Numerical Analysis
Professor R Usha
Department of Mathematics
Indian Institute of Technology Madras
Lecture -2, Part - 1
Mathematics Preliminaries, Polynomial Interpolation-1

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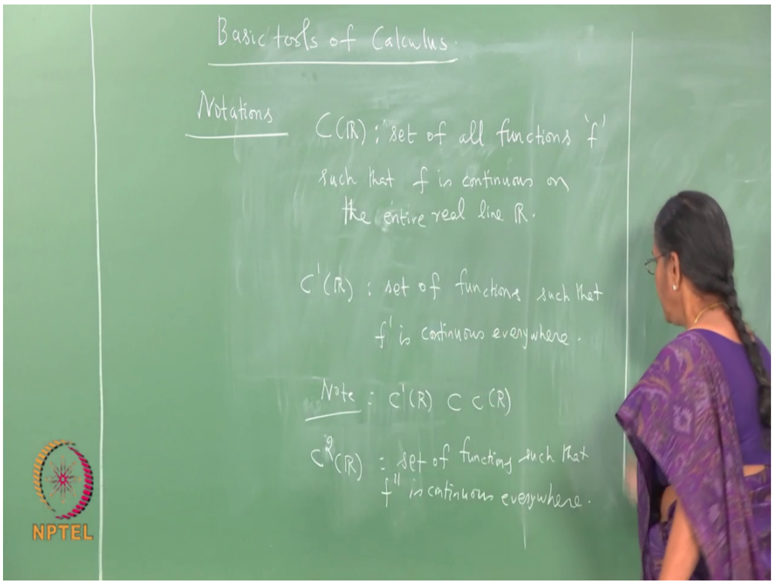

So we introduced some basic tools of calculus. So let us introduce some notations. So by C(R) I mean the set of all functions f such that f is continuous on the entire line R. And by C 1 (R) we do not set of functions such that f prime is continuous everywhere. So we immediately note that C 1(R) is a subset of C(R). So the set inclusion is a proper inclusion since there are continuous functions for which the derivatives do not exists. So we follow this notation by C 2 (R) and denote the set of functions such that f double prime is continuous everywhere.

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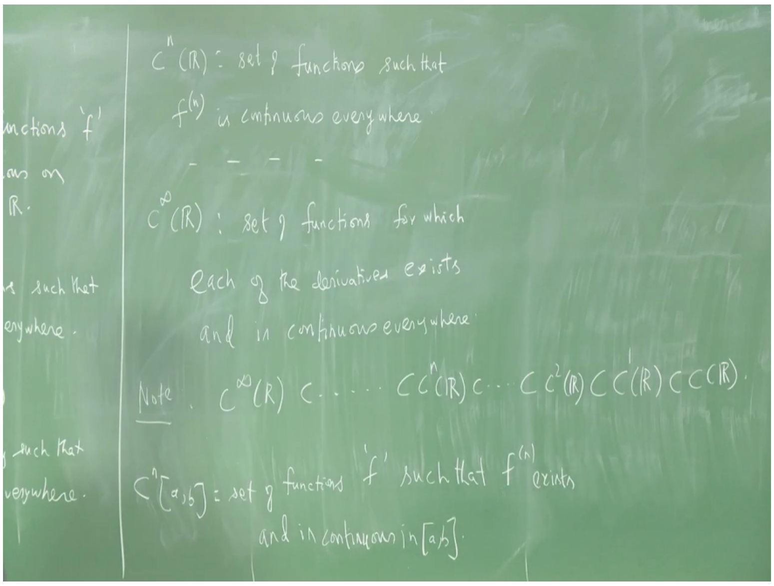

We continue to use this notation and denote by C n (R) by set of functions such that the n th derivative is continuous everywhere and so on. And By C infinity (R) we denote the set of functions for which each of these derivatives exist and is continuous everywhere. So we again note that C infinity(R) is a subset of C n (R) and so on is a subset of C 2 (R) is a subset of C 1 (R) is a subset of C (R). By C n [a,b] post interval [a,b] we need all those functions f such that the n th derivative exist and in continuous in closed interval [a,b]. So we shall use these notations in the course of our discussions.

