{(2h)^4}{4} - h(2h)^3 + h^2(2h)^2$$

$$= 4h^4 - 8h^4 + 4h^4$$

$$= 0$$

So we shall apply the limits that will give you at the upper limit x 0 plus 2h this will give you 2h power 4 by 4 minus h into 2h the whole cube plus h square into 2h the whole square that is at the upper limit at the lower limit all this will vanish. So giving 16 h power 4 by 4 minus 8 h power 4 plus 4 h power 4 turns out to be 0. So this tells us that this method is exact for polynomials of degree upto 2.

(Refer Slide Time: 04:43)

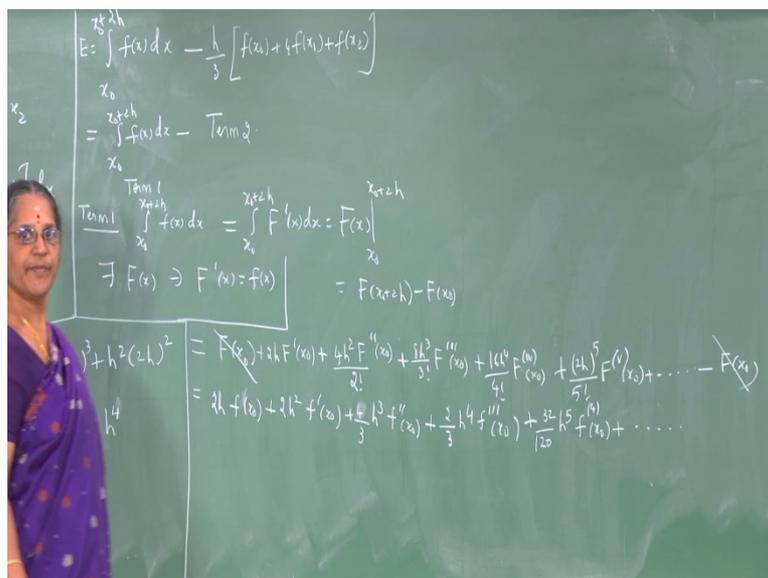


So let us evaluate the error as follows:

Namely we wanted to see what is the difference between integral  $x_0$  to  $x_0 + 2h$   $f(x) dx$  minus the formula that we derived mainly  $h$  by  $3$  into  $f(x_0) + 4$  times  $f(x_1)$  plus  $f(x_2)$ . We approximated this integral by this method. So the error is this minus the method that evaluates this integral approximately. So it is  $x_0$  to  $x_0 + 2h$   $f(x) dx$  minus I shall call this as term 2 and this as term 1 and evaluate one by one and then finally see what the difference is.

So let us first do term 1 it is integral  $x_0$  to  $x_0 + 2h$   $f(x) dx$ . Now this integral exists  $f(x)$  is a continuous function integral exists and therefore there exists a function capital  $F(x)$  such that  $f'(x)$  is equal to small  $f(x)$  by fundamental theorem of integral calculus. And therefore this will be integral  $x_0$  to  $x_0 + 2h$  capital  $F'(x) dx$  and that is  $F(x)$  between  $x_0$  to  $x_0 + 2h$  and that is  $f(x_0 + 2h)$  minus  $f(x_0)$ .

(Refer Slide Time: 06:46)

