Numerical Analysis
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Lecture 10, Part 1
Numerical Differentiation-1
Taylor series Method

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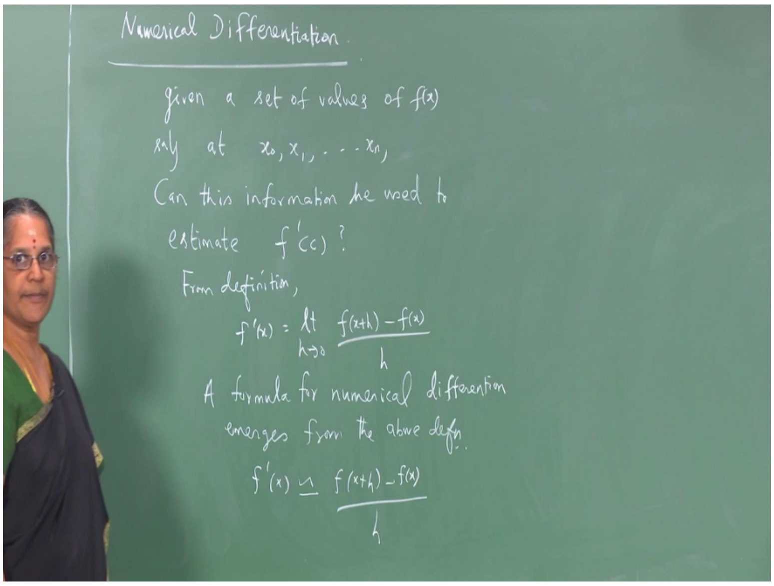

So we shall now consider the next topic of interest to us in this course mainly numerical differentiation. So given a set of values of f(x) say at x 0, x 1, ... x n, The question is can this information be used to estimate the derivative of f at some point say c? This question is answered in numerical differentiation. The answer is yes.

There are a number of methods which we would now develop with the help of which we will be inefficient to present an estimate of various order derivative of this function whose values are given at a set of points, say x 0 ......x n. So we know from the definition of the first derivative that f dash (x) this limit of h tending to 0 of f(x plus h) minus f(x) divided by h.

So this definition is such that a formula for numerical differentiation emerges from the above definition namely fpime (x) is approximately f(x plus h) minus f (x) divided by h. So the right hand side gives you an estimate of the derivative active point x.

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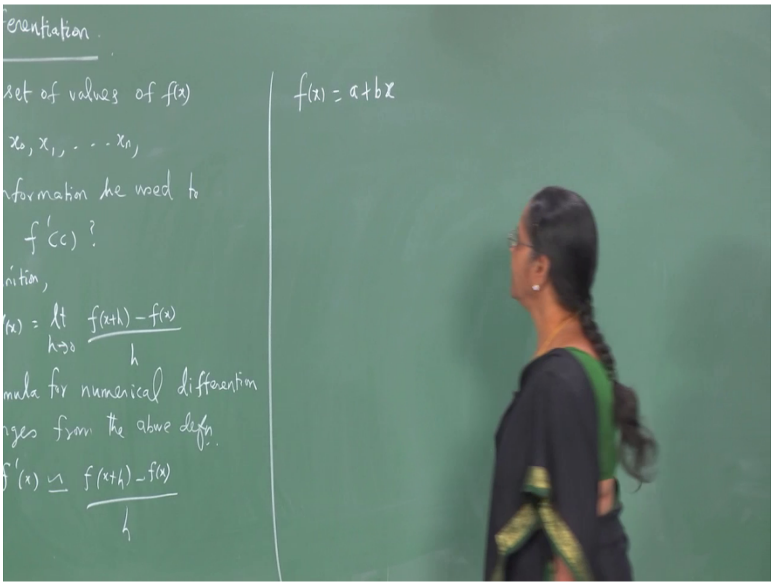

Suppose a f (x) is a linear function a plus bx. And let us see what this formula tells us.