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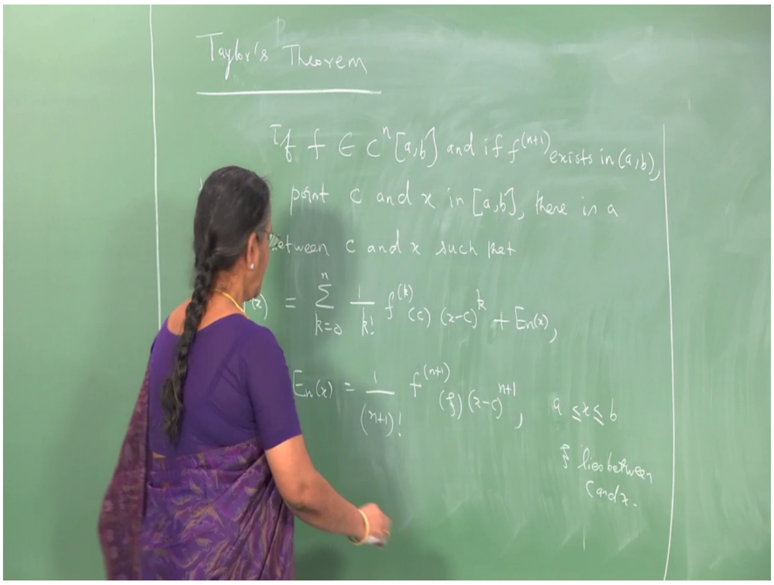

So we said that we will look at approximations of functions to begin with but we already have some knowledge about a function which satisfies certain properties can be approximated by a polynomial namely Taylor's theorem which you have studied all year tells us how to represent a function in an interval and how much of error is incurred in approximating this function by a polynomial.

So let us look at the statement of this theorem. It says that if f is in C n [a,b] we know what we mean by C n[a,b] f is such that all its derivatives upto n th order derivative exist and are all continuous in [a,b] and if the (n plus 1) th derivative exist in open interval [a,b] then the theorem says that then for points c and x in the closed interval [a,b] there is a Psi between c and x such that a function f(x) which is defined in the interval a,b can be represented by sigma a equal to 0 to n 1 by k factorial into k th derivative evaluated at c into (x minus c) power k plus E n (x), where E n (x) is given by 1 by m plus 1 factorial into n plus 1 th derivative at Psi into x minus c power n plus 1, so it is for all values of x which lie between a and b such that Psi lies between c and x.

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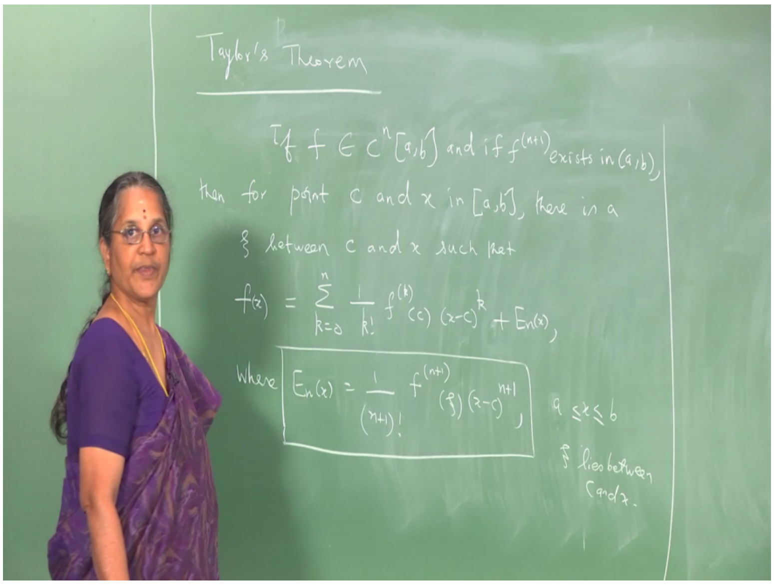

This term E n (x) is referred to as the error term and this statement gives you Taylor's theorem with Lagrange form of remainder. So it states that if f satisfies these properties then for any two points c and x in this interval you will be able to find out Psi such that it lies between c and x and for any interval a,b you can represent this function by means of the first term and you observe that the first term is a polynomial in x minus c of degree n.

So you can represent this function by a polynomial and the error that you incurr is E n (x) you have a control over this error namely you can determine the number of terms n that you should include in the first term in order to obtain the error to be less than some prescribed value. So correct to the desired degree of accuracy you will be in a position to get a representation of the function f (x) in the form of the polynomial. So it is possible to obtain a bound on the error.

