Course on Differential Equations for Engineers
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Lecture 56
Non-homogeneous heat equation

So welcome back in the last video we have seen how to the solve a heat equation for semi-infinite domain that is from 0 to infinity with the initial data and boundary data at x equal to 0. So when you give two different types of boundary data so whether it is insulated at that end or you maintain the temperature 0 temperature at that time, so both the problems are solved by just by extension as extend the functions as even or odd functions depending on the boundary condition as we have done for the wave equation.

So this technique we have already seen for a wave equation and then we see that the solution of initial value problem for the heat equation is actually some integral type of solution. So that integral is actually so this integral we have just seen that kernel of that integral solution is actually nothing but some delta function that is what we have seen, so that is related to the delta function that is what we have seen.

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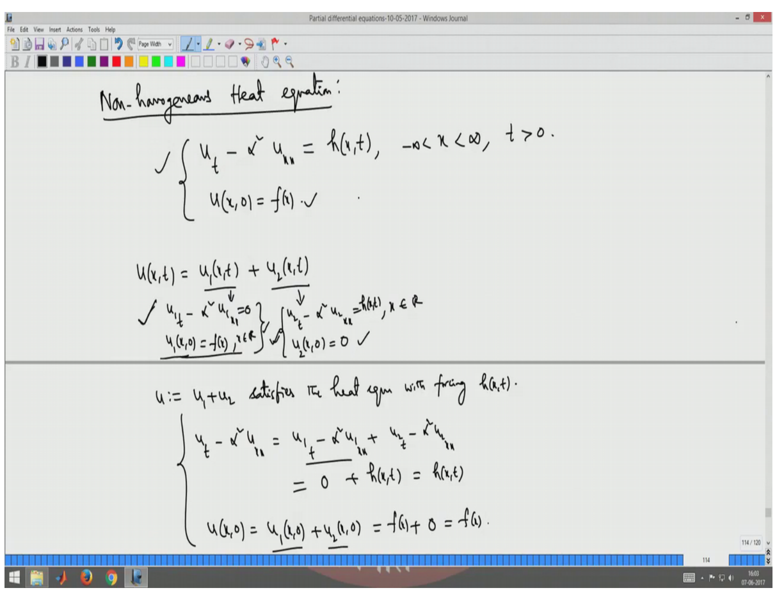

So then we looked at the non-homogeneous heat equation so that is what we have here so this is the non-homogeneous heat equation and this we can solve by splitting into two problems one is the homogeneous problem homogeneous problem this initial data whatever is given for the non-homogeneous problem and the other problem is u2 that satisfies non-homogeneous equation with 0 initial data.

So that is the domain is full domain x belongs to full R so what you have is the full R, so if we know the solution for the initial value problem for the homogeneous equation so that is actually the heat equation that we know that is the solution and what we do not know is this second part so u2 we do not know so if we solve this u2 we can combine this u1 and u2 to the solution of the initial value problem for the non-homogeneous heat equation.

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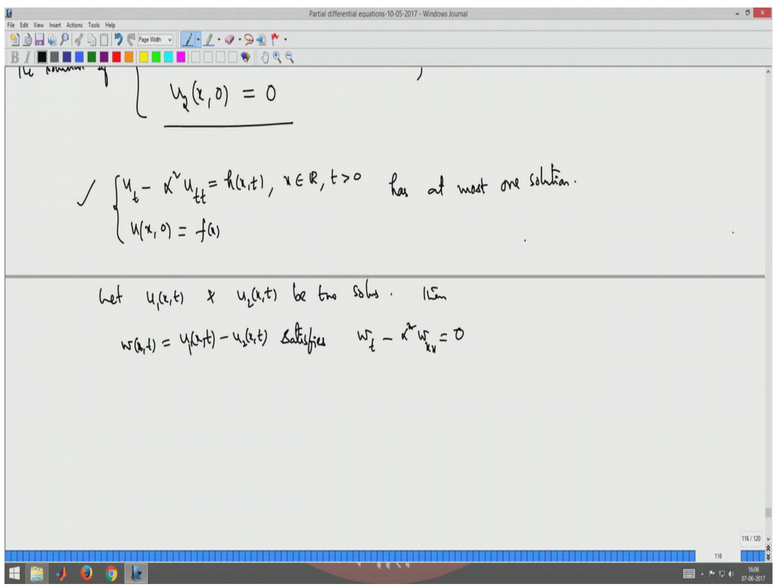