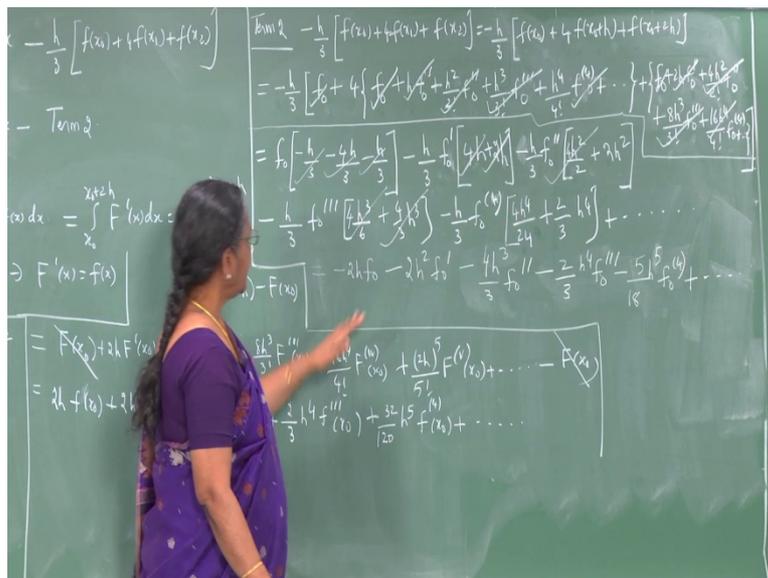


So we shall use the Taylor's theorem and write this as  $f(x_0 + 2h)$  into  $f'(x_0 + 2h)$  the whole square into  $f''(x_0)$  by factorial 2 plus  $(2h)$  the whole cube by factorial 3 into  $f'''(x_0)$  plus  $2h$  the whole power 4 by factorial 4 into the fourth derivative  $(x_0)$  plus  $2h$  the whole power 5 by factorial 5 into the fifth derivative  $(x_0)$  and so on.

So you will see why I have stopped with this later on when we also complete the evaluation of the second term. This minus  $f(x_0)$  which appears here. So  $f(x_0)$  can be cancelled and this gives us  $2h$  into, I know that  $f'(x)$  is  $f(x)$  so  $f'(x_0)$  is small  $f(x_0)$  plus  $4h^2$  by factorial 2. So  $2h^2$  square at  $f''(x_0)$  is small  $f''(x_0)$  then  $8$  by  $6$  so  $4$  by  $3$   $h^3$ ,  $f'''(x_0)$  will be  $f''(x_0)$  plus  $16$  by  $24$  so  $2$  by  $3$   $h^4$ , the fourth derivative of capital  $F$  so  $f^{(4)}(x_0)$  then  $32$  by  $120$  into  $h^5$  into fifth derivative  $(x_0)$ .

So this will be the fourth derivative  $(x_0)$  plus etc. So we have been able to express integral  $x_0$  to  $x_0 + 2h$   $f(x) dx$  in terms of the function values and its derivatives using fundamental theorem of integral calculus and Taylor series. So let us keep term 1 as this.

(Refer Slide Time: 09:16)



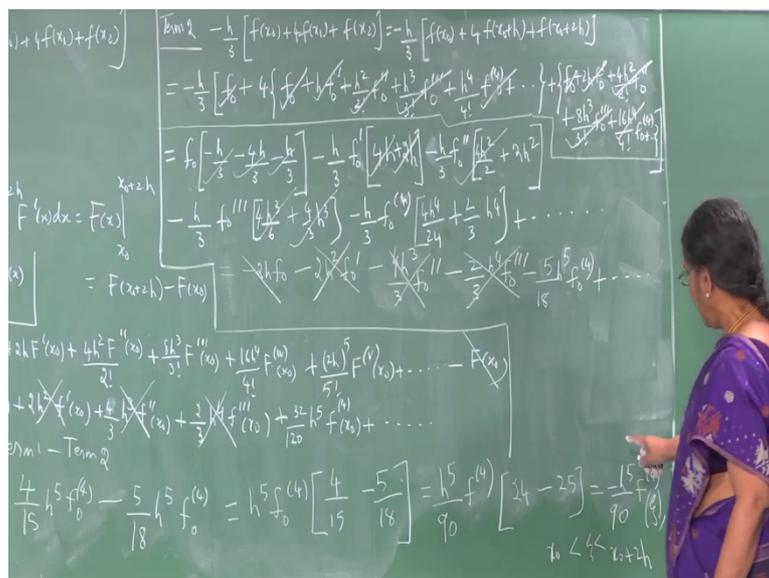
Now we move on to term 2 and then evaluate what it is. So term 2 is minus h by 3 f(x 0) plus 4 times f(x 1) plus f(x 2) so which is minus h by 3 into [ f(x 0) plus 4 times f(x 1) is (x 0 plus h) plus f(x 2) is (x 0 plus 2h)].

So I shall use the following notation namely I shall write f(x 0) as f 0 and then I expand 4 times [ f(x 0 plus h) is f(x 0) plus h into f dash (x 0) plus h square by factorial 2 into f (x 0 plus h ) into f dash (x 0 plus h square by factorial 2 plus f double dash (x 0) plus h cube by factorial 3 plus f triple dash (x 0) plus h power 4 by factorial 4 into fourth derivative (x 0) plus etc ] that is the first term.

The next term is f(x 0 plus 2h into f dash (x 0) plus 2h the whole square by factorial 2 f double dash (x 0) plus 2h the whole cube by factorial 3 into f triple dash (x 0) plus 2h the whole power 4 by factorial 4 into fourth derivative (x 0) plus etc].