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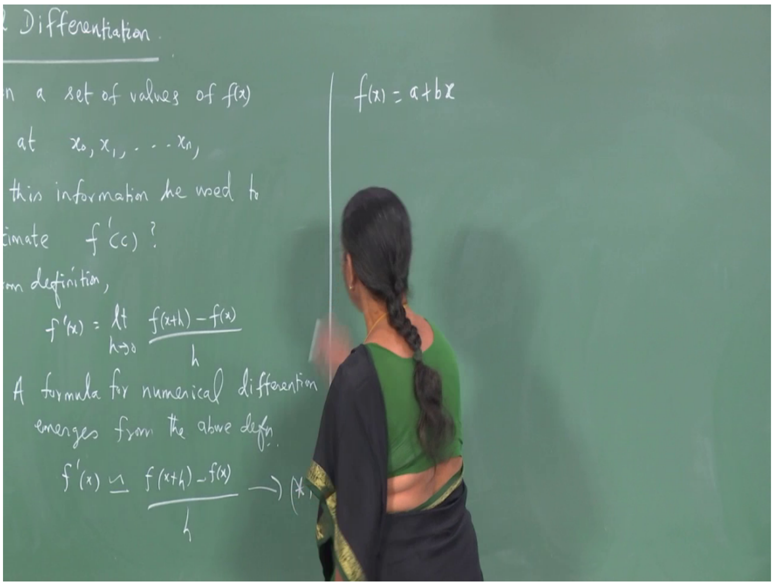

Suppose I call this as star.

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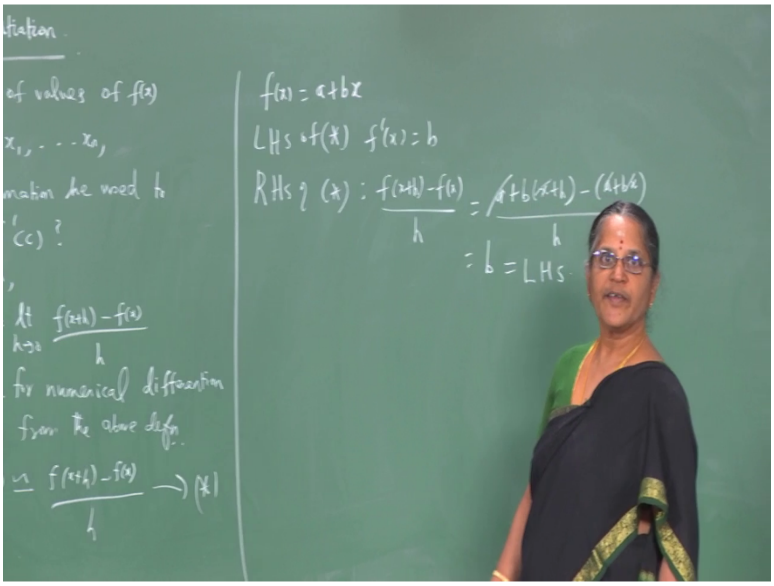

Then Left hand side of star is f dash (x) and that is b. So I compute the right hand side of star for this function. So it is f (x plus h) minus f(x) divided by h. So it is a plus b into x plus h minus a plus bx divided by h. So that gives you the h by h which is b and this is the left hand side of star. So the formula is exact for a linear function or a linear polynomial. So we would like to see the expression for the error that is incurred when we used star to estimate the derivative of a function.

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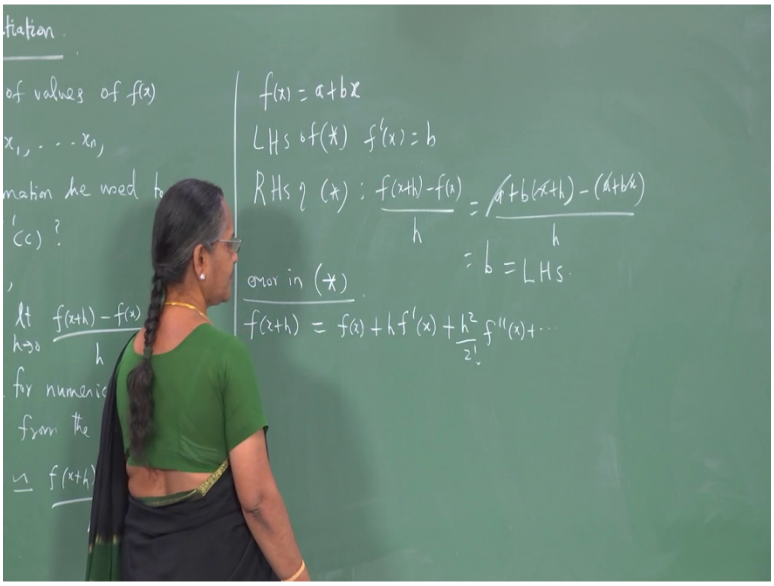

So let us compute the error in this approximation star so to do that let us evaluate f(x plus h). Use Taylor series expansion so this will give you f(x) plus h f dash(x) plus h square by factorial 2 into f double dash of x plus etc.