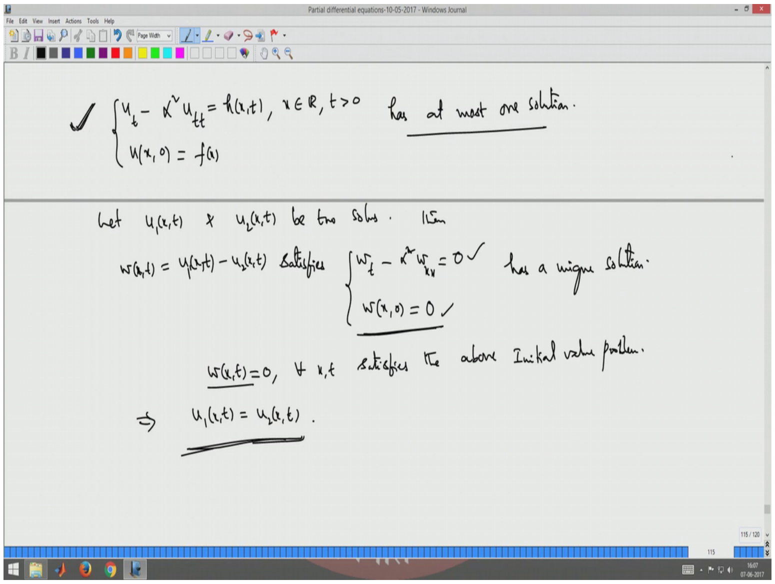

So we will see how to solve this here so look at this so what we need is this one so to solve this so this is the non-homogeneous equation so if you want to solve this we know that if you take let us take non-homogeneous problem simply you take the non-homogeneous problem ut minus alpha square utt equal to 0 equal to non-homogeneous problem that is h of x, t x belongs to R t positive, okay so and u at x, 0 is let us say some fx.

So this is the non-homogeneous problem and uniqueness this has a unique solution has a unique solution has at most one solution, okay at most one solution because we have not find the solution for this non-homogeneous equation so we cannot say whether it has a solution but you can say that it has at most one solution with this so not more than two solutions you have for this problem, okay that we can easily show just by taking u1 is one solution let u1 of x, t and u2 of x, t be two solutions then as usual you consider a w of x, t as u1 of x, t minus u2 of x, t so if you do this satisfies this satisfies the heat equation so wt minus alpha square wxx.

So u1 x u1 t minus alpha square u1 xx is actually h of x, t. And similarly u2 t minus alpha square u2 xx is also h of x so now this becomes homogeneous w satisfies the homogeneous equation and the w at x, 0 is also so that is 0 initial condition is also 0. So we have seen by the energy argument that this problem has a 0 solution has a unique solution, okay homogeneous equation with the initial data has a unique solution, okay.

So we know that this has this homogeneous equation with the initial data has a unique solution and clearly 0 satisfies this, so w x of t is completely 0 for every x, t satisfies this problem, okay the above problem above initial value problem. So that means this should be the unique solution so that implies u1 has to be u1 of x, t equal to u2 of x, t. So this means non-homogeneous equation this one the non-homogeneous equation with the initial data has at most one solution.

So if you can construct one solution so that should be if you can construct a solution that should be the unique solution for this problem. So I construct in a notorious way, okay so some special way I try to construct a solution for this non-homogeneous equation with actually not this one so we try to find non-homogeneous we try to find the non-homogeneous solution we try to find a solution for this non-homogeneous equation with 0 initial data.

So u2 we will try to get it, so once you get this u2 now if you take f equal to 0 here now we know that if you choose in particular f is 0 then this problem has at most one solution now if I construct one solution so that means you have this solution, okay so you have a unique solution, how do we do this? So we try to find in the solution for this u2 in a special way.

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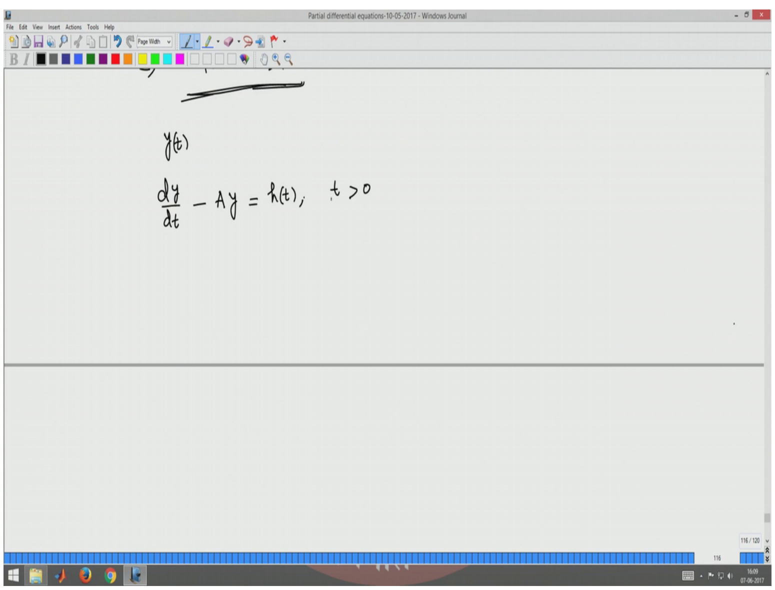

