Numerical Analysis

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Lecture 11

Numerical Differentiation 2

Polynomial Interpolation Method

In the previous lecture we saw some numerical differentiation techniques namely where Taylor expansion and the method of undetermined coefficients. We shall learn some more methods by means of which numerical differentiation formulas can be derived. So we know interpolation by polynomials we can approximate a function f(x) in an interval or next values given at a set of points by means of interpolation polynomials. So let us make use of this idea and then see whether we can get numerical differentiation methods from interpolation polynomials.

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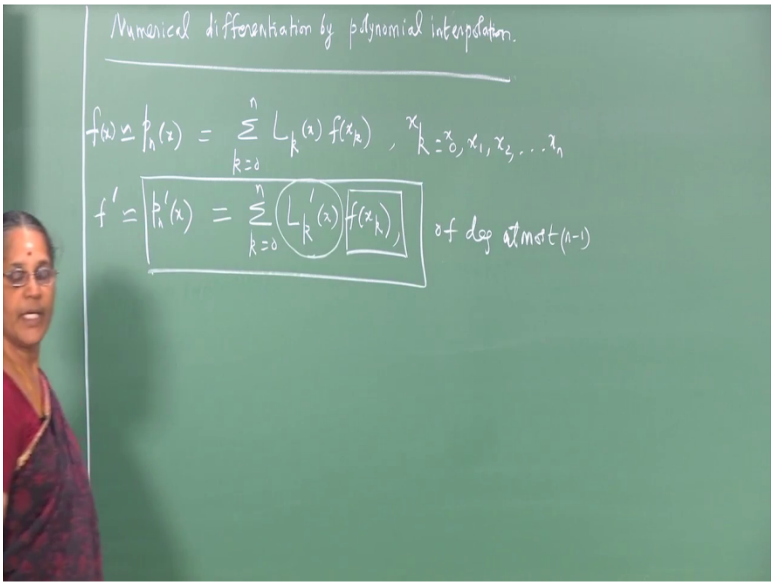

So we shall now discuss numerical differentiation by polynomial interpolation. So we know that p n(x) is sigma k equal to 0 to n L k(x) into f(x k) is the Lagrange interpolation polynomial that approximates the function f where x k are points x 0, x 1, x 2 etc upto x n. So from the interpolation polynomial Lagrange interpolation polynomial that approximates the function one can obtain an approximation for the derivative f dash and it is defined by a derivative of the interpolation polynomial.

And that is defined by sigma k equal to 0 to n L k dash(x) into f(x k), and this polynomial p n dash(x) is of degree at most n minus 1. So here p n(x) is the interpolating polynomial that interpolates the function at a set of n plus 1 points and p n(x) is of degree n.

So p n dash is a polynomial of degree at most n minus 1 that approximates the function f dash which is derivative of f and this p n dash therefore when you look at this expression is a linear combination of these polynomials which are obtained taking the derivative of L k (x), so this expression is a linear combination of the derivatives of L k (x) with coefficients given by the values of the function f at these points.

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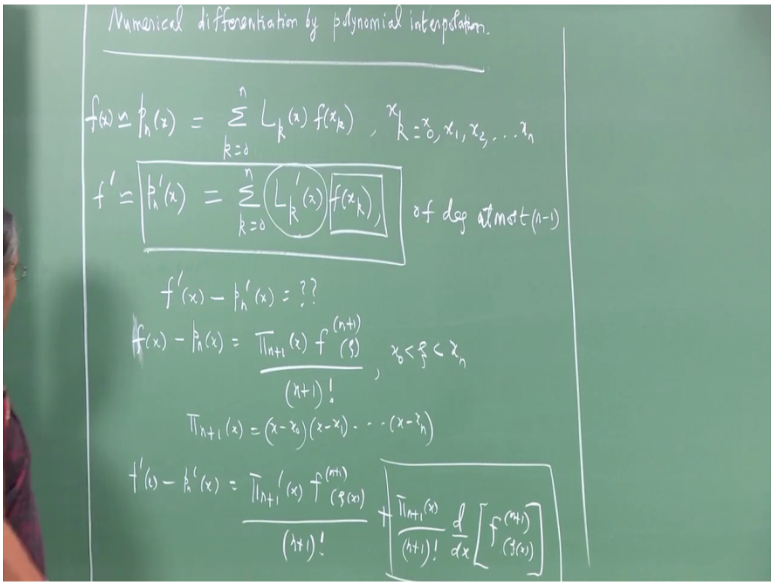

So the question now is what is going to be the error in such an approximation, what is the error? We already have an expression for the error in interpolation namely the f(x) minus p n(x) is given by Pi n plus 1(x) into n plus 1 th derivative at some point Psi by n plus 1 factorial where Psi lies between x 0 and x n where x satisfies certain conditions. We already have obtained the expression for error in interpolation where Pi n plus 1(x) is (x minus x 0) (x minus x 1) etc upto (x minus x n).

So let us compute the derivative and see whether we are able to give an expression for the error f dash (x) minus p n dash(x). So if I compute f dash(x) minus p n dash(x) from the above expression then I get Pi (n plus 1) dash (x) into (n plus 1)th derivative at Psi we call that Psi as a function of x it depends on x by n factorial then plus I have Pi n plus 1(x) by n plus 1 factorial into d by dx of the n plus 1)th derivative of f as a function of Psi(x).

So if I look at this expression I observe that I cannot estimate the second term because I do not know the dependence of Psi on x. And therefore I cannot use this to obtain an estimate for the error in f dash(x) minus p n dash(x). So we must have some methods by means of which we should be able to estimate this error and we have an alternate approach and that is given by the following result in the form of a theorem.

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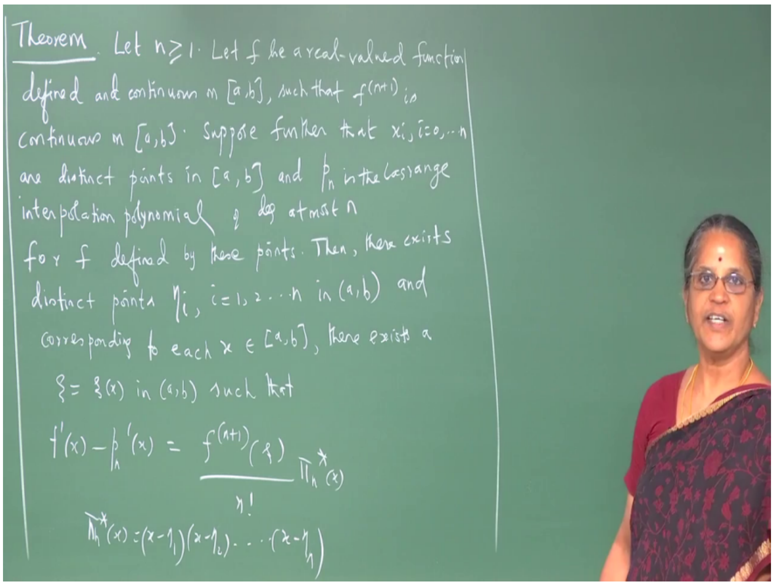

So let us look into the result of this theorem. It says let n be greater than or equal to 1 or let f be real valued function defined and continuous on the interval [a,b] such that its n plus 1)th derivative is continuous on the closed interval [a,b]. And suppose further that x i for i is equal to 0, 1,2,3 etc upto n are distinct points in the interval [a,b] and that p n is the Lagrange interpolation polynomial of degree at most n for f defined by these points.

Mainly the function f is reconstructed by seeking a polynomial p n of the Lagrange interpolation type that interpolates the function at a set of these n plus 1 times so it is a polymial of degree at most n then the theorem says there exists distinct points theta i for i is equal to 1,2,3 upto n.

There are n points where in the open interval (a,b) and corresponding to each x in the interval a,b there exists a Psi which depends on x in the open interval (a,b) such that the difference in f(x) and p n(x) is given by the n plus 1)th derivative at Psi by n factorial into Pi n star (x) where Pi n star (x) is (x minus theta 1) into (x minus theta 2) etc upto (x minus theta n).