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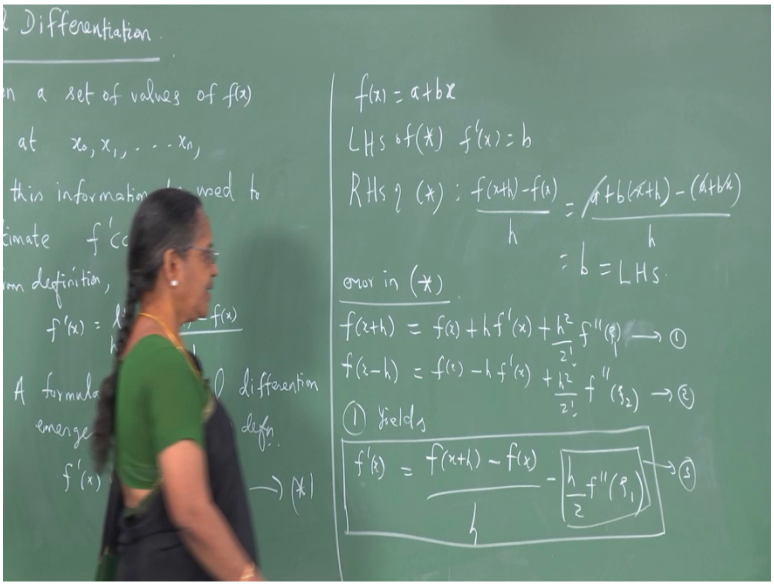

Suppose I truncate this at this stage then this will be f double prime at (Psi). And f at (x minus h) is going to be f(x minus h into f dash (x) plus h square by factorial 2 into f double dashed at say some other point (Psi 2). Let us use the result say 1 in the writing what f dash (x) is?

So f dash (x) will be f (x plus h) minus f(x) divided by h minus h by 2 into f double dashed at (Psi 1). So this tells me that I can approximate f prime (x) by this term which appears as the first term here and the error that I incurred is given by this term which is h by 2 into f double prime at Psi) 1.

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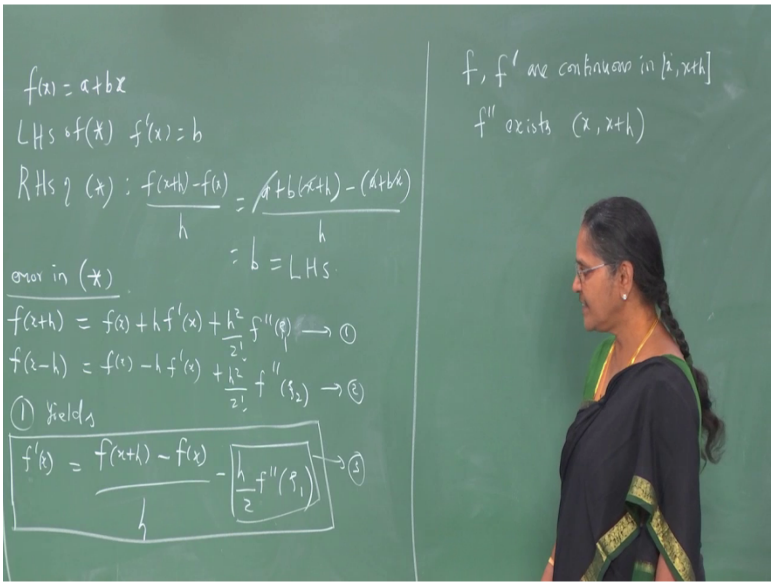

The approximation of f dash (x) by star is valid when I use 3 2 approximated we require that the function f, f prime are continuous in the interval x to x plus h and that f double prime exists in the open interval (x, x plus h). So the error term which consists of these two factors h by 2 and f double prime tell us to what class of functions that f must belong to so that I can estimate its derivative by the formula star.

Mainly this term tells us that f much be such that if it belongs to the class of functions where f, f prime are continuous in x to x plus h and f double prime exists in the open interval x to x plus h then I can estimate the derivatives of all functions belonging to this class by star. Mainly the first derivative can be estimated approximated by the expression on the right hand side in star.