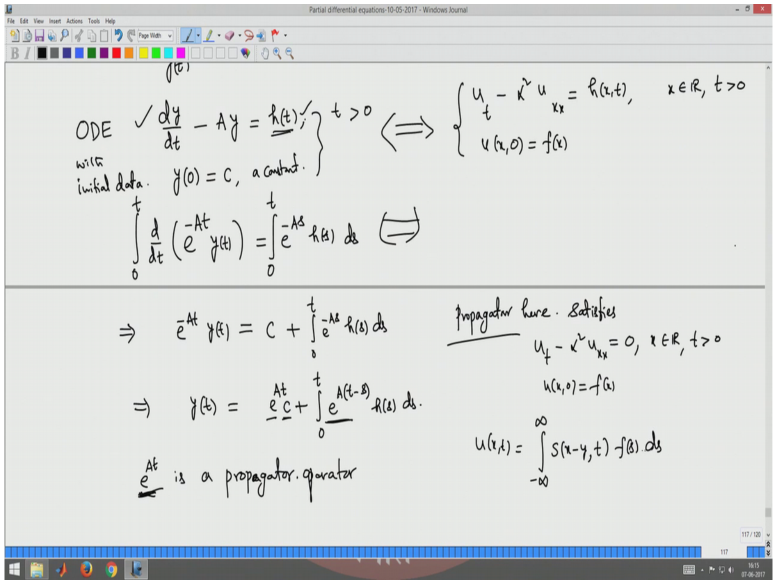

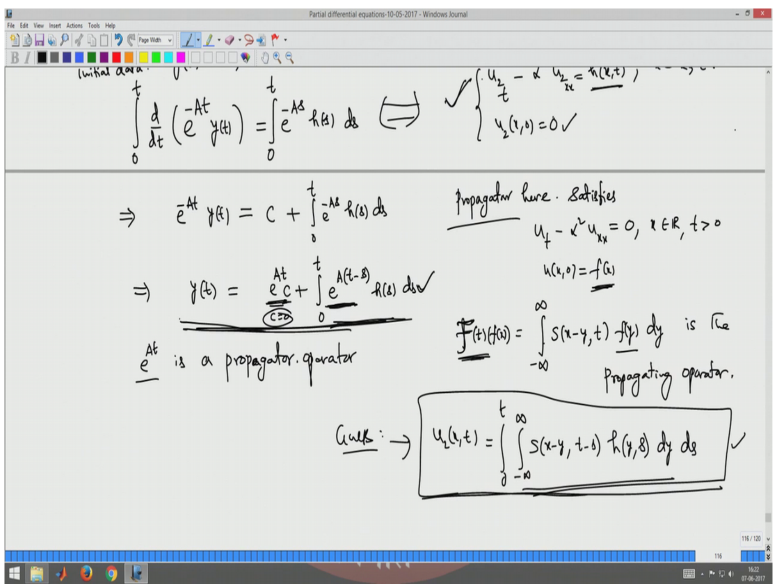

So let me construct something similar similar is you consider du by dt so let us say u is a let us call this not u some other function so let us say f of x let y of x is y is a dependent variable x is independent variable.

Now consider this linear equation dy by dx equal to dy by dx minus A times y, A is a constant, okay equal to and y is a function of x so you have a non-homogeneous term that is h of x. So let us see this is some x positive let us say, okay so if you want positive so you can say if you want t is the time so as a time you can have this t positive so that you have a t positive, okay. So this if you have and initially you have a at t equal to 0 initially you have some constant so let us say C a constant, okay a constant.

So what is the solution here, so how do we find this solution for this ordinary differential equation, so consider this ODE with the initial data with initial condition initial data. So if you see this we know that e power minus At is integrating factor so that you can multiply this y of t so this is the derivative of this you just multiply e power minus At both sides what you get is e power minus At h of t, okay.

So now you integrate both sides this is t, okay so you can integrate now both sides from 0 to t so t is positive so 0 to t now here inside t is a dummy variable so I can write as As h of s ds, okay. So this will give me my solution as e power minus At y of t minus e power A into 0 that is 1 so you have y at 0 that is C, okay so that is C so if you bring it to the other side that is e equal to C times C plus integral 0 to t e power minus As h of s ds, okay.

So what is the solution now? y of t is C into e power At plus e power At is a function that is outside you can take it inside integrant that is only variable functions whose variable As integration variable is s if you combine this e power minus At e power As so what you get is e power minus A s minus t or e power A t minus s h of s ds. So you can see that for this problem this is called a propagator e power At this function is a propagating propagator.

So it takes this initial data plus and it also takes acts on this propagator operator is acting on the non-homogeneous data. So something so along with this we can draw parallel to this kind of whatever is happening here for this ordinary differential equation with the initial data, okay. So if you can do the you can draw the parallel for the heat equation along these lines so what is it corresponding so we just write I will just write it parallel to this what you have is u2 t minus alpha square u2 xx equal to h of x, t so this is what you have so x belongs to R t positive so this is my problem for u2.

And what is the initial data? u2 at x, 0 equal to 0 that is exactly my yeah u2 of x, 0 is 0 so this is what is the initial data, okay. Let us choose some data so some function of x let us not give simply initial so that is for u so u2 so let us consider only u we will solve we will consider this u2 as the special case, okay. So let us say you solve this non-homogeneous problem with this initial data. Now if you draw parallel here with this, okay so if you see here corresponding correspondence here what you have so how does it looks so what is the solution so this is actually e power At is a solution of if h is 0 then this is the solution of homogeneous equation with the initial data.

So I know if h is 0 I know its solution, okay so that solution is that my propagator, okay. So in this case what is so question is what is a propagator? So propagator here is actually if you see this is a propagator this actually solves the homogeneous equation with the initial data, okay it actually solves the homogeneous equation with the initial data, so what is the homogeneous equation here? So if I consider here should be satisfying ut minus alpha square here should satisfies ut minus alpha square uxx equal to 0 I consider homogeneous and the initial data initial data is fx, okay.

So this should be the propagator, propagator here satisfies like this so if I draw the parallel here e power At satisfies homogeneous equation, okay initial data is different so C into C that is, okay. So you can write this is a constant again write it here so this is an operator propagator operator if you view like this e power At acts on the constant the initial data C, okay. So homogeneous equation with the initial data so the propagator is one homogeneous solution acting on the initial data.