I have now collected the terms whose coefficients are x 0 x 0 dash and so on and have simplified and obtained the result at this step and that is term 2. So now let us substitute in this expression for the error.

(Refer Slide Time: 11:41)



So the error is term 1 minus term 2, so we now look at the corresponding coefficients, I have  $f(x_0)$  and have denoted it by  $f$  suffix 0, so the coefficient of  $f(x_0)$  term 1 is  $2h$  and in term 2 it is minus  $2h$ . Then the coefficient of  $f''(x_0)$  in term 1 is  $2h^2$  that in term 2 is minus  $2h^2$   $f''(x_0)$  double dash is  $4 \times 3 h^3$  minus  $4 \times 3 h^3$ .

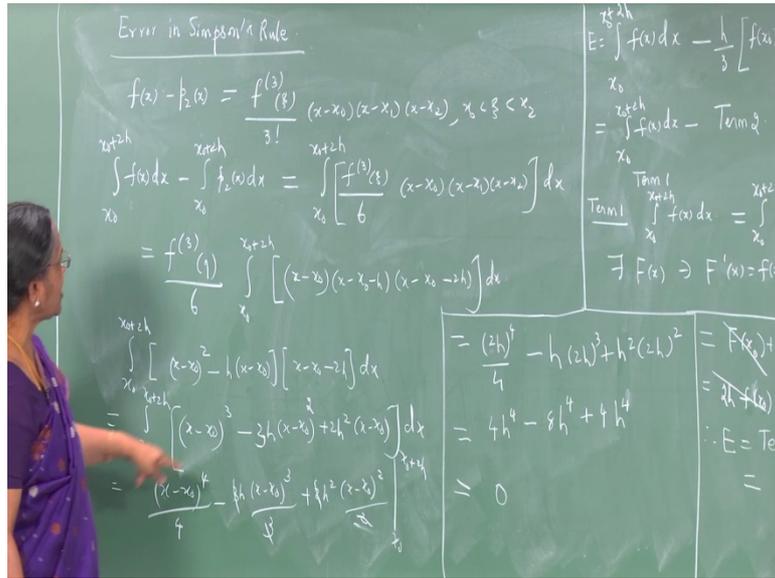
And then the third derivative at  $x_0$  has coefficient  $2 \times 3 \times h^4$  here and in term 2 it is minus  $2 \times 3 \times h^4$ . So now I write down this term which is  $32 \times 120$  and that is  $4 \times 15 h^5$  into the fourth derivative at 0 that comes from term 1. And term 2 gives me minus  $5 \times 18$  into  $h^5$  into the fourth derivative at 0.

So we observe that the first non zero term in evaluating the error in Simpson's rule comes with the derivative of  $f$  namely the first derivative of  $f$  evaluated at  $x_0$ . So this is the first non zero term in the expression for the error. So this is the leading term in error. This gives you  $h^5$  into fourth derivative multiplied by  $[4 \times 15$  minus  $5 \times 18]$ .

So that will give you  $h^5$  so by  $90$  into fourth derivative, so  $24$  minus  $25$  and that gives you minus  $h^5$  by  $90$  into fourth derivative since it is something like the remainder term in Taylor expansion. So I write down the fourth derivative at some  $\psi$  where  $\psi$  lies between  $x_0$  to  $x_0 + 2h$ . So we see that the error in Simpson's rule is of order of  $h$  to the power of 5 and the expression for the error is  $h^5$  by  $90$  into the fourth derivative evaluated at some  $\psi$  between  $x_0$  and  $x_0 + 2h$  and what does this tell?