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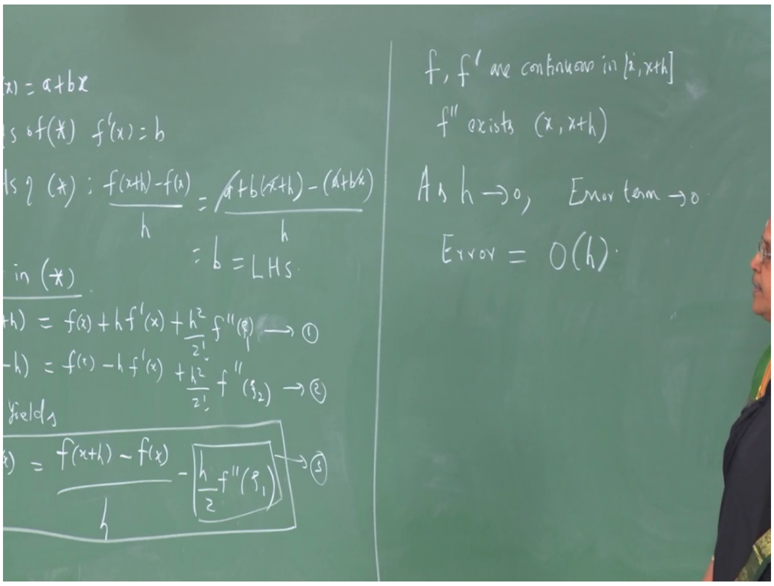

But what about this factor h? That tells me that what happens as h goes to 0 h goes to 0 the error term goes to 0. So this gives me the order of accuracy of the formula that is used in the estimation. Namely the power to which h is raised to gives you the order of accuracy of the method and therefore the method 3 which can be used to estimate f prime x in the interval x to x plus h is of order of h. So the error in 3 is error in star is given by the term here in 3 and it is first order of h.

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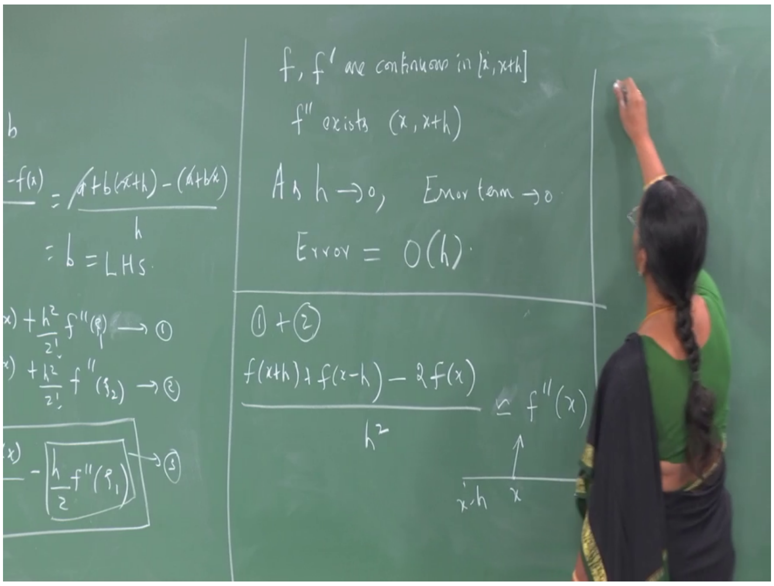

So having obtained a method or a numerical differentiation formula which directly emerges from the definition of the first derivative we would like to obtain some more numerical differentiation formulas. So let us now use result 1 and 2 and see what we get. If suppose I take 1 and 2 and add then f(x plus h) plus f (x minus h) then minus twice f(x) divided by h square is f double prime at x.

So I have a numerical differentiation formula for the second derivative of x so that if x is here then x minus h and x plus h are symmetrically located about the centre point x. So if I want the second derivative at this centre point then it can be estimated by the values at the points which are symmetrically located on either side along with the value of the function at this point given by the expression on the left hand side.

Here again we can compute the error in such an approximation and that can be done by writing down the next term here. So let us do that and write down the error in this approximation.

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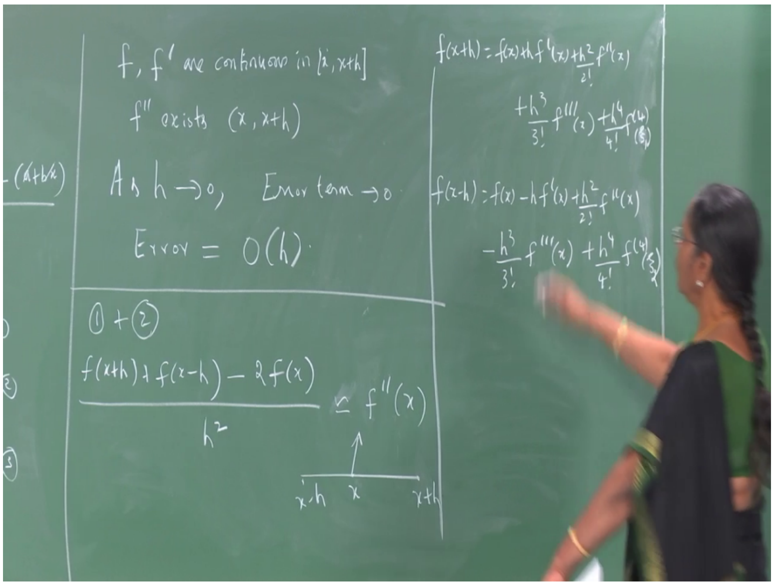