So what happens here I know the solution here u of x, t is actually what is a solution for this initial value problem which we know that is from minus infinity infinity S of x minus y comma t and fs ds or fy dy. So this is what is my homogeneous solution with this initial data so this should be my propagator so propagator here is this one, okay. So this is the propagator here so this is the operator so operator acting on initial data, so I can write this this propagator operator as here some capital F propagator operator on this f of x, t f of x, right? Propagator operator so let us say which is a function of t like here e power At is function of t, okay so let me put it as a script so script is script of function of t and then acting on f of acting on f fx, okay acting on fx if you acting on fx that should be finally function of x, okay.

So acting it takes fx x is from infinity infinity to finally another function of x of course t is also there. So this is the propagating operator I am just guessing, okay so just based on this something here for the ODE we see that this is the propagating operator so in this case this is a which is actually homogeneous solution acting on so e power At C is a propagating operator acting on the initial data C, so we draw the parallel here so this is the propagating operator.

So if this is the case now you look at this one and you can say what is the solution of the non-homogeneous equation. So if I take initial data 0 so this would not be there what is left here is this one, so this should be non-homogeneous solution if you choose this initial data as 0, okay if you take initial data 0 so if you take initial data 0 C equal to 0, okay C equal to 0 if you take this part will not contribute only non-homogeneous terms so if you can write this now the solution of the initial data so you can write u2 u2 solution that is alpha square u2 xx equal to x, t this is a non-homogeneous equation and then with the initial data initial data is 0.

So this one you can get now you can write as this part so that implies u2 of x, t is now what is this one this is actually integral 0 to t here, now what is my propagating operator? So I have a propagating function f of so this f of t minus s acting on h of s so what is h of s here so that is non-homogeneous term, here my propagating operator is acting on the function fx, okay which is x variable in this integral.

So this is t minus s acting on h of x, t, okay d so what is the finally ds so t is the dummy variable so t is actually should be s, right? So t should be t you are integrating so the t variable you are integrating, okay so instead of this let me write completely here so what is this f of so if you see e power A into t minus s that is actually minus infinity to infinity s of x minus y in the place of t I have t minus s.

Now this propagating operator is acting on the initial data f of y so here it should be acting on the non-homogeneous data non-homogeneous data that is h of x, t. So if you act on fx that is f of y here inside the integral if it acts on the h of x, t it should be function of h of y and this should be t, okay this is actually t acting on t and that is dy this should be function of t that you are integrating from 0 to t, okay. So wherever t is there I am just putting s so this is s this should be s ds you consider this one as.

So if you consider this so if you act this you use this propagating operator acting on this non-homogeneous data h of x, t then this operator acting on h of x, t so wherever t is there, okay so t is the derivative here so that is actually so independent variable here so h of s. So wherever t is there you have s, so that is why so operator wherever when propagating operator acting on this h of x x variable will become y and t variable is actually become s as if you see here, okay.

So this is how this is only this is how I guess, okay this is what is my guess is this is what is my guess as a solution of the problem that I am looking for, okay just by looking at this ordinary differential equation with this initial data. So I cannot say this is how I found the solution this is a notorious way of getting it, okay just I am just guessing that this is the solution so if you want to say that this is the solution you just have to verify it, okay. So u2 so what you have to say that it has to satisfy this problem this initial this non-homogeneous equation with this 0 initial data.

So you can easily see that at t equal to 0 0 to 0 this integral it has to be 0. So clearly it satisfies u2 this satisfies this initial data and you just you have to see that it satisfies the non-homogeneous equations.

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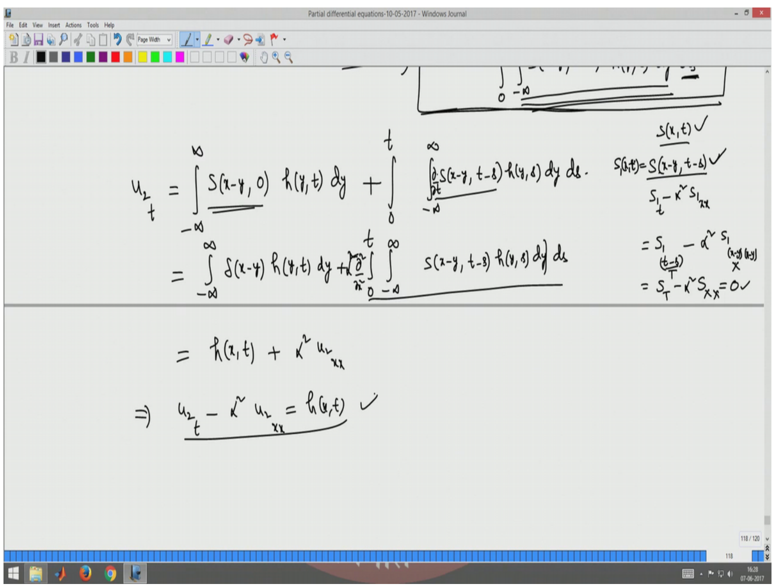