So what does it say f satisfies certain conditions in addition if you have distinct points x i in the interval [a,b] and p n is the Lagrange interpolation polynomial for the function f in that interval that interpolates the function at these n plus 1 points then you will be able to find distinct points then you will be able to find distinct points theta a in the open interval.

So you should be while showing the result you should be able to get what these points eta i are? You should show the existence of these points eta a and in addition the result says corresponding to each x in this closed interval [a,b] there will be Psi depending on x in the open interval (a,b) such that the error in the approximation of the derivative of the Lagrange interpolation polynomial is given by the expression on the right hand side. The preference almost on the same lines as we have done in the case of deriving the error in interpolation mainly this formula.

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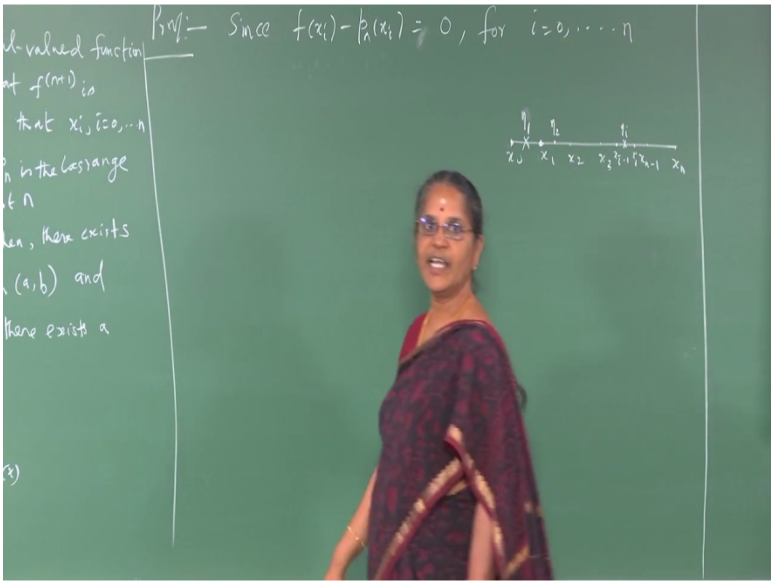

So let us quickly go through the proof of this result. So let us now find out what is f(x i) minus p n(x i) for i is equal to 0 ,1,2,3 etc upto n. So the theorem says p n is the Lagrange interpolation polynomial for the function f defined by these points and so f(x i) minus p n(x i) is 0 for i is equal to 0 to n.

So what does that mean these are the points x 0, x 1, x 2, x 3 etc upto, x n minus 1,x n. It says f(x 0) minus p n (x 0) is 0. f(x 1) minus p n(x 1) is 0. So this f(x) minus p n(x) are x minus p n evaluated at x 0 is 0 and x 1 is 0 and therefore there must be a point between x 0 and x 1 at which the derivative must be 0. Let us call that point as eta 1.

Similarly it says f(x 1) minus p n(x 1) is 0 so there must be a point eta 2 belonging to the open interval x 1 to x 2 at which x dash eta 2 minus p dash eta 2 should be 0. So in general in the interval x i minus 1 to x i there will be a point eta i at which f dash eta i minus p n dash (eta i) is 0. In each of these intervals you will be able to find out points eta 1 eta 2 eta 3 etc eta n such that f dash (eta i) minus p n dash (eta i) will be 0.

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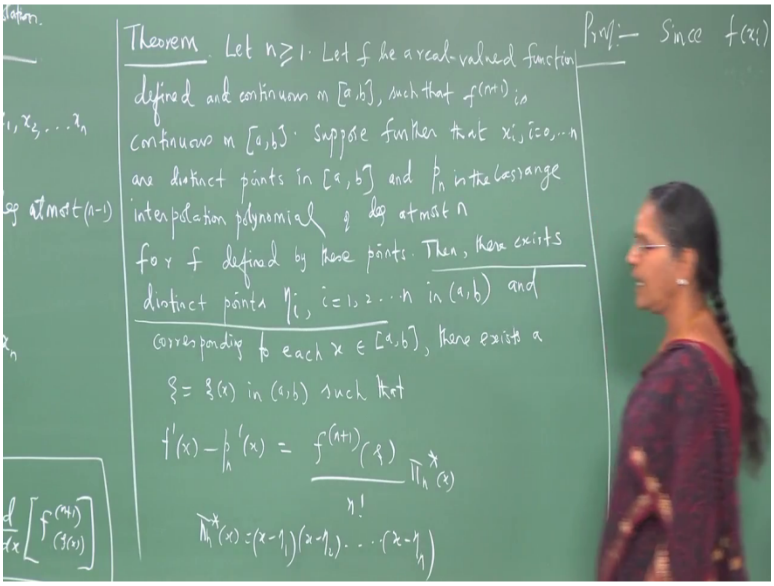

So that immediately shows this result that there exist distinct points eta i in a,b so you are been able to get those points. What are these points these are points at which f dash minus p n dash vanishes.

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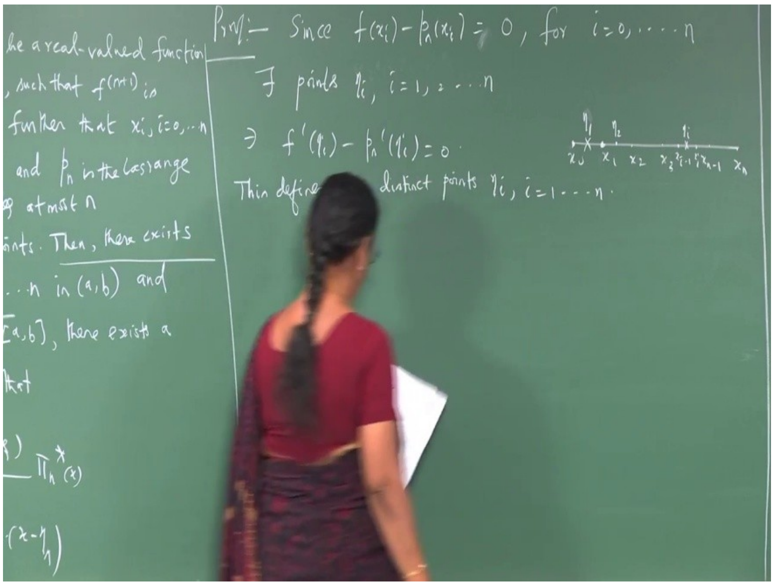

So since this is happening there exists points eta i for i is equal to 1 2 3 upto n such that f dash (eta i) minus p n dash (eta i) is equal to 0. So this defines the distinct points eta i for i is equal to 1 to n. What happens at x is equal to eta i.

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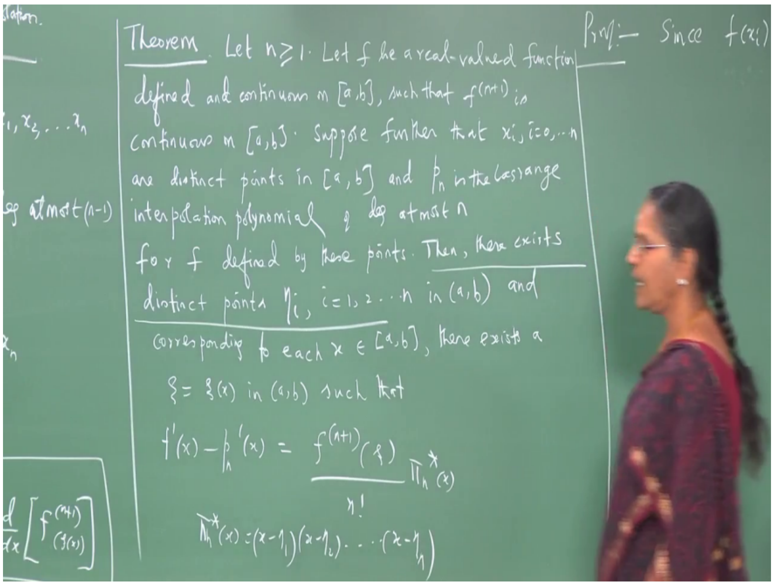

When x is equal to eta i f dash (eta i) minus p n dash(eta i) just now we showed that it is 0, so the left hand side is 0. What about the right hand side? At x is equal to eta i because of the presence of Pi n star(x) which is this expression. At any eta i is equal to 1 to n the right hand side is 0. So there are n point's eta i at which the left hand side is 0 the right hand side is 0 so double star is identically verified that double star is satisfied.