So f(x plus h) is f(x plus h) f dash (x) plus h square by 2 into f double dash (x) plus h cube by 3 into f triple dash at (x). I shall write down one more term you will see why you will do it the fourth derivative at x.

And I am going to truncate here so this will be the remained at. What is f (x minus h) it is f(x minus h) f prime (x) plus h square by factorial 2 f double prime (x) minus h cube by factorial

3 into f triple prime (x) plus h power 4 by factorial 4 into fourth derivative at some say Psi 1 at some say Psi 2.

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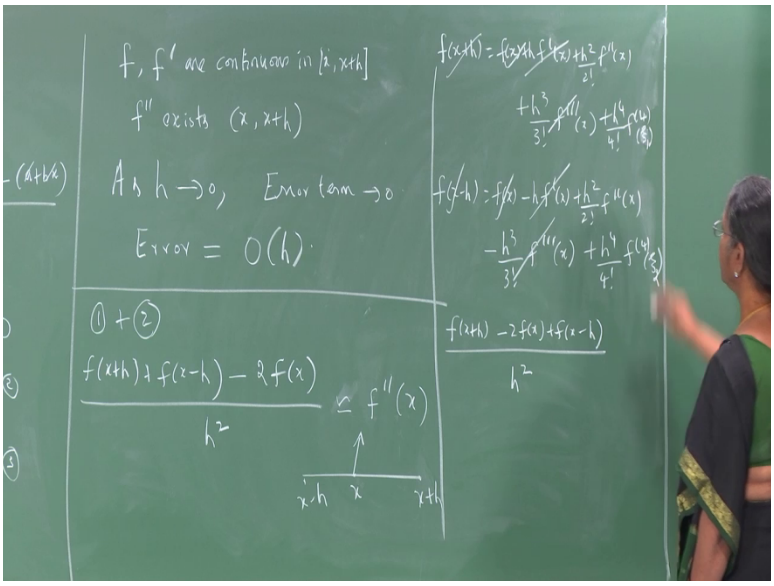

So now what did we do we added the two so that gave us f (x plus h) minus twice f(x) plus f(x minus h). So we observe that when we are this terms cancles with this. And then when we add we get h square f double prime x so I divide by h square. And then when we add these two terms cancel.

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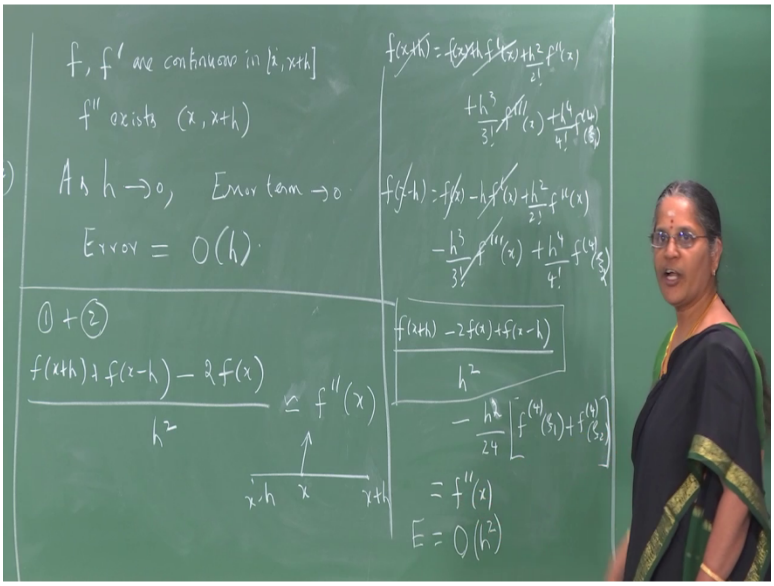

And these two terms right give you minus h power 4 by 24 into the fourth derivative at Psi 1 plus the fourth derivative at Psi 2 and that is f double prime at (x). So you have f double prime (x) to be estimated by the first term and the error that is incurred in such an estimation is given by the term which is this.