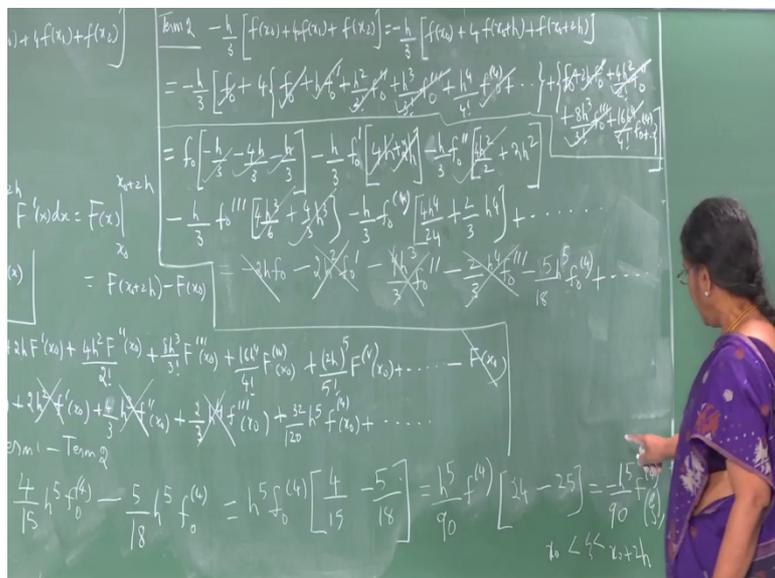


(Refer Slide Time: 15:30)



This tells you that the method is exact upto polynomials of degree 3 when we computed the error using the expression for error in interpolation. We showed that the term which appeared with the third derivative turned out to be 0 and that is what happened here also.

(Refer Slide Time: 15:50)

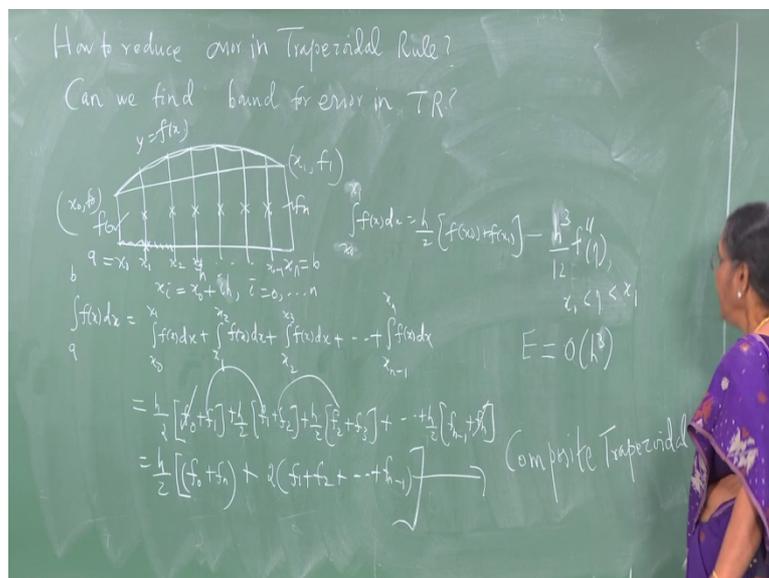


The next term which appears with the fourth derivative survives and it contributes to the error in Simpson's rule. We infer that the error in interpolation is such that the method is exact for polynomials of degree upto 3 and the error is of order of h to the power of 5 and the



So the question is how to reduce error in Trapezoidal rule? Can we find bound for error in Trapezoidal rule? These are the questions to be answered now. So what did we do in Trapezoidal rule we evaluated the integral of the function  $f(x)$  over this interval  $x_0$  to  $x_1$  by approximating the function by means of a straight line passing through the points  $x_0, f_0$  and  $x_1, f_1$  and the  $h$  was the step size which is  $x_1$  minus  $x_0$  we evaluated the Trapezoidal rule as  $\int_{x_0}^{x_1} f(x) dx$  is  $h$  by  $2 \cdot (f(x_0) + f(x_1))$  and the error was of order of  $h^3$ . or if we can make this  $h$  smaller and smaller then the error can be reduced and we would get better accuracy in evaluating this integral.

(Refer Slide Time: 19:12)



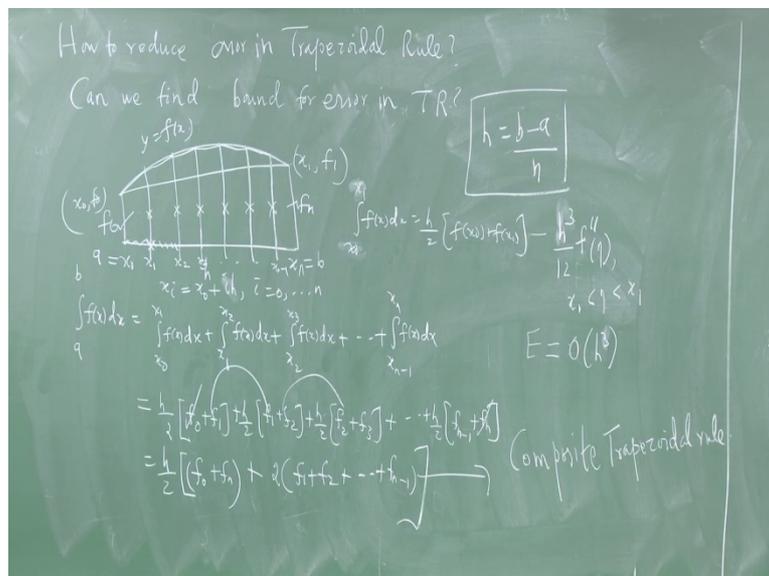
So how can we make the step size  $h$  smaller and smaller? You simply divide the interval  $x_0$  to  $x_1$  into say a number of smallest intervals say of equal width. And then apply Trapezoidal rule in each of these intervals. So what does that mean in this interval I approximate this function by the straight line. In this interval I approximate the function by means of this straight line and so on.