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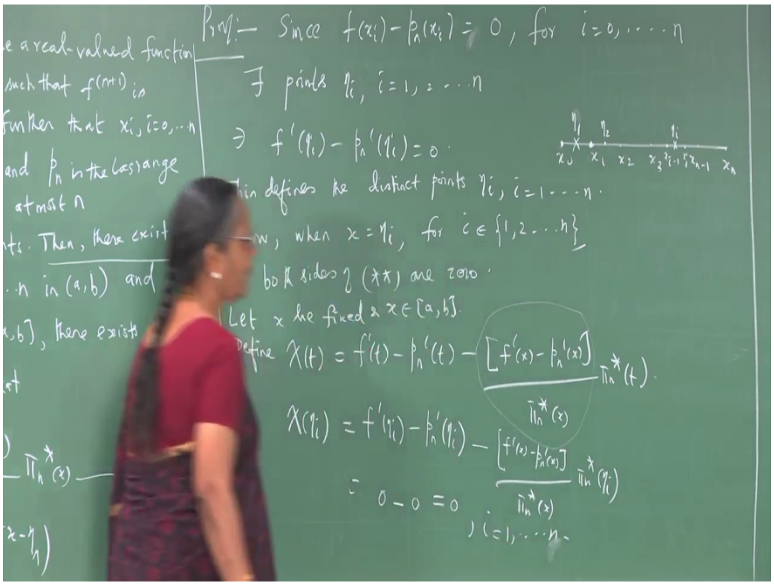

So let us write down this result. So now when x is equal to eta i for i belonging to say the set {1 2 3 upto n}. We observe that both sides of double star are 0. Let x be fixed and x belongs to the closed interval [a,b]. And let us define a function as follows. Say a function kai such that kai(t) is equal to f dash(t) minus p n dash(t) minus f dash(x) minus p n dash(x) divided by Pi n star(x) into Pi n star(t).

Let us evaluate Kai (eta i). Kai(eta i) is f dash(eta i) minus p n dash (eta i) minus this expression f dash(x) minus p n dash(x) by Pi n star(x) which is a scalar into Pi n star(eta i). So this difference is 0 and the second term is 0 because pi n star(eta i) is 0 and therefore this is 0 for what values of i, i is equal to 1 2 3 upto n.

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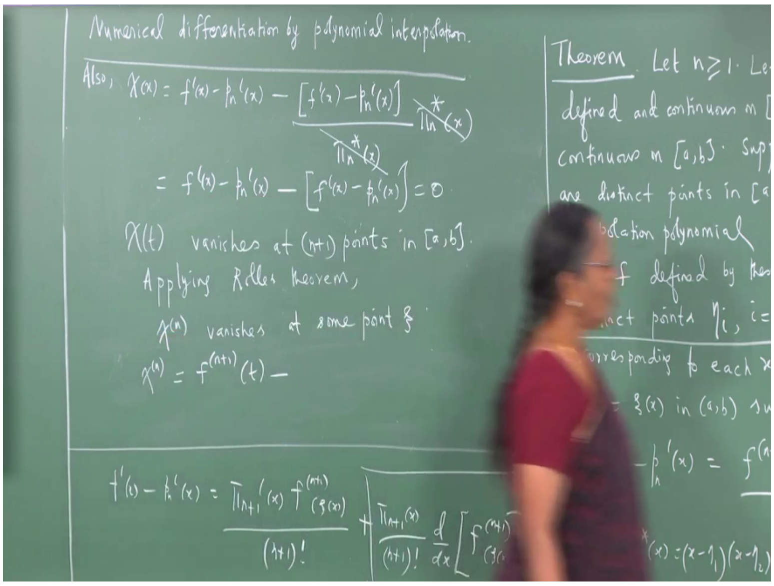

So let us find out what is kai(x) so it is f dash(x) minus p n dash(x) minus [f dash(x) minus p n dash(x)] divided by Pi n star(x) into Pi n star(x) . So I observe that this is f dash(x) minus p n dash(x) minus [f dash(x) minus p n dash(x)] so this is again 0. So you oberve that this function kai vanishes at how many points? At all the eta i which are n in number which are distinct and at x where x belongs to [a,b] so kai (t) vanishes at n plus 1 points in the interval [a,b].

Applying Rolls theorem we will have the n th derivative of Kai vanishes at some points say Psi. And therefore we compute the n th derivative. The n th derivative of Kai will be the (n plus 1) th derivative (t)

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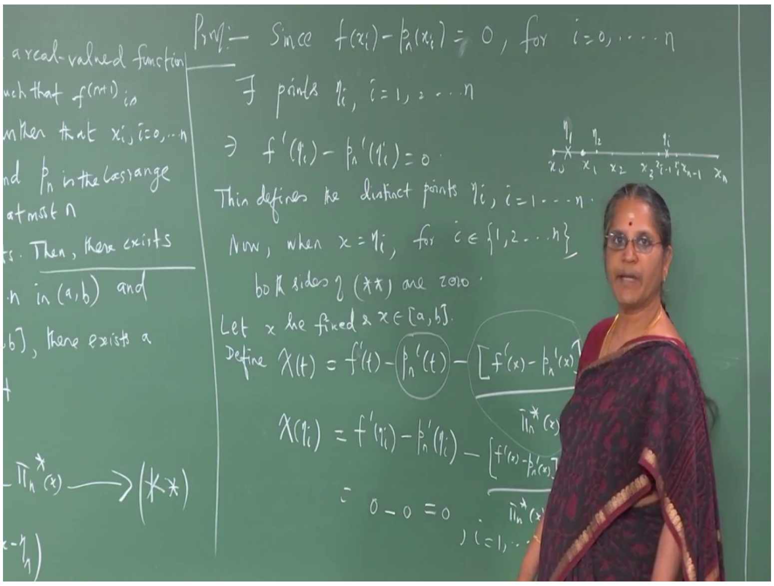

Because I already have a first derivative here, so I will differentiate it n times so I have a (n plus 1) th derivative at t then minus what about this? This is a polynomial of degree at most n minus 1. So when I take the n th derivative this will be 0. This is anyway a scalar.

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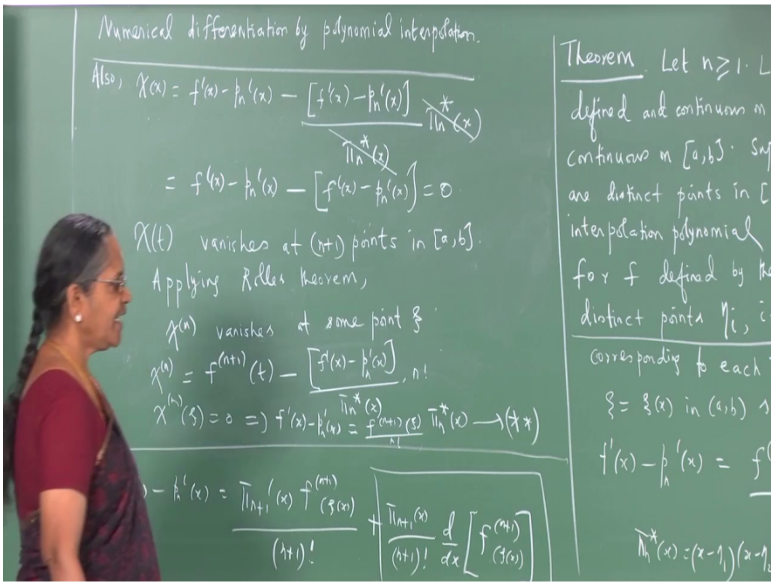

So I have minus f dash (x) minus p dash(x) divided by Pi n star (x) into the n th derivative of Pi n star (t). So what will be the n th derivative Pi n star (t) will be t minus eta 1, t minus eta 2 etc t minus eta n. So it will be of the form t power n plus some coefficient into t power n minus 1 plus etc.