So essentially Taylor's theorem gives you a way of representing a function satisfying these properties in the form of a polynomial with and it also gives you the bound on the error that it has been incurred in such a representation.

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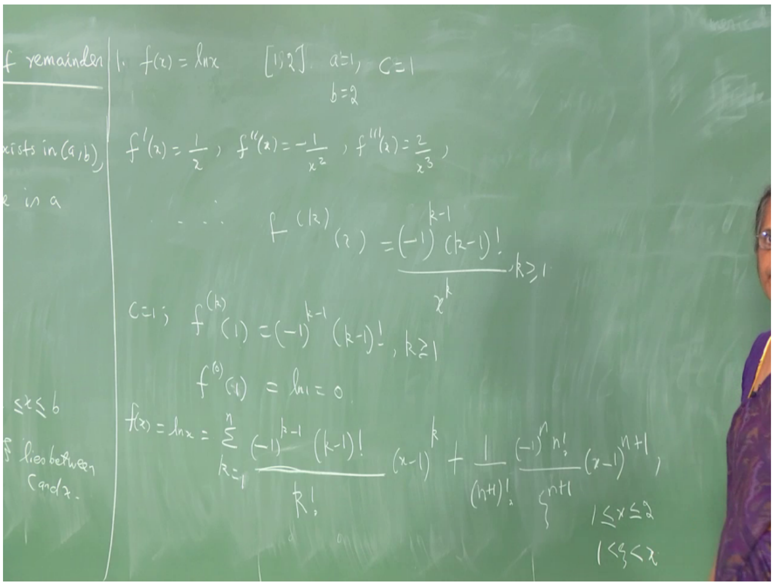

So I shall consider f(x) to be equal to log x in the interval [1 ,2] so i have take a to be 1 and b to be 2 let me take c to be 1. So this says then for points c and x so c is in [a,b]. So I can choose c to a itself. So I have chosen c to be 1 here. So I would like to represent log x by means of a polynomial of degree n. So we compute the various order derivative.

So f prime(x) is 1 by x, f double prime(x) is minus 1 by x square, f triple prime (x) is 2 by x cube and so on the k th derivative of x will be minus 1 power k minus 1 into k minus 1 factorial by x to the power of k. And this is so for k greater than or equal to 1. So have you chosen c as 1 we compute the derivatives at c namely 1. So that will give you minus 1 power (k minus 1) into (k minus 1) factorial. So c is 1 and this is for k greater than or equal to 1.

So let us compute the 0 th derivative at 1 and that will be log 1 so it is 0. So this gives us f(x) is equal to log x and that is sigma k is equal to 1 to n because corresponding to 0 we have value to be 0 so k equal to 1 to n minus 1 power k minus 1 by so I have to substitute here the k th derivative at c, so k th derivative at c is minus 1 power k minus 1 into k minus 1 factorial by I have a k factorial and then x minus 1to the power of k, that is the first term E n (x) that is 1 by n plus 1 factorial into n plus 1 th derivative at Psi so n plus 1) th derivative will be minus 1 to the power of n into n factorial and then divided by so Psi power (n plus 1) into x minus 1 to the power of (n plus 1) that will be E n(x). And this representation is valid for x in 1to 2 and Psi lying between 1 and x.

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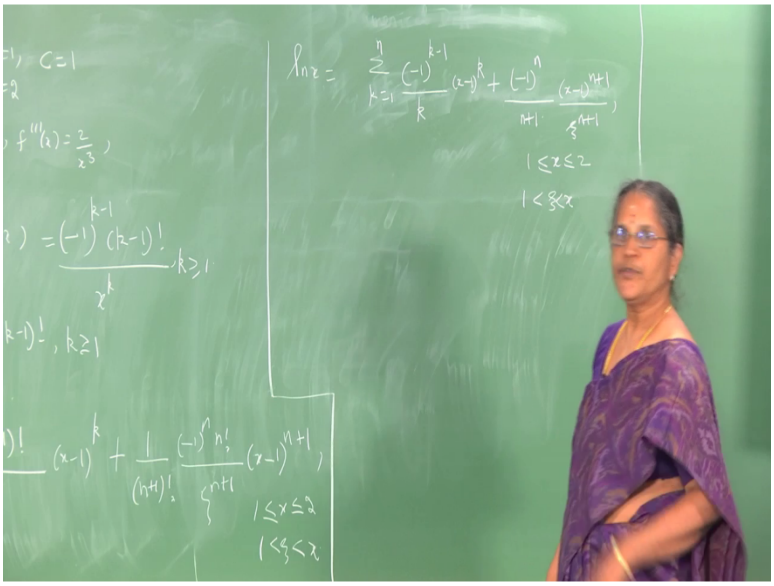