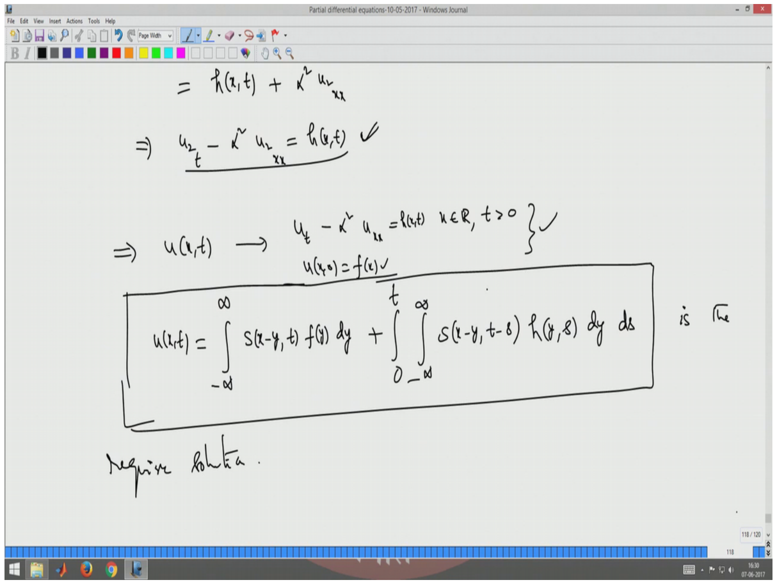

So for that you calculate u2t of x, t u2t you calculate so what is this one u2t if you do I am differentiating this t so if you do this if you differentiate this you differentiate this (())(20:50) rule you differentiate this d dt of this d dt of t that is 1 and wherever s is there s is the variable this is the integrating variable for this integral 0 to t wherever s is there you put t, so if you do that what remains is the first one is s of x minus y t minus t that is 0 and h of y s is t y of t dy that is what you have plus.

Now you can integrate whatever is the integrant so integrant is a function of t so that you can integrate so now you can write integral 0 to t doh doh t of this integration so this integral whole integral you are just integrating with respect to s with respect to t h of y, s dy ds a simple integration and then your, okay so you just took this t derivative inside that is what you have. And you can see that that we know yesterday we have seen that we see that s of x minus y, 0 is nothing but your delta function, so you have minus infinity infinity delta function of x minus y acting on h of y, t dy plus and this now if you see this one so doh doh t of this function s of you see that s of x minus y and t minus s satisfies the heat equation, how do we see this? s, t minus alpha square Sxx is actually equal to S t is now t minus s, okay we know that s of x, t is satisfying heat equation.

Now you want to show that this is a this is also satisfying the heat equation, so for that you need St so S is this one, okay so S is a function of x minus so this you call this S1 S1 of x, t as this one so if you do that S1 t S1 t and you want to calculate S1 xx. So what you get is S1 with respect to t minus s dt minus S by dt that is what is S1 t so that is simply 1 minus alpha square S1 xx so that is S1 xx is x minus y is one variable x minus y is another variable so there is like d square S1 by d x minus y whole square, okay something like this.

So but we know that so this is exactly equal to S1 of this is simply exactly if you replace t1 s by some t variable and this x1 by some x variable this is nothing but St minus alpha square S xx so this we know that is 0 because S of x, t is satisfying the heat equation. So in that sense S of x minus y, t minus S is also satisfying the heat equation, okay. So you can actually take this derivative inside and you can write doh doh t here.

So if you do that, okay so if you do this one because it satisfy the heat equation so you can see that 0 to t minus infinity infinity now St is nothing but alpha square that you can write it as alpha square doh doh doh square by doh x square of S of x minus y t minus s h of y, s dy ds. So this is now if you operate this delta this is nothing but h of x, t plus alpha square now this one what is this one, it is actually equal to alpha square into now I can take this doh doh see this integral is from minus infinity infinity this is 0 to t so I can take this doh square by doh x square outside.

So I can write this outside here so doh square by doh x square I can write it outside. So now whatever is here this dy integral is nothing but your what is this one doh y integral is, what is this doh y integral? Actually you can take it outside so not just here you can write actually so this is that see this St is Sxx so the doh square by doh x square I can even write doh square by doh x square outside. So this whole thing is nothing but your u2, so that means u2 xx what you have.

So this implies u2t minus alpha square u2xx is nothing but h of x, t. So that means u2 that is this this satisfies the non-homogeneous heat equation that is this and it also satisfying when t equal to 0 is simply 0. So it satisfies this one this problem so you have this this problem is solved so u2 satisfies that initial value problem. So once you know this u2 you can write your solution of the non-homogeneous equation with the initial data as fx, so that you can write as u1 plus u2 so this implies u of x, t is u of x, t so that satisfies ut minus alpha square uxx equal to 0 x belongs to R t positive, sorry this is this should be h of x, t non-homogeneous equation and u at x, 0 equal to fx if you have this problem if you are satisfying this what you get is the u of x, t is u1 u1 is the homogeneous equation with the initial data fx.

So that is simply minus infinity infinity S of x minus y, t acting on fy dy so that should be your solution plus this whatever you found as u2 so u2 is this one integral 0 to t now integral minus infinity infinity S of x minus y t minus s now this is a the non-homogeneous term h of y s ds dy first and then ds s is the integrating variable from 0 to t, so this is my solution for the non-homogeneous problem, okay this is the required problem so this is the required problem required solution solution for this non-homogeneous heat equation with the initial data fx, okay.