So that in each of these intervals you would be approximating the curve by means of a straight line passing through those two end points and apply the Trapezoidal rule. So I shall use the notation that I am interested in evaluating integral  $a$  to  $b$   $f(x) dx$  and I do it by dividing the interval  $a$  to  $b$  by means of equally spaced points  $x_i$  which are given by  $x_0$  plus  $ih$  for  $i$  is equal to  $0, 1, 2, 3$  upto  $n$ .

So that my points are  $x_0, x_1, x_2, x_3$  and so on  $x_{n-1}$  and  $b$  will be  $x_n$ . And apply Trapezoidal rule in each of these sub intervals. So the first intergral is  $x_0$  to  $x_1$   $f(x) dx$  plus integral  $x_1$  to  $x_2$   $f(x) dx$  plus integral  $x_2$  to  $x_3$   $f(x) dx$  and so on the last integral will be  $x_{n-1}$  to  $x_n$   $f(x) dx$ . So when I apply trapezoidal rule it will be  $h$  by  $2$  into  $[f(x_0) + f(x_1)]$  plus  $h$  by  $2$   $[f(x_1) + f(x_2)]$   $h$  by  $2$   $[f(x_2) + f(x_3)]$  and so on plus  $h$  by  $2$  into  $[f(x_{n-1}) + f(x_n)]$ . Because I have equally spaced points the distance between any two points namely the upper and the lower limit will be  $h$  so I have used Trapezoidal rule in each of these intervals.

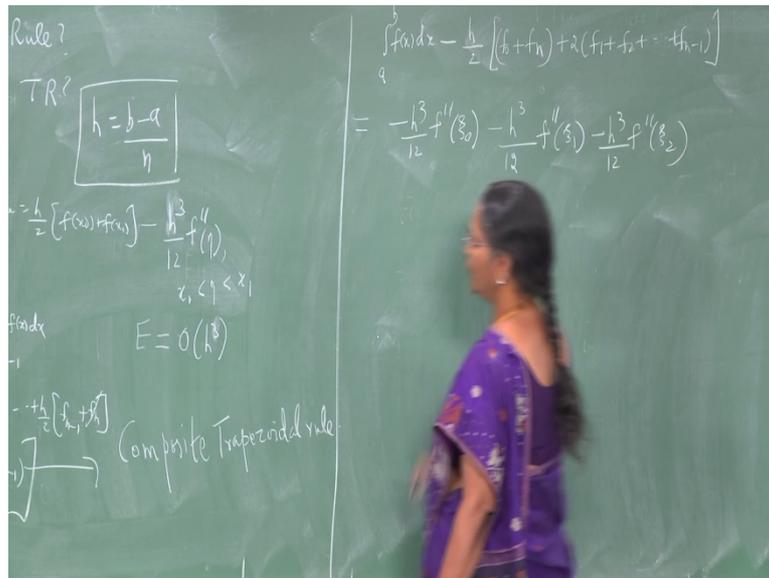
So I observe that it is  $h$  by  $2$  into I see that  $f_0$  appears once  $f_n$  also appears once but  $f_1, f_2$  etc they all appear twice. So twice  $(f_1 + f_2 + \dots + f_{n-1})$ . So what do I have now integral  $a$  to  $b$   $f(x) dx$  is the step size by  $2$  multiplied by the sum of the end ordinates  $f_0$  plus  $f_n$  plus the intermediate ordinates I have taken twice. So the formula gives me integral  $a$  to  $b$   $f(x) dx$  is  $h$  by  $2$  times sum of the end ordinates plus twice the intermediate ordinates. The resulting formula is what is known as composite Trapezoidal rule.