So let us simplify this, so log x will be equal to sigma k equal to 1 to n minus 1 power k minus 1 by k into x minus 1 power k and then the next term is minus 1 power n by n plus 1 into x minus 1 power n plus 1 by Psi power n plus 1 for x lying between 1 and 2 for Psi between 1 and x.

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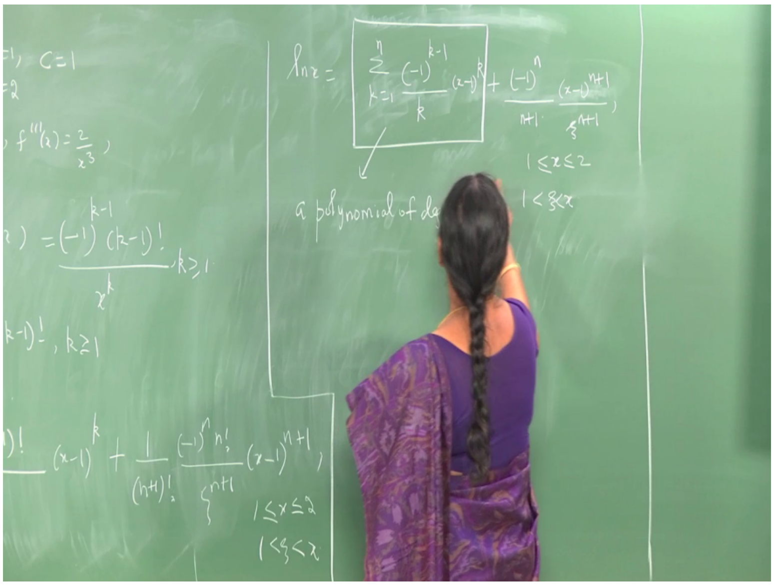

So we observe that the first term is a polynomial of degree n.

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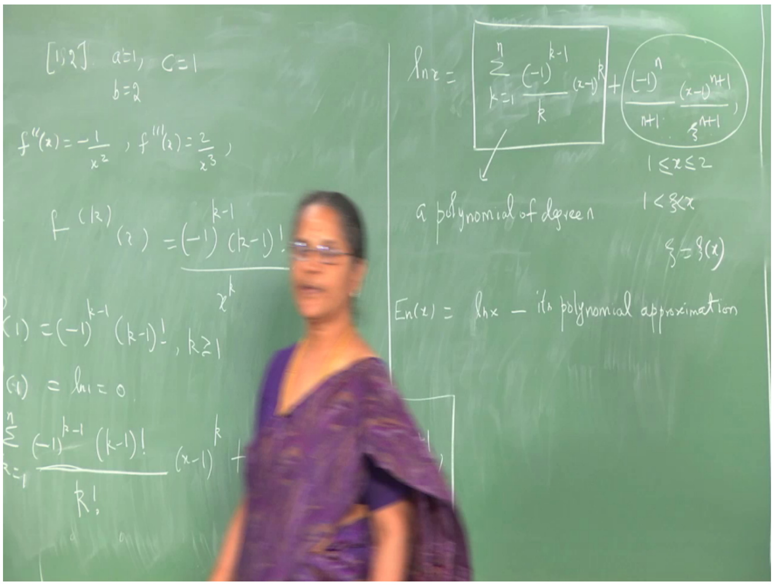

And the second term, is it a polynomial? No why? We have a factor x minus 1 to the power of (n plus 1) in the numerator and we have Psi power n plus 1 in the denominator, and Psi is a function of x. So it's not a polynomial.