Remember we had divided by h square and so the term here will be h square. So the error is of the order of h square. And you now can give the class of functions which can be approximated whose second derivative can be approximated by this formula they stand the error term namely it involves the fourth derivative and the factor which involves h square tells you that the error is of order of h square.

So let us try to see how we can present this error term. So it is the fourth derivative at Psi 1 plus the fourth derivative at Psi 2 which should be now expressed in such a way that we make you solve some properties of the function that we have considered.

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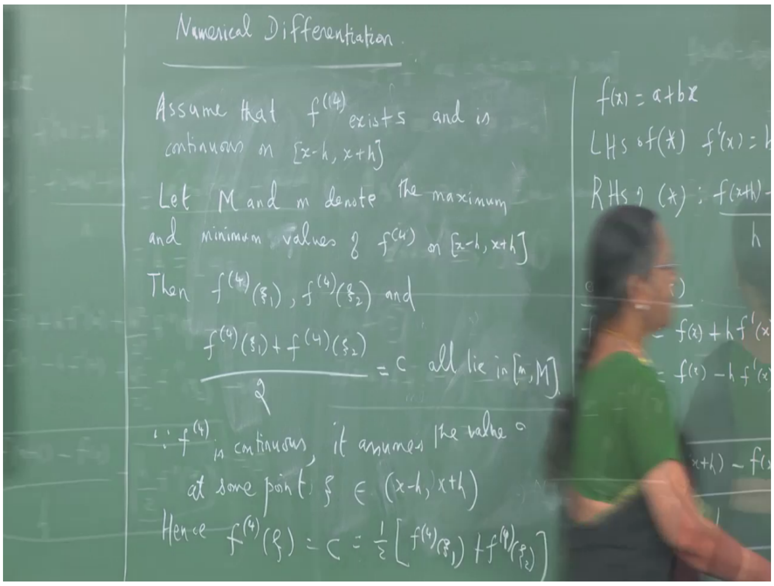

So we assume that the function belongs to a particular class such that the fourth derivative exist and is continuous and where all the interval (x minus h) to (x plus h). And let capital M and small m denote the maximum and minimum values of the fourth derivative on this interval x minus h to x plus h then we know that the fourth derivative at Psi 1 fourth derivative at Psi 2 and the fourth derivative at Psi 1 plus the fourth derivative at psi 2 by 2, let us call this as some c.

They all lie in the interval small m to capital m. Because M and m are the smallest and the largest values of the fourth derivative in this interval. So these lie in the interval m to M. And since this fourth derivative is continuous which assumes the value c at some point say psi which lies in the interval (x minus h) to (x plus h). So hence f fourth derivative at Psi is c which is half of the [fourth derivative at Psi 1 plus the fourth derivative at Psi 2].

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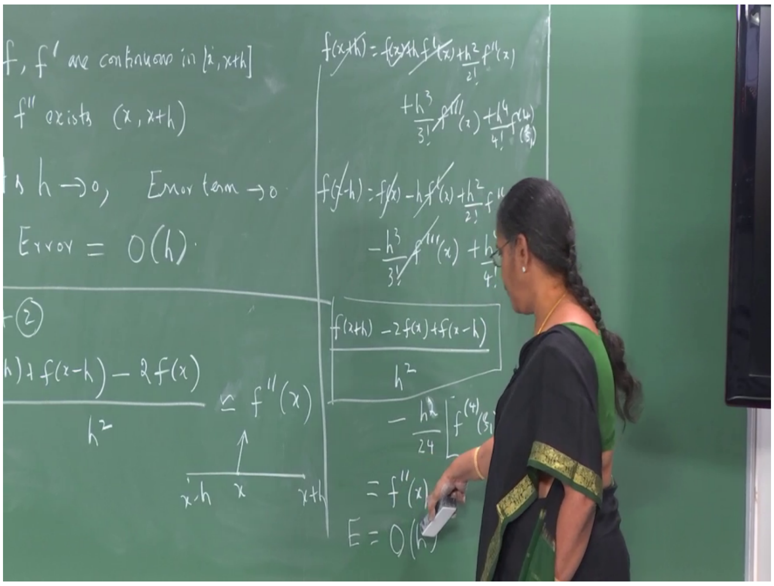

And therefore now we can give the numerical differentiation formula for this second derivative in terms of what we have obtained.