(Refer Slide Time: 23:12)



So it is obtained by dividing the interval of integration say  $N$  equal sub intervals of width  $h$  which is  $b$  minus  $a$  by  $n$  and the formula gives you  $h$  by  $2$  times the sum of the  $n$  ordinates plus twice the intermediate ordinate. Again the question is what is the error that is incurred in composite trapezoidal rule?

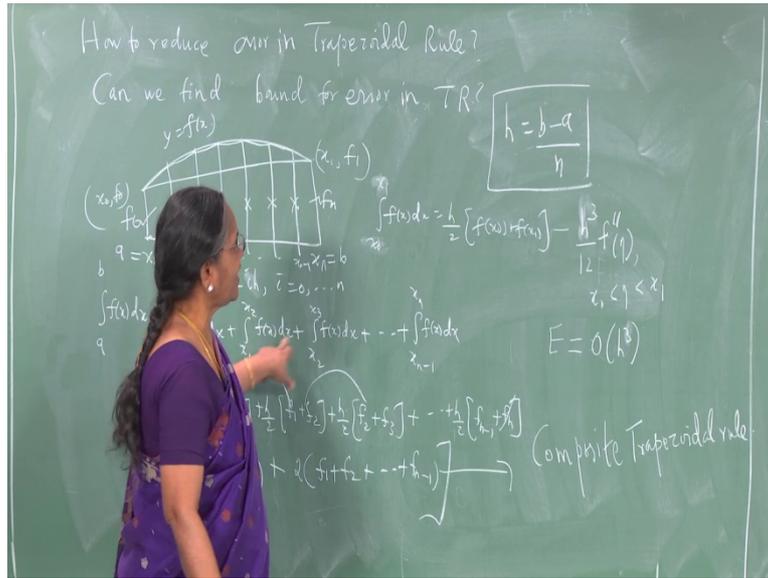
(Refer Slide Time: 23:47)



So that can be obtained by computing the error that is incurred in evaluating each of these integrals in the  $n$  sub intervals. And we already have determined the error in trapezoidal rule when it is applied to an interval so we shall make use of that.

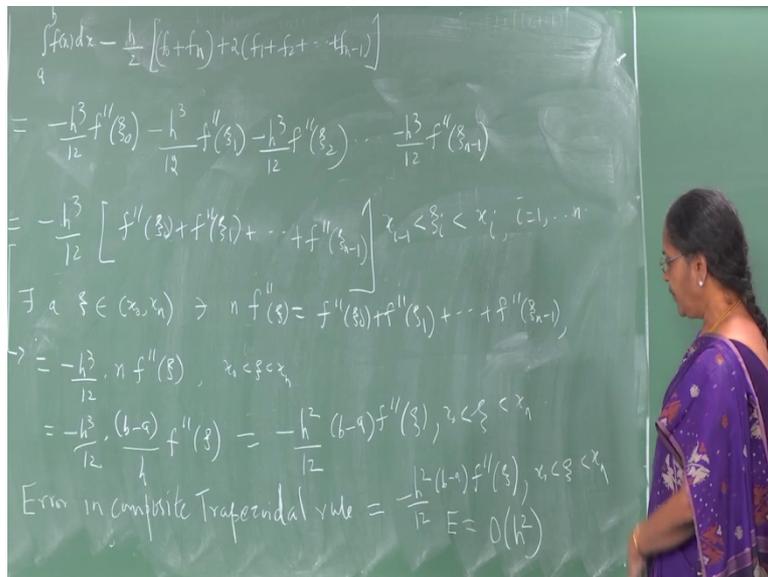
So  $\int_a^b f(x) dx$  minus the Trapezoidal rule  $\frac{h}{2} [f_0 + f_n + 2(f_1 + f_2 + \dots + f_{n-1})]$  will be equal to the error that is incurred in evaluating the first integral between  $x_0$  and  $x_1$ . What is it? We have this expression  $-\frac{h^3}{12} f''(\xi_0)$  where  $\xi_0$  lies between  $x_0$  and  $x_1$ . Then the next interval is  $x_1$  to  $x_2$  so the error incurred is  $-\frac{h^3}{12} f''(\xi_1)$  minus  $\frac{h^3}{12} f''(\xi_2)$ .

(Refer Slide Time: 25:10)



So between  $x_0$  and  $x_1$  first term gives the error  $x_1$  to  $x_2$  the second term gives the error  $x_2$  to  $x_3$  the third term gives you the error.