So log x has been represented as a polynomial plus some term which we call as error term which is not a polynomial. And this tells you how much is this function log x deviated from its approximation namely a polynomial of degree n. So the error E n(x) is this function log x minus its polynomial approximation. So it tells you how much this function log x is deviated from its polynomial approximation. So we have illustrated Taylor's theorem with lagrange form of remainder.

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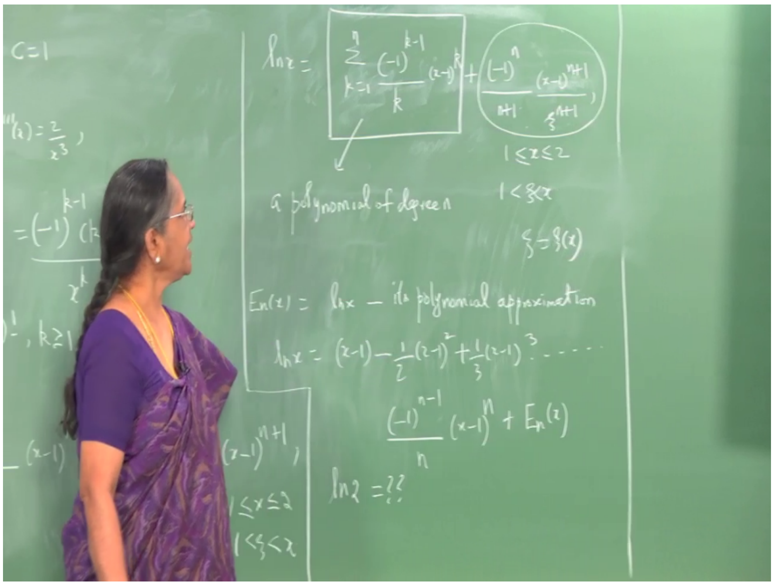

So let us write out this polynomial explicitly, so when k is 1 so this gives you x minus 1 then 1 by 2 (x minus 1) the whole square plus 1 by 3 x minus 1 the whole cube and so on. The last term minus 1 to the power of n minus 1 by n into x minus 1 to the power of n plus the error which is E n (x).

So our goal now is to use this polynomial and the error term to compute what log 2 being obtained and expansion of log x in this form along with the Lagrange form of remainder given by E n(x). Our goal is to determine log 2 and see how much error has been incurred in approximating log x by a polynomial of degree n. Or we would like to determine how many terms that we will have to include in its expansion so that the error is less than some prescribed quantity. So let us do all this now.

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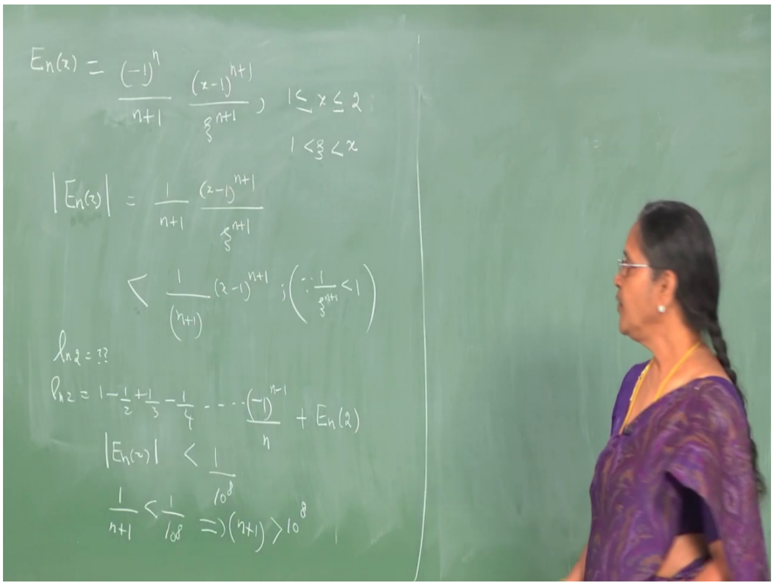

So let us first write down what E n(x) is so E n (x) is minus 1 power 10 by n plus 1 into x minus 1 power n plus 1 by Psi power n plus 1. So this is for x in 1 to 2 and Psi less than x and greater than 1. So let us find the bound on this error so its 1 by n plus 1 x minus 1 power n plus 1by Psi power n plus 1and it is less than 1 by n plus 1 into x minus 1 power n plus 1. Why Psi is greater than 1 so 1 by Psi power n plus 1 is less than.