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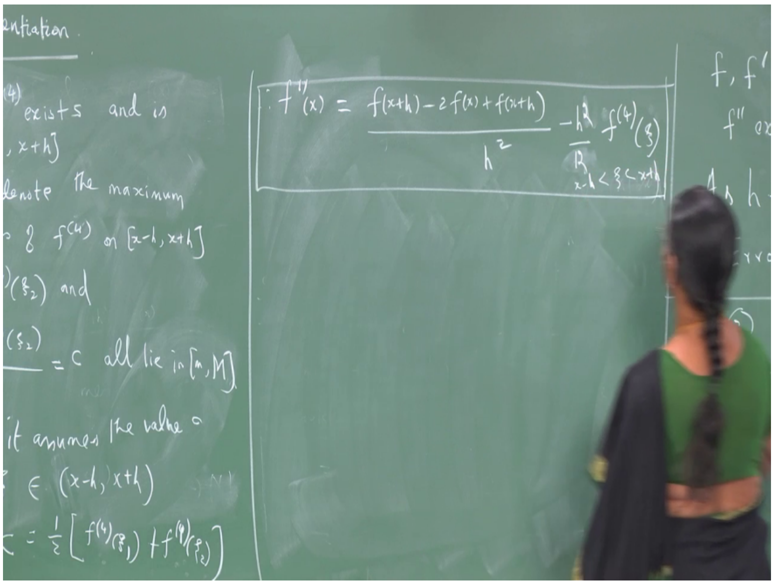

So f double prime at x will be f(x plus h) minus twice f(x) plus f(x plus h) by h square then minus h square first derivative at Psi so this gives you a numerical differentiation formula for f double prime x where Psi lies between minus h and x plus h.

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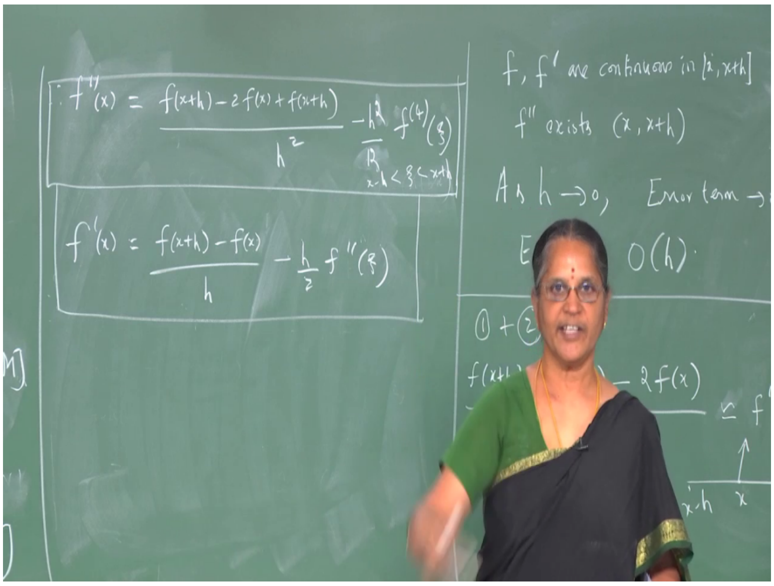

So one can use error expansions of the function at different points like (x plus h) (x minus h) etc. and obtain various numerical differentiation methods involving say two points for f prime x three points for f double prime x. and you can also obtain numerical differentiation formula involving more number of points for estimating f prime (x) similarly for estimating f double prime (x) or higher order derivatives.

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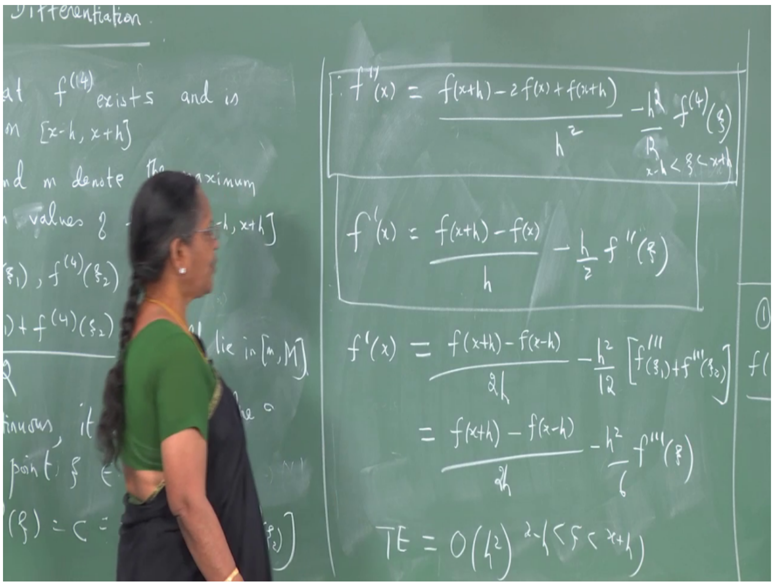