So when you take the n th derivative of the first term t power n that will give you n factorial then the n th derivative of the next terms which appear their they are all of degree less than n. And therefore they will not contribute to the n th derivative. So we get kai n th derivative of kai to be this. And the result that we have got says there is a point Psi which depends of course on x such that the n th derivative vanishes.

So the n th derivative equal to 0 will immediately give us the expression that f dash(x) minus p n dash(x) is equal to n plus 1) th derivative evaluated at Psi multiplied by Pi n star(x) into Pi n star(x) divided by n factorial. And you observe that this is essentially the result that we have given in double star where Pi n star(x) is the product of n factors of the form x minus eta 1 etc x minus eta n.

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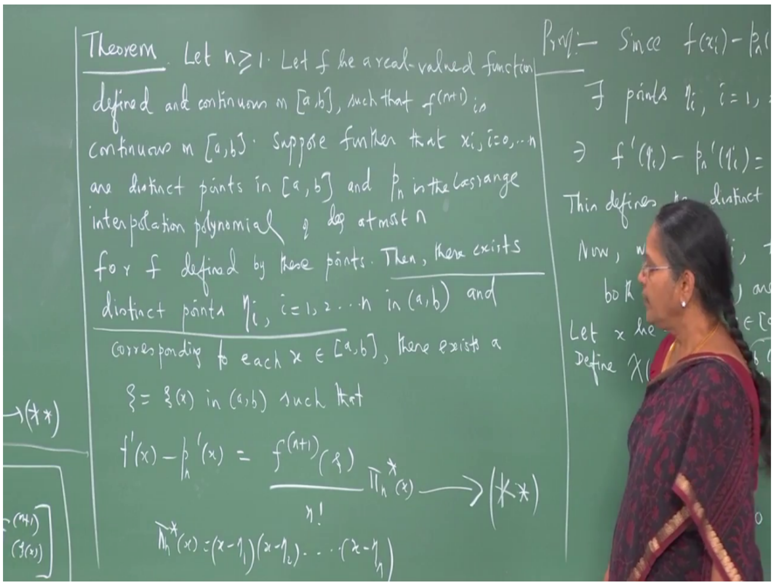

So the theorem tells us that when P n is the Lagrange interpolation polynomial of degree at most n for the function f then its derivative P n dash approximates the derivative of f namely f dash and the error in such an approximation is given by this expression. There exists points which are distinct eta 1 eta 2 etc eta n in the open interval (a,b) such that this happens. That is what the result says so this tells us that the derivative of the function f can be reconstructed by the derivative of the Lagrange polynomial that reconstructed the function f at a certain set of distinct points that is what the information obtained from this result.

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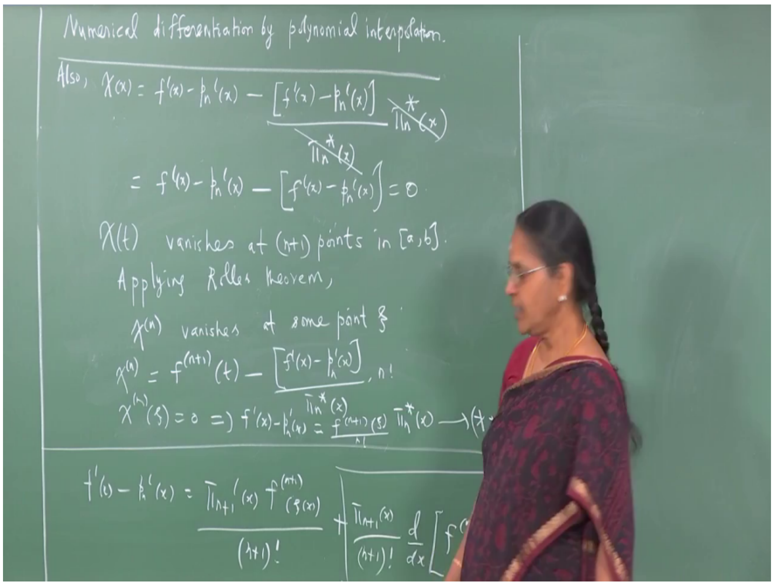

Let us now go back to the result that we wrote earlier. Starting from the Lagrange interpolation polynomial an the expression for the error in interpolation we took the derivative and set f dash(x) minus p n dash(x) is this expression. And we also remarked that we cannot get an estimate for this error because the second term involves the (n plus 1) th derivative (Psi) which depends on x and we do not have an explicit expression for the dependence of Psi on x.

However this result is useful at the nodal points, let us see what happens at the nodes namely at the x i what happens to this formula.

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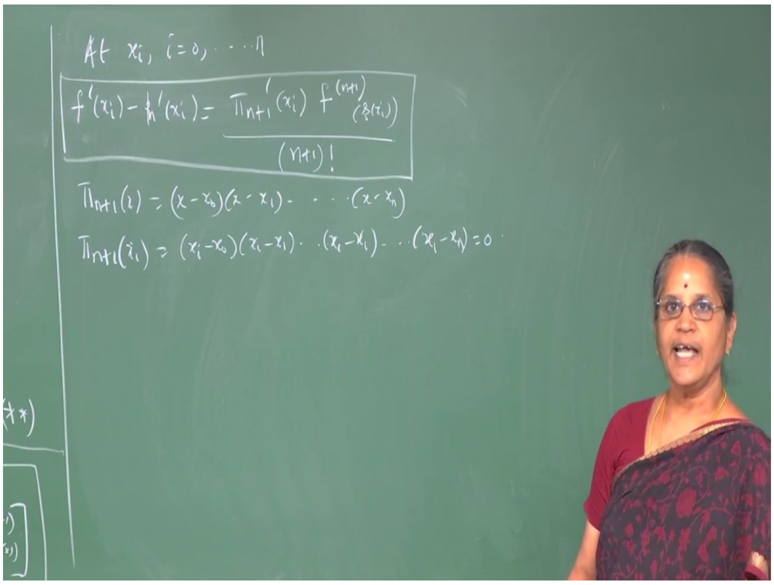

So at x i for i is equal to 0,1,2,3 etc n we compute what is f dash(x i) minus p n dash(x i) we observe that it is Pi n plus 1 dash (x i) into (n plus 1) th derivative (Psi (x i)) divided by (n plus 1) factorial then let us look into the second term. The second term involves Pi (n plus 1)(x). What is Pi (n plus 1)(x)? Pi (n plus 1)(x) is(x minus x 0) (x minus x 1) etc upto (x minus x n).

So Pi n plus 1 evaluated at x i will be (x i minus x 0)(x i minus x 1) and so on there will be a factor x i minus x i into x i minus x n and so this is 0. And therefore at the nodes x i the second term vanishes and therefore we have an estimate of the error in approximation of the derivative f dash by the derivative of the Lagrange interpolation polynomial as given by the right hand side. So at the nodes we have an expression by means of which we will be able to obtain an estimate on the approximation that we have made in computing or in estimating what f dash is.