So E n (x) is bounded by 1 by n plus 1 into x minus 1) power (n plus 1). So now I would like to compute log 2 with the help of the expansion that we have derived namely I call this as star so use star to get log 2. So log 2 will be equal to 1 minus half plus 1 by 3 minus 1 by 4 and so on. Minus 1 power n minus by n then plus E n(2) so error evaluated at 2. We already have the expression for the bound on E n (x) for any x in the interval 1 to 2.

Now my question is determine log 2 corrective accuracy of 1 (())(18:52) 10 to the power of 8 namely you want E n(2) to be such that it is less than 1 in 10 power 8. So 1 by 10 to the power of 8.which means you want 1 by n plus 1 to be less than 1 by 10 power 8, so which tells you that n plus 1 is greater than 10 to the power of 8.

So Taylors theorem has provided us with a nice way of representing a function satisfying certain properties by means of a polynomial and it also helps us to determine the number of terms that we may have to include namely the degree of the polynomial that we need to use in order to represent this function so that the error that we incur can be made as small as we please.

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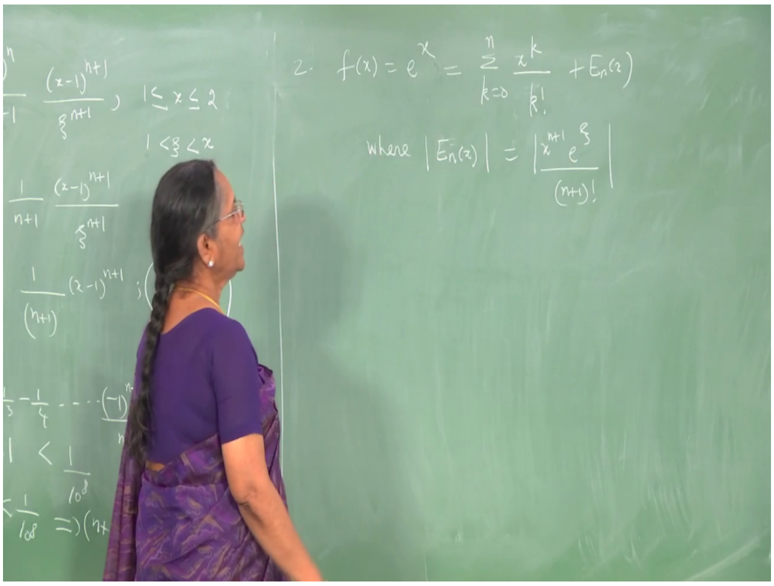

Let us now take another example mainly I consider f(x) is equal to E to the power of x which can be expanded as k equal to 0 to n x power k by k factorial plus the error term where this error E n(x) is x power n plus 1 into E power Psi by n plus 1 factorial. So I have used the expansion for E to the power of x and written down a representation of E power x plus a polynomial of degree n and then the corresponding remainder.

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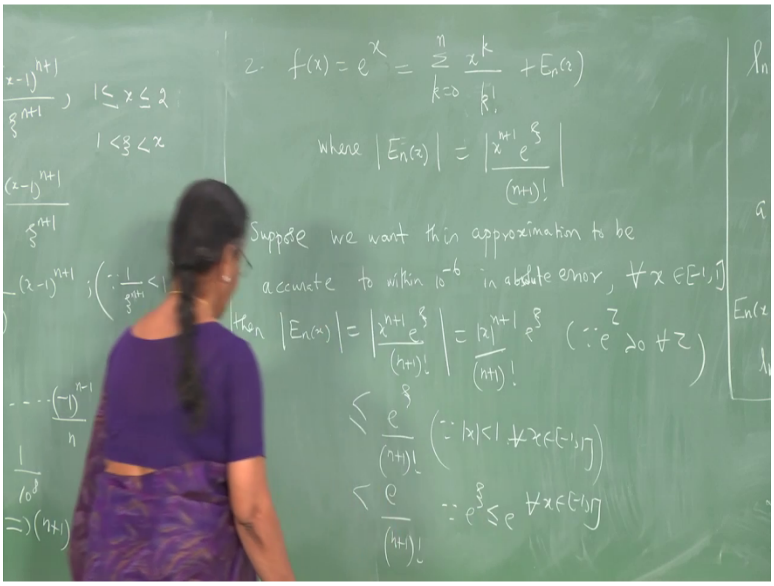
3 

Suppose we want this approximation to be accurate to within 10 to the power of minus 6 in absolute error. We require that this should happen for every x say in the interval minus 1 to 1. Then this tells you that modulus of E n(x) which is 1 x power n plus 1 into E power psi by n plus 1 factorial, which is mod x power n plus 1 into E power psi by n plus 1 factorial. Since E z is greater than 0 for every z so this will be less than or equal to E power psi by n plus 1 factorial. Since mod x is less than 1 for every x in minus 1 to 1.