f prime (x) can also be obtained using the estimation f at (x plus h) minus f at (x minus h) by 2h and the error involved is minus h square by 12 into f triple prime at (Psi 1) plusf triple prime at (Psi 2). And by arging out in a way similar to this while deriving an approximation of f double prime you can show that f prime (x) can be estimated using the first term in this formula and the second term is minus h square by 6 into f triple prime at (Psi) where Psi lies between x minus h and x plus h. So that the truncation error in this case is first order of h square.

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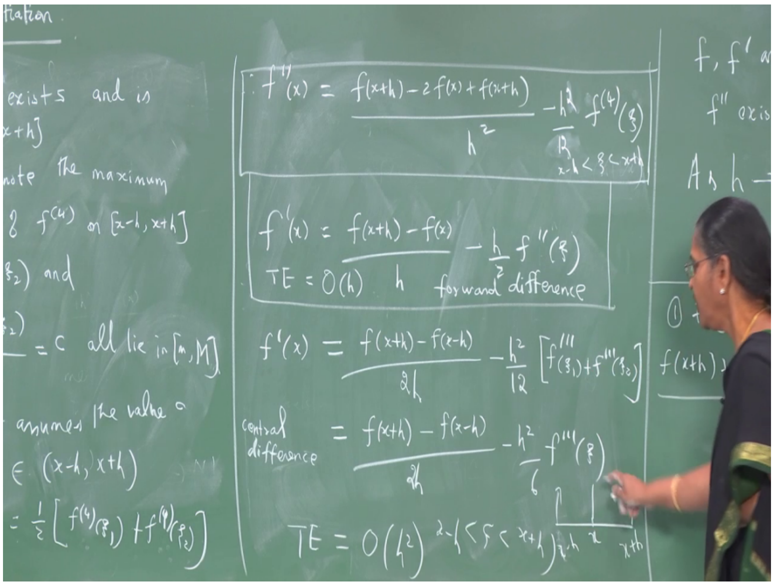

And you observe that while the truncation error in this method is of order of h in this method you are able to get the truncation error to be of the order of h square. these are called forward difference approximations for the derivative and this is called a central difference approximation for the first order derivative because the derivative at the central point x is approximated in terms of the values of the function at x minus h and x plus h here.

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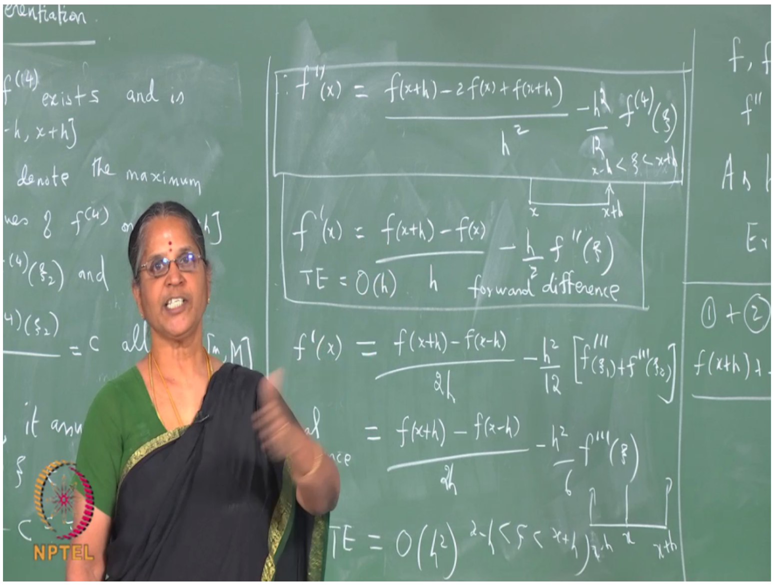

Whereas here the derivative at any point x is given in terms of values of the function at a point which is x plus h and it also involves the function value at x. So one can also derive a backward difference approximation to the first order derivative. And that will be given by f prime x is equal to f (x) minus f (x minus h) by x with error term which will be similar to this expression.

So we shall now consider numerical differentiation methods which can be obtained in case we are given the set of information about f at points which are equally spaced the method is known as the method of undetermined coefficients.