(Refer Slide Time: 25:25)



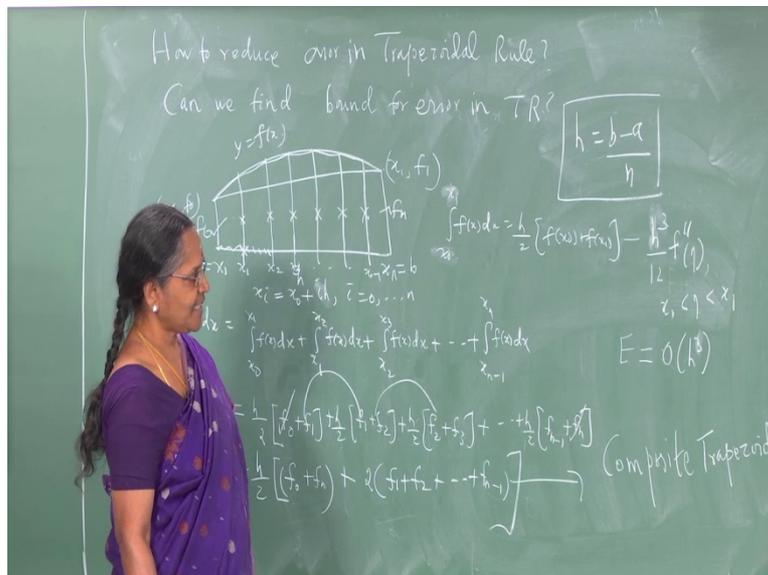
So  $x_{n-1}$  to  $x_n$  will give you minus  $h^3$  by 12 into the second derivative at some  $\xi$  ( $\xi_{n-1}$ ). So Where this  $\xi_i$  lies between  $x_{i-1}$  and  $x_i$  for  $i$  is equal to 1, 2, 3 upto  $n$ . So this is the total error that is incurred in any composite Trapezoidal rule. So that is minus  $h^3$  by 12 into double dash ( $\xi_0$ ) plus  $f$  double dash ( $\xi_1$ ) plus etc plus  $f$  double dash ( $\xi_{n-1}$ ).

So there exists a  $\Psi$  in the interval  $x_0$  to  $x_n$  such that  $n$  times  $f''(\Psi)$  will be equal to  $f''(\Psi_0)$  plus  $f''(\Psi_1)$  plus etc plus  $f''(\Psi_{n-1})$  where  $\Psi$  is in the interval  $x_0$  to  $x_n$ .

Namely I can find the  $\Psi$  in this interval such that the average of this namely the sum by  $n$  will be the second derivative evaluated at  $\Psi$ . So I use that fact and write down this to be equal to minus  $h^3$  by 12 into  $n$  times  $f''(\Psi)$  for  $\Psi$  lying between  $x_0$  and  $x_n$ . So this will be minus  $h^3$  by 12 into I know what is  $n$  how did I obtain this formula by dividing the interval  $a, b$  into  $n$  equal sub intervals of width  $h$  is equal to  $b - a$  by  $n$ .

So I substitute for  $n$  as  $b - a$  divided by  $h$  into the second derivative ( $\Psi$ ) and that gives me minus  $h^2$  by 12 into  $b - a$  into the second derivative evaluated at  $\Psi$  where  $\Psi$  lies between  $x_0$  and  $x_n$ . So the error in composite Trapezoidal rule is minus  $h^2$  by 12 into  $b - a$  into the second derivative ( $\Psi$ ) for  $\Psi$  lying in this interval  $x_0$  to  $x_n$ . So we observe that this error is of order of  $h^2$  while the error in Trapezoidal rule is of order of  $h^3$  the error in composite Trapezoidal rule is of order of  $h^2$ .