So this is how you can find so this is what you the technique that you used to draw parallel to this initial value problem for the ordinary differential equation and just guess one solution and you just verified that it is a solution because it is a solution is unique and you cannot have more than 1 solution, okay and I have a solution I see that it is solution you cannot have more than one solution, okay we have shown that it has only at most one solution whatever it is so I give you that solution by guessing it and by verifying it is actually a solution so that means it is the unique solution.

So you have the unique solution u2 and you know that this is the unique solution for the u1 problem. So finally the u is for this problem this has a unique solution, okay.

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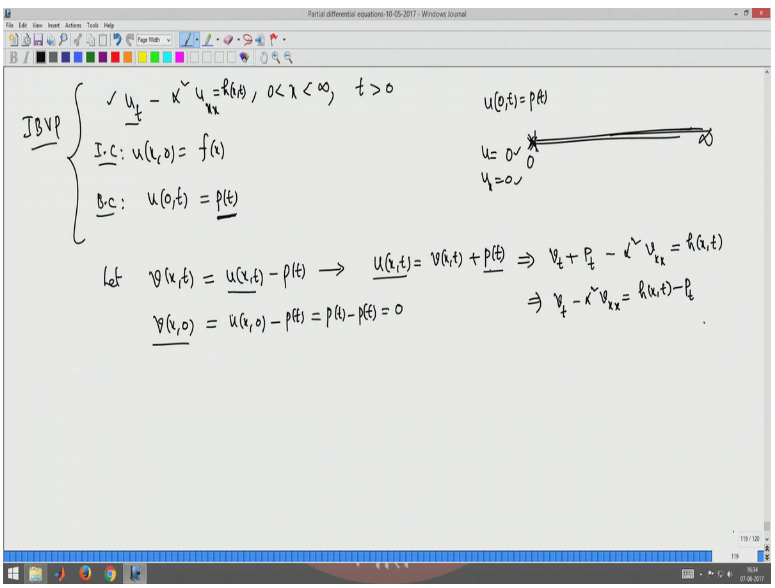

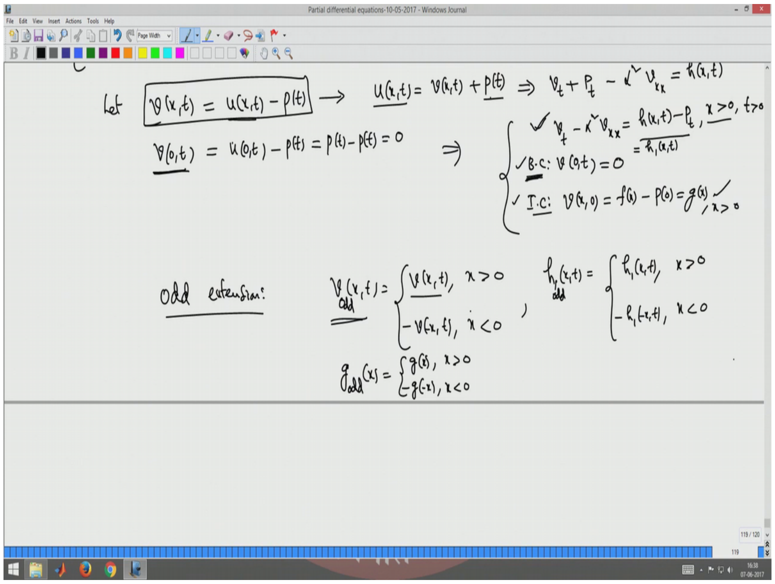

So far we have considered infinite rod minus infinity infinity for this non-homogeneous equation, so you can also do same thing on the semi-infinite line 0 to infinity semi-infinite rod you can have non-homogeneous equation so let us consider ut minus alpha square uxx equal to 0 x is positive so x is this is a semi-infinite rod and t is positive.

So now you have the initial data as fx and let us say here so far we have considered only 0 u is 0 or ux equal to 0 either you insulate or you keep the temperature 0, why should I do that? So you can have a general condition u at 0, t as some p of t. So if you have that boundary condition general boundary condition still you can show that you can solve this problem, okay. So this is the initial boundary value problem for the heat equation non-homogeneous heat equation you can also have the non-homogeneous equation some source is also there h of x, t, okay. How do I do this one? So what you do is let us consider I want to make this boundary as 0 if I have the boundary 0 then I know that its solution, so let us solve that one first, okay.

So if you do this so let v of x, t be the u of x, t minus this p of t if you take this one so that v of x, 0 equal to 0 is pt minus pt that is 0, okay. So u at x, 0 minus p of t so but u at 0, t is p of t minus p of t so that is 0, okay. So you can see that initial data is this and this satisfies if because u satisfying the non-homogeneous equation what happens to so this gives me u of x, t is v of x, t plus p of t. So since this satisfies the non-homogeneous equation and what you see is this implies vt, okay vt plus Pt that is the first term, okay vt plus Pt P is a function of t so it is simply P dash of t, okay Pt means P dash of t times derivative P of t this is this because if you put in at ut minus alpha square uxx uxx is because once you differentiate this Pt with respect to x it will become 0 or simply vxx equal to h of x, t.