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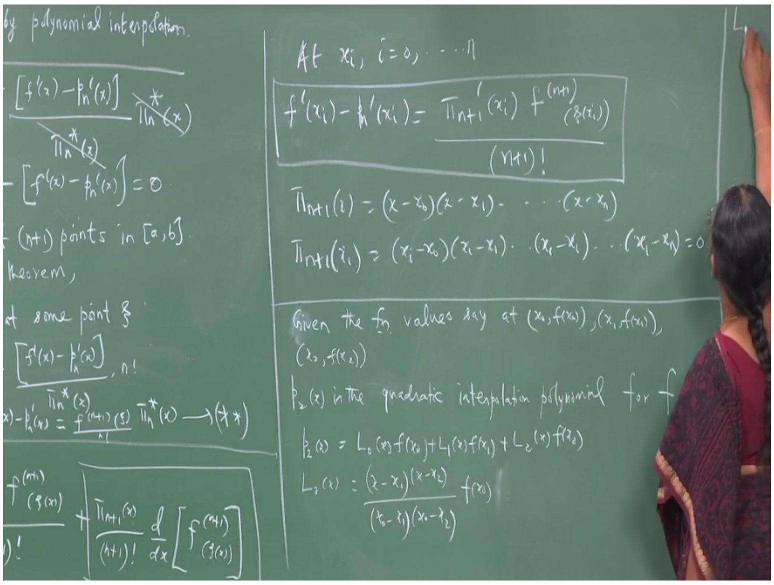

So let us illustrate this by means of say a quadratic interpolation polynomial and obtain what this error is? Given the function values say at (x 0 f (x 0)) (x 1 f(x 1)) (x 2 f(x 2)) p 2(x) is the interpolating polynomial which is the quadratic interpolating polynomial that interpolates the function at the set of points x 0 x 1 x 2. So it is the interpolation polynomial for the function f. So what is p 2(x)? p 2(x) is we know is L 0(x) into f(x 0) plus L 1(x) into f(x 1) plus L 2(x) into f(x 2) that is the interpolation polynomial. What are L 0, L 1, L 2 so L 0(x) will be equal to x minus x 1 into x minus x 2 into x 0 minus x 1 into x 0 minus x 2 multiplied by f(x 0) and what about L 1(x)?

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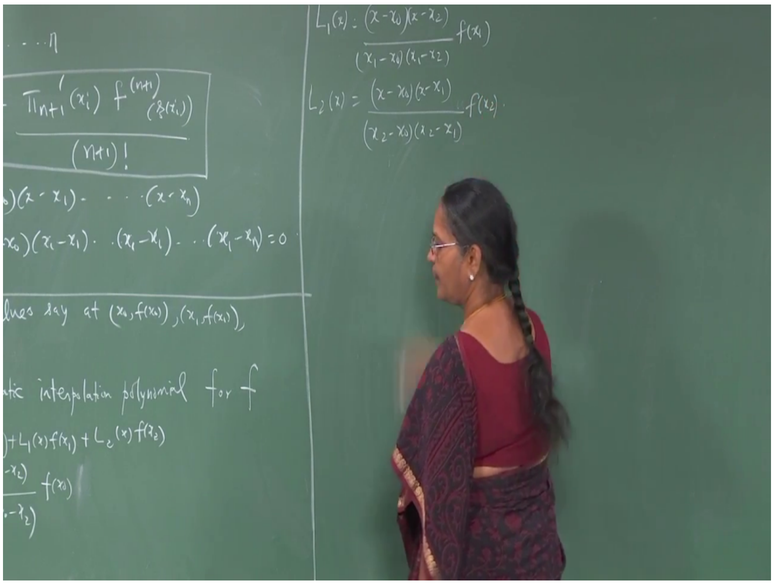

L 1(x) will be (x minus x 0) into (x minus x 2) divided by (x 1 minus x 0) into (x 1 minus x 2) into f(x 1). What about L 2(x)? (x minus x 0) into(x minus x 1) by (x 2 minus x 0) into (x 2 minus x 1) into f(x 2).

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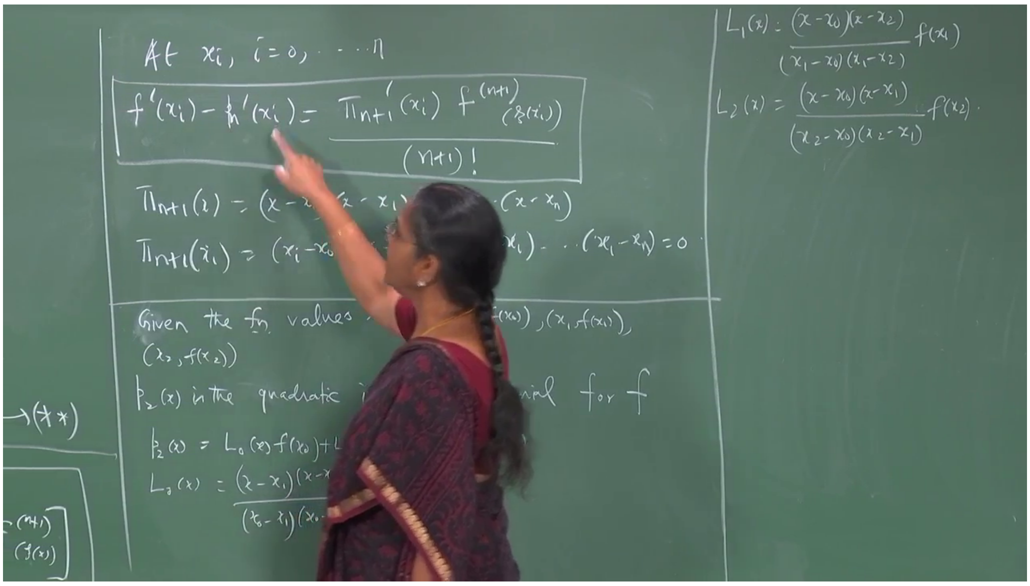

So we require now in order to estimate this we require p 2 dash at each of the x i.

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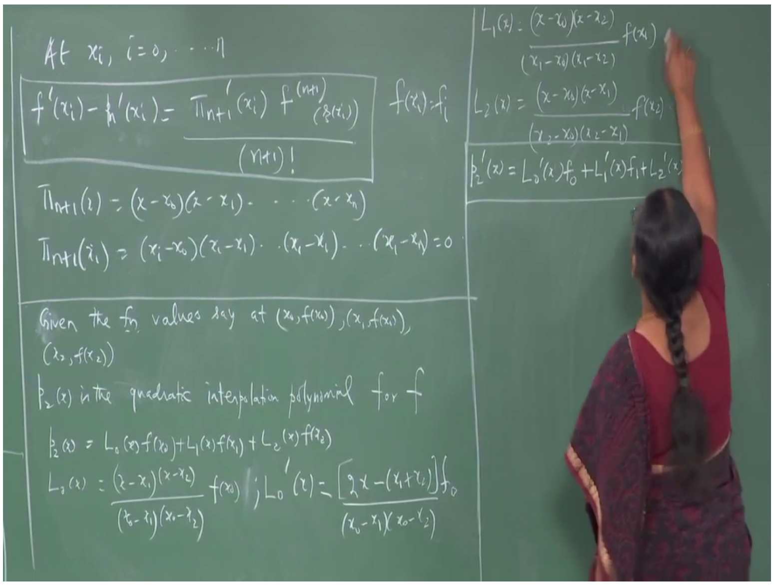

So we write down the expression for p 2 dash. So P 2 dash will be L 0(x) into f 0 I shall use the notation that f(x i) is f i so f(x 0) is x 0 plus L 1dash (x) into f 1 plus L 2 dash (x) into f 2. So we must find this at the nodes x 0, x 1 and x 2 and then use this to get an estimate on the error. So let us work out the details for L 0(x).

So what is L 0 dash (x) it is going to be so I have a x square minus x into x 1 plus x 2 in plus x 1 x 2, So when I take the differentiation then 2x minus x 1 plus x 2 divided by (x 0 minus x 1) into (x 0 minus x 2) and multiplied by f 0.