So this will be less than e b n plus 1 factorial since E power Psi is less than or equal to e for Ez x in 1 to 1.

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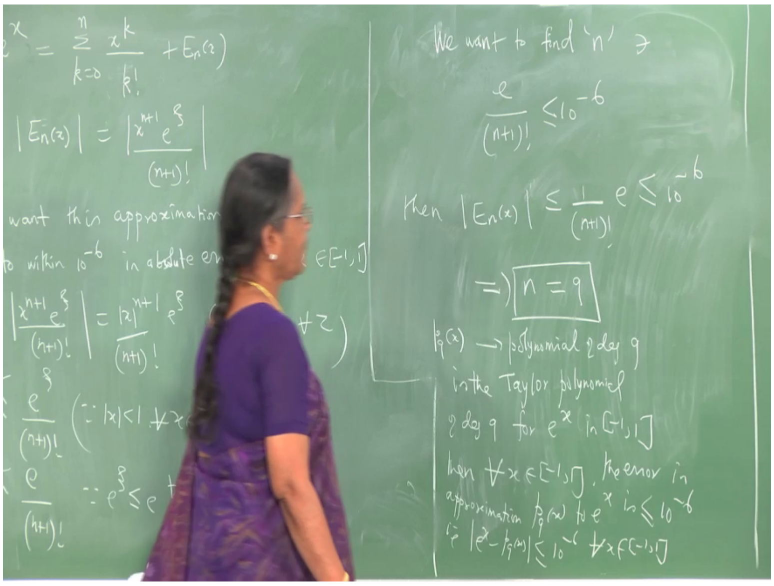

We want to find the number of terms to be included such that the error e by n plus 1 factorial must be less than 10 to the power of minus 6. THis tells you that E n (x) would be less than or equal to 1 by n plus 1 factorial into e and that will be less than 10 to the power of minus 6. And when you do these computations you will be able to determine that n is equal to 9.

So we have shown using Taylor's theorem with Lagrange form of remainder that if p 9 (x) that is a polynomial of degree 9 because n turns out to be 9. So p 9 of 6 which is a polynomial of degree 9 is the Taylor polynomial of degree 9 for the function E power x in the interval [minus 1 to 1] then for every x in this interval [minus 1 to 1] the error in approximation p 9 (x) to e to the power x is less than 10 to the power of minus 6. That is the difference e power x minus p 9 (x) is less than or equal to 10 power minus 6 for every x in this interval [minus 1 to 1]

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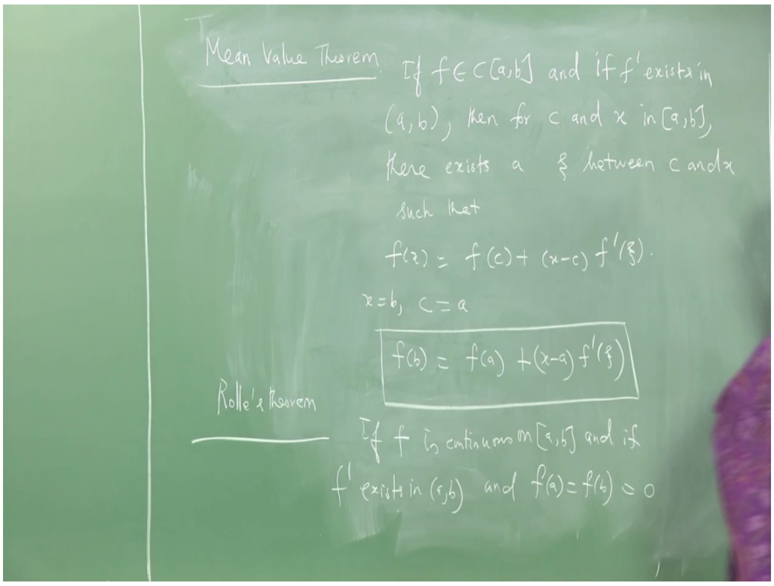