So this implies vt minus alpha square vxx equal to h of x, t minus Pt, okay. So v satisfies non-homogeneous term becomes this x is again x is positive t is positive, so domain is not changing so what you have is this one. So v satisfies this equation and I see that this initial data is v at x, 0 equal to 0, okay initial not initial data this should be boundary data, right? v at 0, t this is 0, t u at 0 t is Pt. So the boundary data becomes this one boundary condition is this, what happens to the initial condition? Initial condition is v initial condition u at x, 0 is fx so v at this should be a boundary so you have 0, t is this and v at x, 0 is nothing but u at x, 0 that is fx minus P at 0 that is simply a constant, okay.

So you call this some gx, okay this you call it some h1 non-homogeneous term as some h1 of x, t so this is what is now the problem. So if you use this transformation what you get is the transformed initial boundary value problem, okay. So the transformed initial boundary value problem is it satisfies this non-homogeneous equation boundary data is 0 now and initial condition is initially you have the velocity some gx satisfying, okay.

So because this boundary condition is 0 you can extend this functions involved h1 and v and g you extend as an odd functions, okay and then make use of the solution of the non-homogeneous solution with the initial data, okay that is what we derived here so use this one this is now this is true for every x belongs to R and t positive, what we need is for only x positive, okay. So for x positive if you want to have the solution this boundary condition is automatically satisfied if you actually extend this functions v and h1 and g you extend as an odd functions this is the technique I have done even for wave equation and earlier heat equation with for the homogeneous heat equation, okay.

So if you just do this extension odd extension of these functions unknown function v of x, t and the non-homogeneous function h1 of x, t you extend as a odd function on the from 0 to infinity to minus infinity infinity, okay. So you can do that so if you do this one so odd extension odd extension is I simply give you the outline odd extension is let us call this v1 of x, t is simply v of x, t if x is positive so that is what we have that is the function what we want and you have v of minus x, t if x is negative, so you extend it like this. Similarly you do it for h1 of x, t odd extension let me call this odd v odd, okay. So similarly h1 odd extension is something like h1 of x, t if x is positive and minus h1 of minus x, t if x is negative, so do it like this so that x equal to 0 is actually value is 0, okay for all time t.

So that you need not worry just one point so you have a g also you extend so g odd you extend as a function of x as g of x if x is positive that is what we have for x positive, okay and now minus g of minus x as x negative. Now what happens v odd also satisfies this non-homogeneous equation now because for x positive this satisfies non-homogeneous equation and you can easily see that x negative this is minus v of minus x, t that also satisfies this equation, okay.

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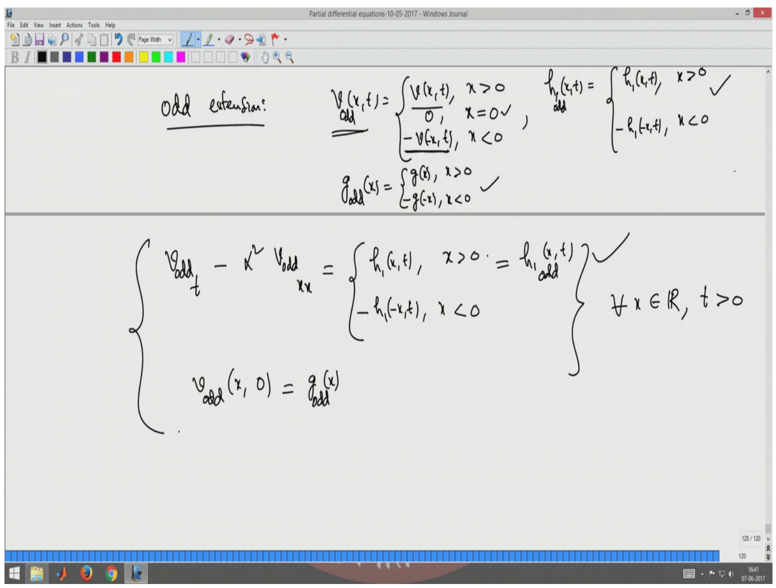

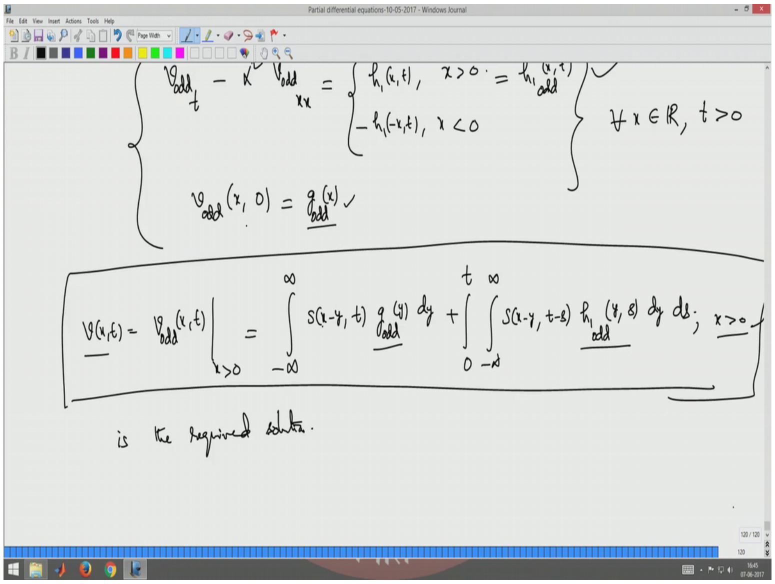

But of course you are what happens your this satisfies different so we will what v odd satisfies v odd actually satisfies v odd t minus alpha square v odd function v odd is a function of x, t so that is xx is actually equal to h1 of x, t for x positive, okay that is what is clearly you can see because it satisfies for x positive v satisfies this non-homogeneous equation, otherwise what is that satisfies? Minus h of minus h1 of minus x, t, okay so if you see that one if you actually substitute and see this one minus vt that is minus is there so I have a minus common, so if you simply substitute this (())(38:01) into this equation, okay if you simply substitute into the vt minus alpha square vxx for this function you calculate time derivative and x to x derivatives and take this vt whatever for this function time derivative minus alpha square 2 x derivatives of this function what you see is minus is common and what you get is this is a simply satisfying minus of h of in the place of x you have minus x.