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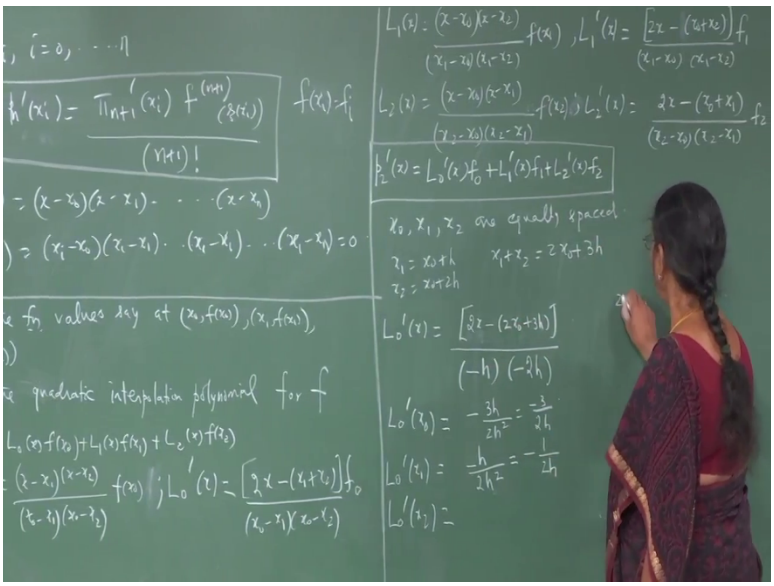

Similarly we compute L 1 dash (x), so that is 2x minus (x 0 plus x 2) by (x 1 minus x 0) into (x 1 minus x 2) and multiplied by f(x 1). And here L 2 dash (x) will be again 2x minus x 0 plus x 1 by x 2 minus x 0 into x 2 minus x 1 into f 2. Let us work out the details in a simple case. Suppose the points are equally spaced. The result can be (())(29:35) for an arbitrary located points x i also but in order to simplify our computations we consider the case that the points x 0 x1 x 2 are equally spaced.

So which means x 1 is x 0 plus h and x 2 is x 0 plus 2h. So we use that information and find out what is L 0 dash (x). So it is 2x minus x 1 plus x 2, so x 1 plus x 2 will be equal to 2 x 0 plus 3h. So L 0 dash (x) 2x minus 2x 0 plus 3h and divided by (x 0 minus x 1) so minus h into x 0 minus x 2, so minus 2h but where do I require L 0 dash at each of these nodes. So let us evaluate L 0 dash (x 0), L 0 dash (x 1) and L 0 dash (x 2). So x is x 0 so 2x 0 2 x 0 will go so I am left with minus 3h by 2h square so it is minus 3 by 2h.

Let us work out L 0 dash (x 1). So this will be 2x 1, so 2 x 1 minus 2 x 0 will give you 2 times h minus 3h so this will give you minus h divided by 2h square and therefore this will give you minus 1 by 2h. Alright? At x is equal to x 1 I require L 0 dash (x 1) so this is 2x 1 minus 2 x 0 minus 3h. So it is 2 into x 1 minus x 0 minus 3h so this will be numerator is minus h the denominator is 2h square and so you will be left with minus 1 by 2h. Fine.

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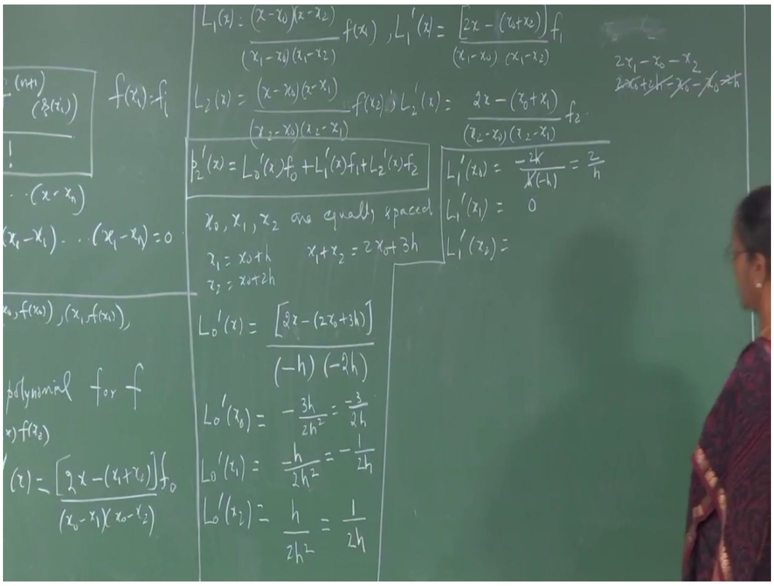

So let us now L 0 dash (x 2). So that will give you 2 x 2 minus 2 x 0 minus 3h so it is 2 into x 2 minus x 0 which is 2h minus 3h so the numerator will be h divided by the denominator is

2h square and so that will give you 1 by 2h so we have the results for L 0 dash evaluated at all the nodes. So this must be done for the other derivatives also. So let us find out what is L 1 dash(x 0), L 1 dash(x 1) and L 1 dash(x 2).

So L 1 dash(x 0) willl be 2 x 0 minus x 0 so that will give me x 0 and minus x 2, so it is minus 2h divided by x 1 minus x 0 into x 1 minus x 2, so that will give you 2 divided by h. What about L 1 dash(x 1) will be 2 x 1 minus x 0 minus x 2. So 2x 1 is 2x 0 plus 2h minus x 0 minus x 0 minus 2h, so 2x 0 cancels 2h cancels so the numerator is 0 so L 1 dash(x 1) is 0.

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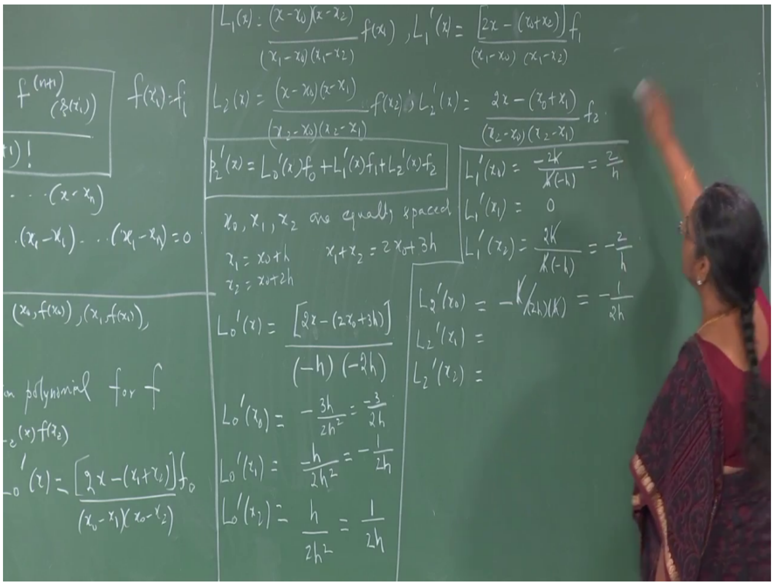

So let us now compute L 1 dash(x 2). So 2x 2 minus x 0 minus x 2 so that will give you x 2 minus x 0 which is 2h and the denominator is h into minus h so h into minus h and that will give you minus 2 by h. So the derivatives are also computed at the node points from L 1 so we require L 2 dash(x 0) L 2 dash (x 1) and L 2 dash(x 2).

So let us see what we have? L 2 dash(x) is given here so at x 0 , 2 x 0 minus x 0 minus x 1 so that will give you x 0 minus x 1 and so that is minus h by what is the denominator? x 2 minus x 0 that is 2h into x 2 minus x 1 that is h so that gives you minus 1 by 2h. What about L 2 dash(x 1)?