So now we move on to the mean value theorem this is what we have learnt earlier. And what does the mean value theorem state? It states if f is a continuous function in the closed interval [a,b] and if f dash exist in open interval (a,b) then for c and x for any two points in the closed interval [a,b], there exists a Psi between c and x such that f(x) is f(c) plus x minus c into f dash of Psi. So in particular if I choose x to be b and c to be a then this tells me f(b) is f(a) plus x minus a into f dash of Psi.

Roll's theorem which is another important result which states that if f is continuous on the closed interval [a,b] and if f dash exist in the open interva (a,b) and f (a) equal to f(b) equal to 0.

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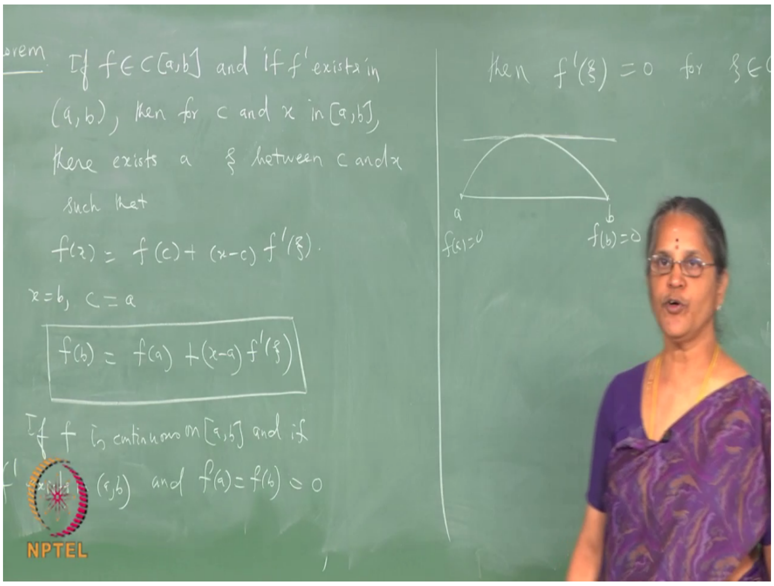

Then f dash (psi) is 0 for psi belonging to the open interval (a,b). The result says that if you have at a f(a) is 0 and at b f(b) is 0 then there is a point Psi between a and b at which f dash of Psi is 0 namely you have the tangent which is parallel to the x axis. The tangent at this point Psi has its slope to be 0 that is what the Rolls theorem is and the result is immediate from the mean value theorem.

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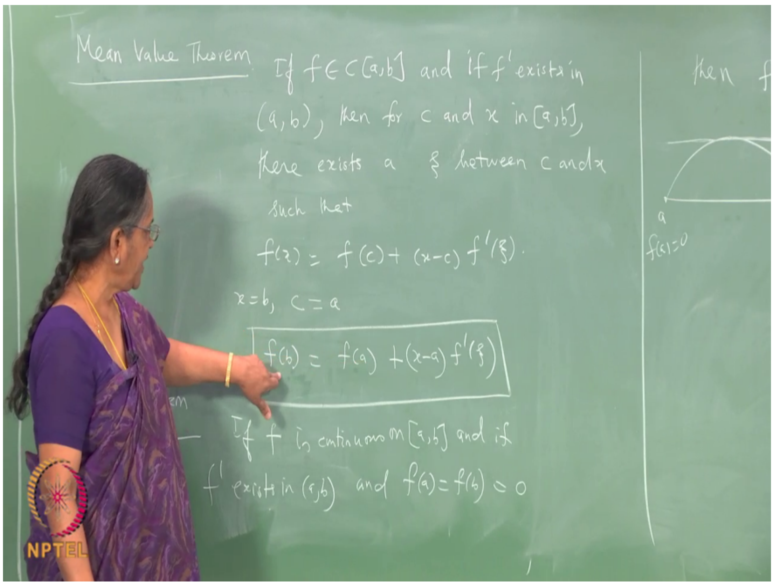

If f(a) is 0 f(b) is 0 then there is a Psi between a and b such which f dash (psi) is 0.