So you have h1 that is nothing but h1 of minus x, t and x is negative this is we already extended as an odd function that is nothing but h1 odd of x of t. So this satisfies this non-homogeneous equation and you have this boundary condition is automatically satisfied now, okay this is 0 because it is an odd function extension is odd so at x equal to 0 for all times it is 0 so boundary condition is automatically satisfied now we have to see what is this v x, 0.

So v odd at x time t equal to 0 is nothing but if you see this one so this is nothing but v odd is if you see t equal to 0 this becomes gx and minus g of minus x that is nothing but g odd. So you extended as an functions and what you got is this, now this is for every x belongs to full real line and t positive. This problem we solved just now, okay so this is the solution for such problem, so this is exactly this one you have the non-homogeneous equation with non-homogeneous data so this is your solution once you get you write in this as a solution there and then restrict to restrict x to only positive side so x positive whatever is that is your solution.

So we can write we can write that so v of x, t what we are looking for is only for x positive is v odd of x, t restricted to x positive what is this one v of what is v odd x of t this is you can have minus infinity infinity s of x minus y t acting on this g odd of y dy this is my earlier problem so it is a u1 homogeneous solution with this initial data plus non-homogeneous solution with the 0 initial data that is 0 to t minus infinity infinity S of x minus y t minus s now you have h1 odd of y s dy and ds you simply substitute now what is your h1 g odd and h1 odd and you can simply see that so this is what is your solution and wherever x is there x is always positive so x is always positive, okay this is what is a solution this is what you want for x positive only you want this problem, okay.

So you restrict this this is actually this expression is valid for even x negative but your solution is only for x positive, okay. So this x positive wherever x is that is x is only positive and g odd this integral you can split from 0 to infinity g odd is g of y minus infinity is 0 g odd is minus g of minus y and you can do the simplification (())(41:41) this is your then this should be your required solution, okay this is the required solution after simplification you can do by just substituting g odd and splitting by splitting this integral g to 0 to infinity and then minus infinity to 0 make use of this g odd function g odd is gx if it is positive g of minus g of minus x as a negative.

So if you substitute it will take a different form that you can do yourself, okay is the required solution. So this is actually what is the solution this is the solution for this general non-homogeneous equation with boundary condition is the general data. So this is at that point you need not be simply you need not maintain the 0 temperature, so the semi-infinite rod at one end you need not maintain the 0 temperature or you insulate it you can maintain a constant temperature or you can actually change the temperature over the time so as the function P of t in general.

If you do that this problem has a solution and that is we have seen that it is also unique, okay so one can show that it is also unique because you reduce this problem into this type, okay so this one and this problem has a unique solution, okay. So this implies this also has a unique solution so because this problem this is what we have seen this problem is having a unique solution the boundary data is 0 that is what we have seen already.

So that energy argument with one end when you kept 0 temperature or insulate it you know that solution unique. So you reduce this general problem into this type and this has a unique solution that implies finally if v is unique u is also unique, okay so this is how this problem general problem is also you have a unique solution the solution is constructed here in this fashion, okay. So this is we can solve some of the boundary value boundary and initial value problem for the heat equation heat equation is 1 dimensional, okay.

So when the domain is unbounded that is minus infinity infinity or semi-infinite, so in the next video we can have we can solve the heat equation in a finite interval so that is you can consider that is like a situation like you have a infinite if I have a finite rod and is initially you give the temperature on it you and then you have a two boundaries both the ends you can maintain a 0 temperature or same temperature or you can insulate one end you can keep other end you can keep maintain some temperature or you can actually in general you can maintain P of t one side P of t one end you always maintain as a temperature as P of t other end Q of t for all times it is varying over a time, okay. And then what is a diffusion what is the how the temperature varies over a time so that is what that is exactly solving the boundary value initial boundary value problem for the heat equation in a finite domain 0 to L that is a rod, okay 0 to L with the boundary data that is u at 0, t is P of t u at L, t is Q of t.

So this and the initially you give what is the temperature at t equal to 0 what is the temperature of the rod that is some function of x. So this general problem you can solve it by using the by extracting a Sturm liouville problem out of it because it is a finite domain that is what we have seen for the wave equation also, okay. So this heat equation boundary value problem boundary value problem boundary initial value problems initial boundary value problem for the heat equation in a finite domain we can always extract a Sturm liouville problem and then finding eigen values and eigen functions and then make a superposition principle we can find the solutions separation of variables solution, okay.

Separation of variable solutions for this partial differential equation that is the heat equation here that satisfies the initial and boundary conditions that is what we will see in the next video, okay thank you very much.