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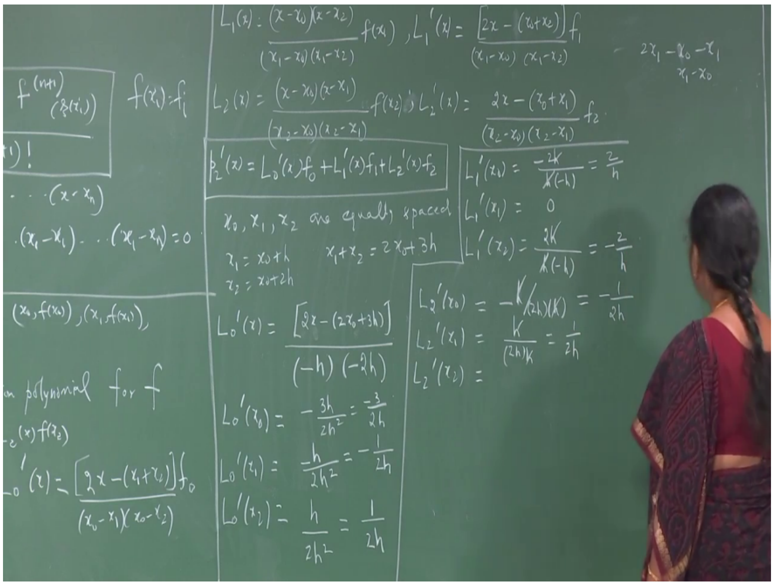

I will give you 2x 1 minus x 0 minus x 1 so it is x 1 minus x 0 which is h by 2h into h so it is 1 by 2h.

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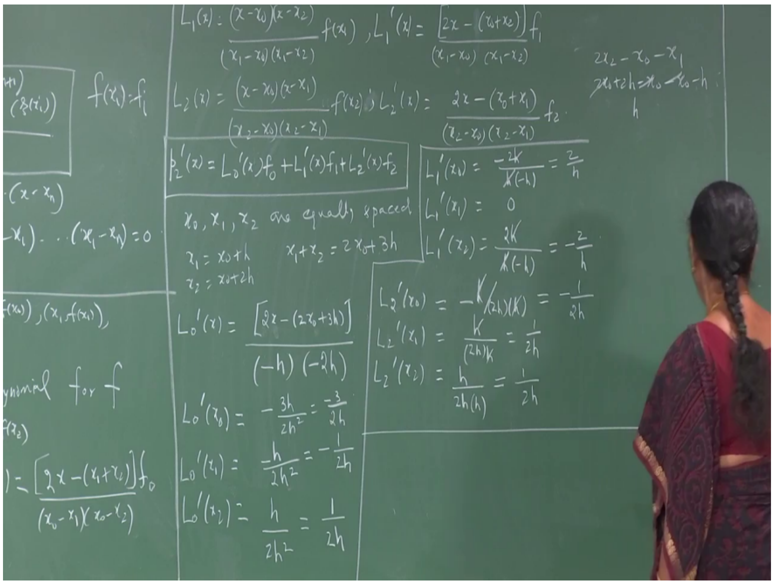

So what is L 2 dash(x 2)? 2 x 2 minus x 0 minus x 1, so it is 2x 0 plus 2h minus x 0 minus x 0 minus h so that will give you h so h divided by 2h into h and that will give you 1 divided by 2h. So let us just check what is L 2 dash(x 2) 2x 2 minus x 0 minus x 1 so 2 x 0 plus 2h minus x 0 x 1 is x 0 plus h so minus x 0 minus h and that gives you h and therefore L 2 dash(x 2) has been computed. So we have evaluated all these derivatives at the nodal points.

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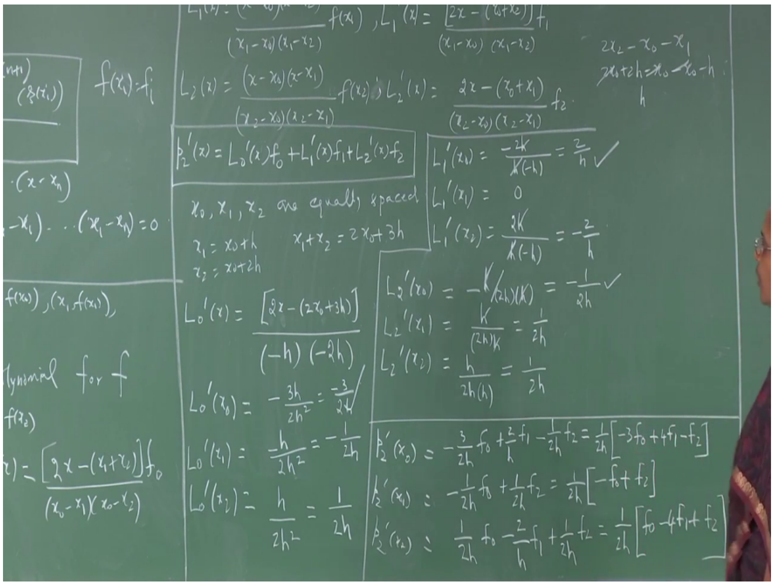

So we can now write down what is p n dash in this case p 2 dash at the point x 0 p 2 dash(x 1) and p 2 dash (x 2). So we have computed p 2 dash as a linear combination of the derivatives, so p 2 dash at x 0 will be L 0 dash(x 0) so minus 3 by 2h into f 0. Then plus L 1 dash(x 0) so 2 by h into f 1 and then plus L 2 dash(x 0) so minus 1 by 2h into f 2.

And so if I write 1 by 2h into minus 3 f 0 plus 4 f 1 minus f 2 then p 2 dash (x 1) so L 0 dash(x 1), what is it? It is minus 1 by 2h into f 0 plus L 1 dash(x 1) that is 0 plus L 2 dash(x 1) that is 1 by 2h into f 2 so it is 1 by 2h into minus f 0 plus f 2. Then p 2 dash(x 2) is L 0 dash(x 2) so 1 by 2h into f 0 plus L 1 dash(x 2) minus 2 by h into f 1 then plus L 2 dash(x 2) so plus 1 by 2h into f 2 so this again gives you 1 by 2h into f 0 minus 4 f 1 plus f 2 so you have evaluated the derivative p 2 dash at the nodes.

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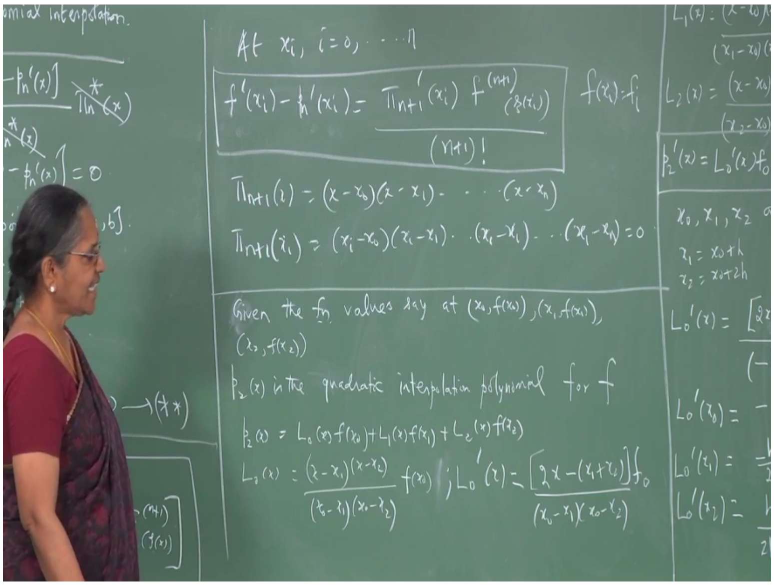

So we now have what is f dash(x 0) minus p 2 dash(x 0) so let us write down the terms on the right hand side and then get the estimate for this difference f dash minus p 2 dash at the nodes. So I have to now consider the right hand side.