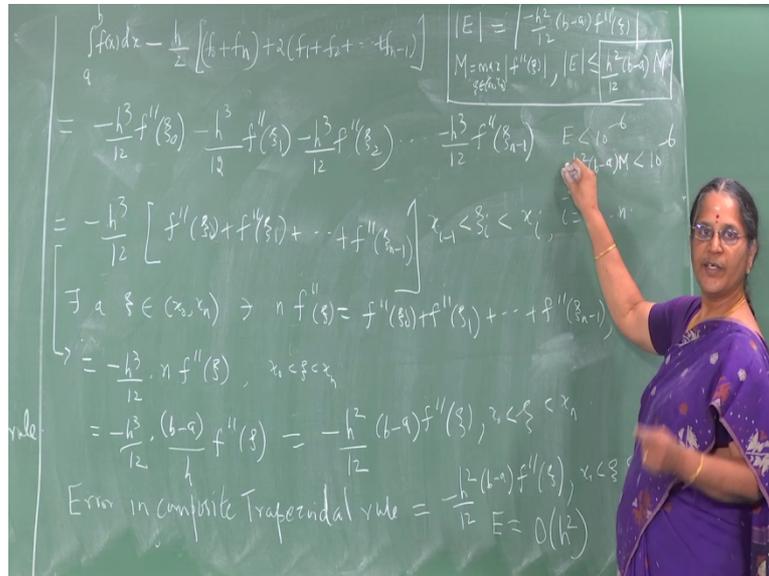
(Refer Slide Time: 28:51)



So we have been able to see that the step size  $h$  has been reduced and we have now obtained a formula which is composite Trapezoidal rule so that the given interval of integration is divided into number of smaller sub intervals of width  $h$ . And resulting integration method has been got and we have obtained an expression for error in composite Trapezoidal rule and we

know that the error is of order of h square. The question now is can we find the bound for the error in this composite Trapezoidal rule.

(Refer Slide Time: 29:36)

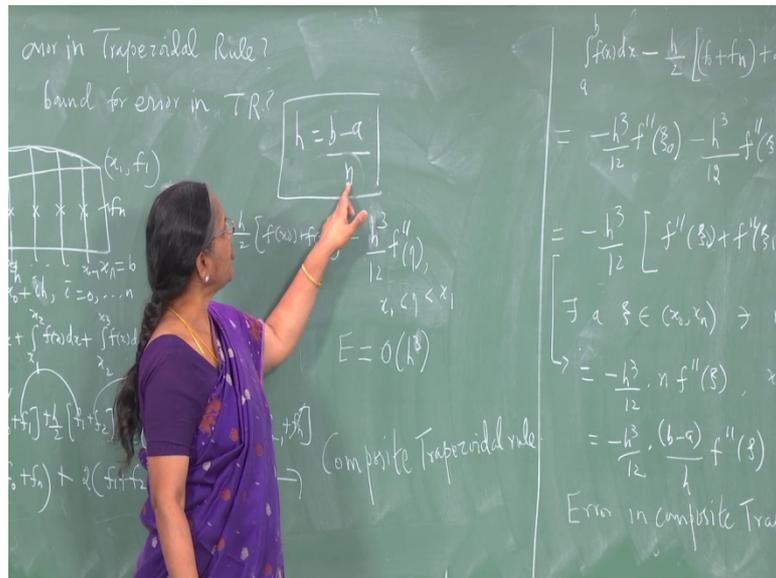


So what is the bound on the error it is modulus of e which is this. So let us obtain the error bound. So this will be absolute value of minus h square by 12 into b minus a into f double dash (psi). So suppose capital M is the maximum of f double dash (psi) for psi in the interval x 0 to x n. Then I have e to be less than or equal to h square by 12 into b minus a in the capital M.

So I can get the bound in such a way that the error can be made as small as this. This gives me an information as to how many sub intervals into which I should divide the interval a, b into in order that my error should be less than some prescribed value. Namely if say I want my error to be less than 10 to the minus 6 then I must have my h to satisfy the condition that h square by 12 into b minus a into M to be less than 10 to the minus 6.

Given the integral a and b are known so I know b minus a the function will be known so I can compute the maximum of the second derivative for values of x in that interval and get what M is. So everything is known in this x(h). So h can be computed. Namely we know the step size of each of these sub intervals.

(Refer Slide Time: 31:44)



So when that information is used here then I know I must divide that interval a to b into n equals sub intervals where n is b minus a by h where h will come from here. So the information on the bound on the error tells you if depending upon what your accuracy is or what you demand for accuracy you can obtain the number of sub intervals n into which this interval a to b can be divided into. So that you arrive at a value of the integral a to b  $f(x)$  dx correct to the desired degree of accuracy.

So that completes our discussion on the Trapezoidal rule the composite Trapezoidal rule the order of accuracy of both the Trapezoidal rule and the composite Trapezoidal rule. And how we can get the number of sub intervals which an interval should be divided into depending upon the desired degree of accuracy. So one thing that remains to be done at this stage is what happens to the Simpson's rule if you perform some similar discussion.