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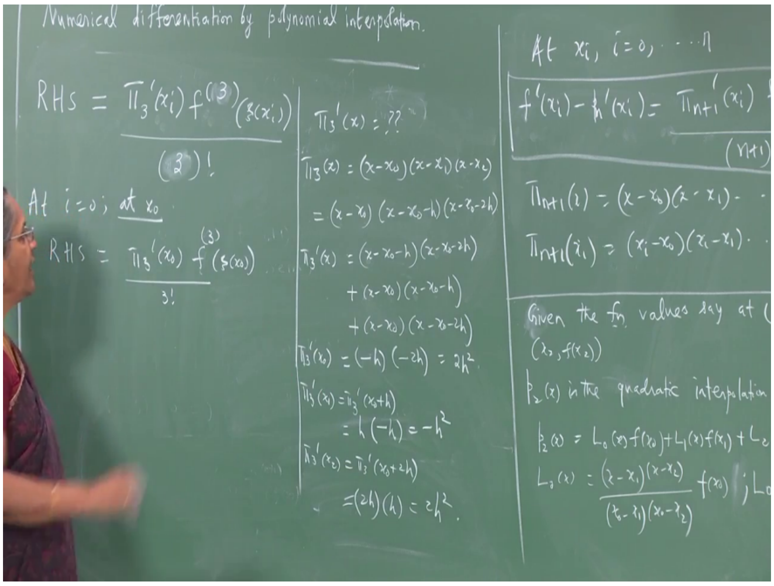

So the right hand side will be Pi n is 2 so Pi 3 dash (x i) into (n plus 1) th derivative (Psi (x i)) divided by n plus 1 factorial. So let us evaluate at i is equal to 0. Namely at x 0 the right hand side will be Pi 3 dash (x 0) into third derivative so here again n is 2 so i can write the third derivative so here again n is 2 so 3 factorial. Anyway the for any n the result is available here.

So I substitute n as 2 so I have to compute this so Pi 3 dash (x 0) into third derivative (Psi (x 0)) divided by 3 factorial. So we require Pi 3 dash (x) and then we shall evaluate at the nodes. What is Pi 3(x)? Pi 3(x) is x minus x 0 into x minus x 1 into x minus x 2, since we work out the details for equally spaced points so we can substitute the details so x minus x 0 into x minus x 0 minus h into x minus x 0 minus 2h.

So everything is in terms of x 0 and h. So we require the derivative Pi 3 dash(x) so that will be x minus x 0 minus h into x minus x 0 minus 2h into derivative of x minus x 0 which is 1 plus x minus x 0 into x minus x 0 minus h into derivative of this and thirdly x minus x 0 into x minus x 0 minus 2h into derivative of this term which is again 1.

So we evalluate this at x 0. So when x is x 0 you get minus h into minus 2h from here when x is x 0 this is 0 x is x 0 this is 0 so it is simply 2h square. And then let us find Pi 3 dash (x 1) what is it? It is Pi 3 dash(x 0 plus h). So I substitute x as x 0 plus h this goes and then x as x 0 plus h this term also goes so from here x 0 plus h minus x 0 so h and x 0 plus h minus x 0 minus 2h so minus h so this will give me minus h square.

And then let us work out Pi 3 dash(x 2) which is Pi 3 dash (x 0 plus 2h) so we observe that at (x 0 plus 2h) this part vanishes (x 0 plus 2h) this also vanishes so I have x 0 plus 2h minus x0 so 2h, x 0 plus 2h minus x 0 what is that? This is (x 0 plus 2h minus x 0) so I will get 2h minus h so which is h and so it is 2h square.

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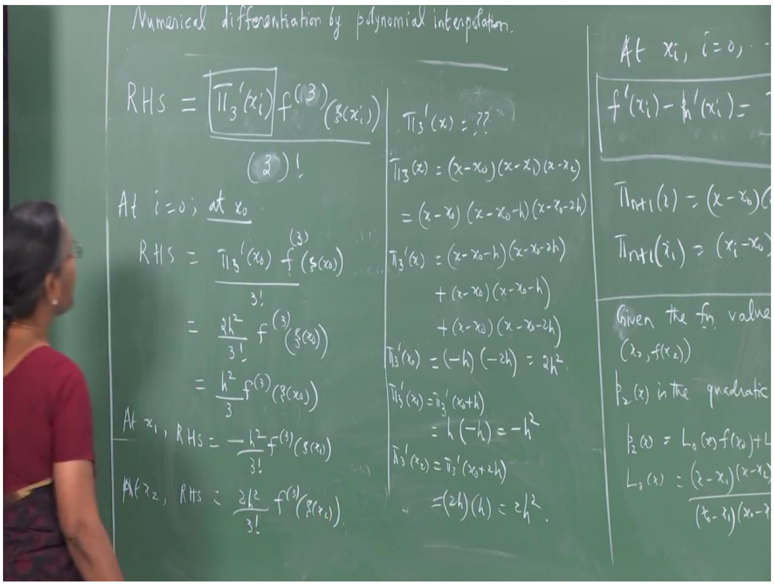

So in the right hand side we have this term and we have evaluated that term at three nodes x 0 x 1 x 2. So we shall now substitute what is P 3 dash (x 0) it is 2h square by 3 factorial into the third derivative evaluated at Psi (x 0) so it is simply h square by 3 into the third derivative Psi (x 0) then at x 1 the right hand side is going to be Pi 3 dash (x 1) so minus h square by 3 factorial into the third derivative Psi (x 1) and then at x 2 we have the right hand side to be given by 2h square by factorial 3 into third derivative Psi(x 2). So we have the expression for the right hand side we computed the derivatives of P 2 at the nodes so we can now write down this estimate at all the nodes.

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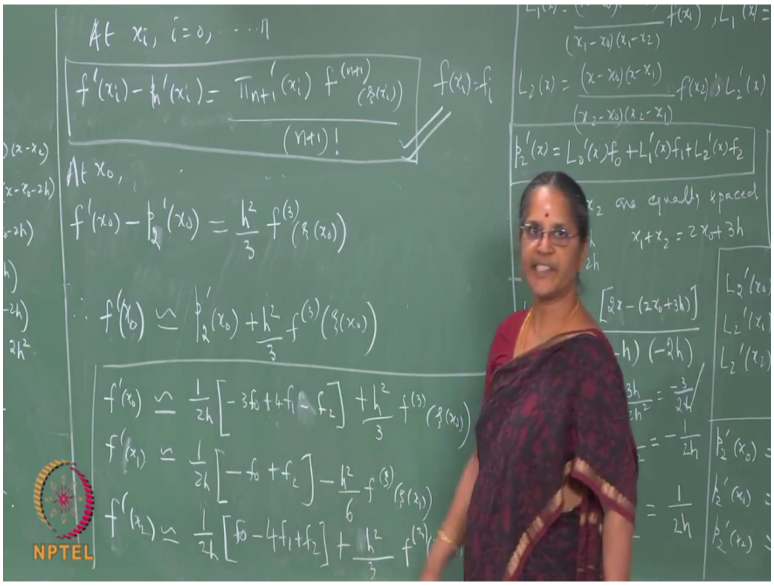

So at x 0 what do we get f dash (x 0) minus p n dash(x 0) is the right hand side which is h square by 3 into the third derivative of f so this tells you f dash (x 0) the derivative of f at the node x 0 can be approximated by here I have approximated by a quadratic polynomial.

So I shall write this as P 2 dash, so p 2 dash (x 0) plus h square by 3 into the third derivative Psi(x 0) and we have obtained an expression for p 2 dash (x 0) that is 1 by 2h so by 2h into minus 3f 0 plus 4 f 1 minus f 2, and error in this approximation is h square by 3 into third derivative evaluated at Psi (x 0) so that is the approximation for f dash (x 0).

So now I can write down the approximation for f dash (x 1) so it is p 2 dash which is 1 by 2h into minus f 0 plus f 2 and the error has been computed here which is minus h square by factorial 3 into the third derivative Psi(x 1) and finally the estimate for f dash(x 2) is going to be 1 by 2h into f 0 minus 4 f 1 plus f 2 and error in this approximation is 2h square by factorial 3 so we can simplify and write h square by 3 into the third derivative at Psi(x 2).

So we used this expression which was obtained from the `expression for error in interpolation by differentiating it and evaluating that at the nodes x i, i is equal to 0 to n and we illustrated this for the case of a quadrating interpolating polynomial that interpolates the function f at the set of three points x 0 f 0, x 1 f 1, x 2 f 2 and we now have approximations or estimates of the derivative at the nodal points.